THE FAST FOURIER TRANSFORM AND ITS APPLICATION TO TIDAL OSCILLATIONS *
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SYNOPSIS
This paper proposes a new way of tidal spectral analysis based on the Cooley-Tukey algorithm, known as the Fast Fourier Transform. The Fast Fourier Transform analysis is used to compute both the harmonic constants of the tide and the power spectrum. The latter is obtained by means of a weighted sum. A new way is also derived to obtain the formula giving the number of the degrees of freedom, on which is based the confidence interval corresponding to the noise spectrum.

1 - INTRODUCTION
Old methods of tidal analysis were developed in order to permit manual calculations with desk calculators. Such analyses can now be considerably improved with the use of electronic computers.

Computers have also facilitated the use of time series analysis of mean sea level fluctuations as influenced by tidal oscillations.

Horn's (1960) least square method and the very similar one developed by Cartwright and Cotton (1963), based on discrete Fourier analysis, are among the earliest methods of tidal analysis entirely dependent upon electronic computers. Both of these

* This is the reprint with minor corrections and improved programs. The former reprint is obsolete.

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methods included approximately the same number of constituents used in Doodson's and other methods of analysis (61 constituents), for which computer programs have been prepared. However Doodson himself (1957) recognized that many more constituents would be necessary to improve predictions, as is observed in his method of computing corrections for predictions based on the classical list of constituents. The corrections were found through analysis of the differences in time and height of recorded and predicted curves. All these methods are lengthy (30 to 60 minutes computer time) and not diagnostic.

A decade ago spectral analysis came into common usage as a diagnostic tool but only quite recently found an application in tidal analysis. Subsequently, it was confirmed that many more shallow water constituents were necessary to represent tidal curves accurately. This was Zetler and Cummings' (1967) conclusion from a study on the port of Anchorage (Alaska), and Lennon and Rossiter's conclusion from the port of London (Lennon, 1969). Both researches showed, independently, that some 50 additional shallow water constituents were necessary to improve predictions.

Munk and Cartwright (1966) provided a completely different approach to the problem considering the "response" of a tidal basin to the driving forces of the equilibrium tide. In addition, shallow water terms are included to take into account the non-linear response of the tidal basin to the principal constituents. The number of such terms, required for an accurate prediction is very much smaller than the ones necessary in the harmonic prediction method. The method, a generalization of the old Laplace method (Franco, 1967), is a neat and thorough approach to tidal prediction. Munk and Cartwright give no quantitative estimation of the computational effort involved; but it is thought to be considerable.

If the harmonic method is preferred, accurate predictions are conditioned by the inclusion of shallow water constituents, which differ from port to port. Unfortunately, work already done shows that results are individual and cannot be extended or transferred from area to area. An example can be found in the comparison between the results from Zetler-Cummings (op. cit.) and Lennon-Rossiter (Lennon, 1968) researches and the figures calculated by Rock (personal communication). However, the diagnosis of
the shallow water effects may be considerably simplified if a method is devised to permit the full use of the Fast Fourier Transform (FFT) (Cooley and Tukey, 1965). The procedure for spectral analysis can be completely changed if means are available to correct, according to the natural angular frequencies of the tidal harmonics, the various Fourier coefficients obtained from the FFT. To show this is one of the aims of this paper.

To give an idea of the advantages of the method, let us examine a flow diagram of the operations involved (Fig. 1). One of the main characteristics of the method is the possibility of using the Fourier coefficients to obtain the power spectrum and the harmonic constants. One can use the tidal values of \( a_j = R_j \cos \phi_j \) and \( b_j = R_j \sin \phi_j \) computed from the Fourier coefficients to correct these coefficients for the tidal effect, and thus isolate the noise contribution to the Fourier series. This is carried out in the tidal frequency bands. The power spectrum obtained through these corrected Fourier coefficients will then be the noise spectrum plus the tidal oscillations not considered in the harmonic analysis.

Although the use of the Cooley-Tukey algorithm (FFT), as a means of simplifying calculations in spectral analysis, was suggested by Zetler (1969), the authors are not aware of any publication on its use as applied to tides. Thus we believe that our method is original.

In order to compare the proposed method with Lennon-Rossiter's a flow diagram is presented for the latter (Fig. 2). The procedure involves a Doodson harmonic analysis, a prediction of hourly heights which is in itself a long task even for electronic computers, and a very lengthy high resolution Fourier analysis for a whole number of lunations. Besides the length of the initial calculation, an additional drawback is that the whole procedure must be repeated to obtain the final spectrum.

2 - FOURIER ANALYSIS

Classical Fourier analysis is one of the methods for determining tidal harmonic constants. In addition to the above

* See list of mathematical symbols annexed.

FIG. 1 - FLOW DIAGRAM OF THE PROPOSED METHOD - ISO CODE FOR POWER SPECIFICATION BY FLIGHTING CLASSICAL DOLDSON ANALYSIS FOR 355 DAYS PREdictION OF ROV REL HTS

FIG. 2 - FLOW DIAGRAM OF THE LENNON-ROSSIER METHOD.
mentioned Catton-Cartwright method, the Miyasaki (1958) method is well known. However, these studies were attached to a whole number of lunations and their analysis cannot take advantage of the Cooley-Tukey algorithm based on sampling at \( N = 2^7 \) points. To show how the harmonic analysis of a tidal curve can be undertaken via the Cooley-Tukey algorithm is the aim of this section.

Suppose that the tidal height at instant \( s \) is given by

\[
y(t) = R_0 + \sum_{j=1}^{Q} R_j \cos (q_j t - r_j) + v(t)
\]

where \( v(t) \) is a "gaussian noise with zero mean". In complex notation we have

\[
y(t) = R_0 + \frac{1}{2} j \sum_{j=1}^{Q} R_j \left[ e^{i(q_j t - r_j)} + e^{-i(q_j t - r_j)} \right] + v(t)
\]

Now if we admit that \( r_j \) and \( q_j \) are negative for negative values of \( j \), and that \( r_j = q_j = 0 \) for \( j = 0 \) it can be written

\[
y(t) = \sum_{j=1}^{Q} R_j \left( e^{-i(q_j t - r_j)} + e^{i(q_j t - r_j)} \right) + v(t)
\]

Finally, if we put

\[
\begin{align*}
\left\{ \frac{1}{2} j R_j e^{-i(q_j t - r_j)} = c_j \quad j \neq 0 \\
R_j = c_j \quad j = 0
\end{align*}
\]

it results

\[
y(t) = \sum_{j=1}^{Q} R_j e^{iq_j t} + v(t) \quad (2a)
\]

If we analyse this curve by using the Fourier technique, it is not possible to adopt the exact angular frequencies \( q_j \). All the discrete angular frequencies of the Fourier analysis are given by

\[
q_n = \frac{2\pi n}{NT} \quad (n = 0, 1, 2, \ldots, \frac{N}{2} - 1) \quad (2b)
\]

where \( t \) is the sampling interval (usually 1 hour). For the exact number of cycles in the same time interval corresponding to the angular frequencies \( q_j \), we have

\[
q_j = \frac{2\pi j}{NT} \quad (2c)
\]
Now the complex Fourier coefficients for curve (2b) are

\[ c_n = \frac{1}{N} \sum_{t=0}^{N-1} y(t) e^{-in\omega t} \quad (n = 0, 1, 2, \ldots, N-1) \]

But if we call \( c_n \) the complex Fourier coefficient obtained from the analysis of \( v(t) \), it follows that

\[ c_n = \frac{1}{N} \sum_{t=0}^{N-1} v(t) e^{-in\omega t} \quad (2e) \]

Then we can write

\[ c_n - c_n^* = \frac{1}{2\pi} \int_{0}^{2\pi} [y(t) - v(t)] e^{-in\omega t} dt \quad (2f) \]

However expression (2e) shows that, for \( T=1 \) hour

\[ q_n N = 2\pi n \]

hence

\[ e^{-in\omega} = e^{-i2\pi n} = 1 \]

Thus (2f) can be modified as follows:

\[ c_n - c_n^* = \frac{1}{2\pi} \left[ \sum_{t=0}^{N-1} [y(t) - v(t)] e^{-in\omega t} - \frac{1}{N} [y(N) - v(N)] \right] \]

But \( v(N) \) is usually small as compared to \( N \), consequently \( v(N)/N \) can be neglected. In addition we can replace \( y(t) - v(t) \) by its value taken from (2b). Consequently, the last expression can be changed into

\[ c_n - c_n^* = \frac{1}{2\pi} \sum_{t=0}^{N-1} e^{-i(t_n - q_j)\omega t} \quad (2g) \]

Now

\[ e^{i\omega t} = \frac{e^{i(u+1)/2} - e^{-i(u+1)/2}}{e^{iu/2} - e^{-iu/2}} \]

but, since

\[ e^{ix} - e^{-ix} = 2i \sin x \]

and

\[ u = (q_n - q_j) \]

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it results

\[
\sum_{q=0}^{N} e^{-i(q - q_j)N/2} \sin \left( \frac{(q - q_j)(N+1)/2}{n} \right) / \sin \left( \frac{(q - q_j)/2}{n} \right)
\]

But

\[e^{-i(q - q_j)N/2} = e^{-i(q_jN/2) - iq_jN/2}\]

or, according to (10), for \(r=1\) hour,

\[e^{-i(q - q_j)N/2} = e^{-i(1)N/2} = (-1)^{n} e^{i(q_jN/2)}\]

Thus

\[\sum_{q=0}^{N} e^{-i(q - q_j)N/2} = e^{i(q_jN/2) = (-1)^{n} e^{i(q_jN/2)}\]

Hence expression (2g) can be changed into

\[
\sum_{q=0}^{N} e^{-i(q - q_j)N/2} = e^{i(q_jN/2) \sin \left( \frac{(q - q_j)(N+1)/2}{n} \right)} / \sin \left( \frac{(q - q_j)/2}{n} \right)
\]

If only positive values of \(j\) are considered, according to (2a) we have

\[
\sum_{q=0}^{N} e^{-i(q - q_j)N/2} = e^{i(q_jN/2) \sin \left( \frac{(q - q_j)(N+1)/2}{n} \right)} / \sin \left( \frac{(q - q_j)/2}{n} \right)
\]

Consequently, if we call

\[(-1)^{n} \sin \left( \frac{(q - q_j)(N+1)/2}{n} \right) / \sin \left( \frac{(q - q_j)/2}{n} \right) = \delta_{nj}\]

and

\[(-1)^{n} \sin \left( \frac{(q - q_j)(N+1)/2}{n} \right) / \sin \left( \frac{(q - q_j)/2}{n} \right) = \delta_{nj}\]

and recall that \(\delta_{j0} = \delta_{n0}\) for \(j=0\), then we have from (2a) and (2b) to (2j)

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\[ c_n = \frac{1}{N} \left( y(n) - \bar{y} \right) \] 
\[ + \frac{1}{2} \beta_j \left[ e^{i \left( (r_j - q_j) \pi / 2 \right)} a_{aj} \right] \] 
\[ + e^{i \left( (r_j - q_j) \pi / 2 \right)} b_{bj} \] 
\[ (2k) \]

Now it is well known that the pair of trigonometric Fourier coefficients \( (a_n, b_n) \), for the total oscillation, and \( (c_n, \eta_n) \) for the noise oscillation, are related to the complex Fourier coefficients, respectively by

\[ c_n = \frac{(a_n - ib_n)}{2} \] 
\[ \eta_n = \frac{(c_n - i\eta_n)}{2} \]

Thus we can develop the exponentials of (2k) and write

\[ a_n = \frac{1}{N} \left[ y(n) - \bar{y} \right] - \frac{1}{2} \beta_j \left[ e^{i \left( (r_j - q_j) \pi / 2 \right)} (a_{aj} + b_{bj}) \right] \] 
\[ - i \beta_j \sin \left( (r_j - q_j) \pi / 2 \right) (a_{aj} - b_{bj}) \]

Now, equating the real and the imaginary parts and considering the \( N/2 \) values of \( a_n \) and \( b_n \) obtained from the Fourier analysis, the following independent systems can be written:

\[ (a_n + \frac{1}{N} \left[ y(n) - \bar{y} \right] - c_n) = \| A_{aj} + B_{bj} \| (a_j) \] 
\[ (b_n - c_n) = \| A_{aj} - B_{bj} \| (b_j) \]

where

\[ a_j = B_j \cos \left( (r_j - q_j) \pi / 2 \right) \]
\[ b_j = B_j \sin \left( (r_j - q_j) \pi / 2 \right) \]

Since \( Q < N/2 \) the systems (2m) and (2n) are redundant and can be solved by the least square method according to the conditions

\[ \| A_{aj} \| = \text{minimum} \] 
\[ \| B_{bj} \| = \text{minimum} \]

respective. But this can be simplified because constituents of different species practically do not contaminate each other. Consequently, the systems (2m) and (2n) may be split into sub-systems, one pair for each species. The frequency band containing the diurnal constituents, for example, is limited by 290 ≤ n ≤ 380 cycles per 8192 hours (2^13), and therefore, about 90 equations exist for computing about 20 values of both \( a_j \) and \( b_j \). These redundant systems are of the form:

\[
M[X] = [L]
\]

The corresponding normal equations are

\[
N^T M[X] = N^T [L]
\]

where \( T \) indicates transposition of matrix \( M \). These systems can be solved by inverting the square matrix \( N^T M \) which gives \( (N^T M)^{-1} \) and pre-multiplying both members of (2p) by \( (N^T M)^{-1} \):

\[
(X) = (N^T M)^{-1} N^T [L]
\]

If unknowns are to be determined directly in terms of \( L \) then the matrix

\[
(N^T M)^{-1} N^T = M^T
\]

must be found in order to give

\[
(X) = M^T [L]
\]

It must be pointed out that the separation of unknowns \( a_j \) and \( b_j \) into two independent systems results from adding the correction \( \frac{1}{2} \{ y(N)-a_0 \} \) to all values of \( a_n \). Although no fixed central time has been established beforehand, it is interesting to note that expressions (2o) contain the phase correction \( \frac{\pi}{2} \), which is an adjustment of \( \omega \) to the central time \( \pi \).

In order to give a clear idea of the results to be expected with an analysis of 8192 samples, the inverse of the normal matrices \( (N^T M)^{-1} \) is presented in Table 2-1, for the semidiurnal constituents. The dominant diagonal indicates that good results may be obtained.

Once known \( a_j \) and \( b_j \), the Fourier coefficients \( y_n \) and \( y_n' \), corresponding to the noise, can be found by taking from (2e) and

\[ y_n \]


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Appendices II and III, respectively, contain the computer programs to solve systems (2q) and compute the residuals $\xi_n$ and $\eta_n$.

3 - SPECTRAL ANALYSIS

Let us take a normalized oscillation expressed by

$$x(t) = \frac{1}{Q} \sum_{j} \epsilon_j e^{i\xi_j t}$$

(3a)

where

$$\epsilon_j = \begin{cases} \frac{1}{Q} \theta_j e^{-i\xi_j} & \text{for } j \neq 0 \\ \epsilon_0 = 0 & \text{for } j = 0 \end{cases}$$

(3b)

Suppose that another normalized oscillation has harmonic terms with the same angular frequencies but with different amplitudes and phases. Such an oscillation can be expressed by

$$y(t) = \frac{1}{Q} \sum_{k} \epsilon_k' e^{i\xi_k t}$$

(3c)

where

$$\epsilon_k' = \begin{cases} 0 & \text{for } k = 0 \\ \epsilon_k = \frac{1}{2} \theta_k e^{-i\xi_k} & \text{for } k \neq 0 \end{cases}$$

(3d)

For a time $(t-\delta)$ where $\delta$ is any time lag counted from $t$, formula (3c) can be changed into

$$y(t-\delta) = \frac{1}{Q} \sum_{k} \epsilon_k' e^{i\xi_k(t-\delta)}$$

The product of this expression by \( (3a) \) gives

\[ x(t)y(t-G) = \sum_{j} \sum_{k} c_j e^{ikG} e^{i(j-k)t} \]

or, if we call

\[ c_j e^{ikG} e^{i(j-k)t} = \delta_{jk} \quad (3e) \]

and

\[ \delta_{jk} = \nu_{jk} \quad (3f) \]

\[ x(t)y(t-G) = \sum_{j} \sum_{k} \delta_{jk} e^{iujkt} \]

The mean value of this product over \(-T/2 < t < T/2\), is

\[ K(G) = \langle x(t)y(t-G) \rangle = \sum_{j} \sum_{k} \delta_{jk} \int_{-T/2}^{T/2} e^{iujkt} dt \]

\[ = j \sum_{k} \delta_{jk} \left( e^{iu_{jk}T/2} - e^{-iu_{jk}T/2} \right) \]

\[ = j \sum_{k} \delta_{jk} \left( 2 \sin u_{jk}T/2 \right) \]

Now, since \( \delta_{jk} = \nu_{jk} \), we conclude from expression (3f) that \( \nu_{jk} = 0 \) for \( k = j \). Consequently, since \( (\sin x)/x = 1 \) for \( x = 0 \), it follows that

\[ K(G) = j \sum_{j} \delta_{j} \left( 2 \sin u_{j}T/2 \right) \]

But the function \((\sin x)/x\) decays very quickly when \( x \) increases. Thus \( (\sin u_{j}T/2)/u_{j}T/2 \) will be small for large values of \( u_{j}T/2 \). Hence, for the usual values of \( u_{j} \), the second term of the above expression is negligible when \( T \) is large. Consequently, if we take (3e) into account, we have as a good approximation of (3g)

\[ K(G) = j \sum_{j} \delta_{j} e^{iu_{j}T/2} \]

However, expressions (3b) and (3d) show that we can write:

\[ e^{j\omega t} - e^{-j\omega t} = 2j \sin(\omega t) \quad (j \neq 0) \]

\[ e^{j\omega t} - e^{-j\omega t} = 0 \quad (j = 0) \] (3b)

Thus

\[ K(0) = \sum_{j=1}^{\infty} \gamma_j e^{-jk\theta_0} \] (31)

From (3b) and (31) we can derive the trigonometrical form of (31):

\[ K(0) = \sum_{j=1}^{\infty} \frac{1}{2} \frac{1}{2} \sum_{n=-N/2}^{N/2-1} \gamma_n e^{-jk\theta_0} \cos(j+n)(\theta_0 - \theta_j) \] (3i)

If a Fourier analysis exists for discrete values of \( x(t) \) and \( y(t) \) with \( N \) hourly heights, then, a good estimate of \( K(0) \) can be expressed by

\[ K(0) = \sum_{n=-N/2}^{N/2-1} \gamma_n e^{-jk\theta_0} \] (3j)

where

\[ \gamma_n = \frac{1}{2} \sum_{k=1}^{\infty} \gamma_k e^{-jk\theta_0} \quad (n \neq 0) \]

\[ \gamma_n = 0 \quad (n = 0) \] (3k)

If values of \( K(0) \) are known by averaging the products \( x(t)y(t+\theta) \) for continuous values of \( \theta \), the Fourier analysis of \( K(0) \) will be the usual way of obtaining an estimate of amplitudes \( \frac{1}{2} \sum_{k=1}^{\infty} \gamma_k e^{-jk\theta_0} \) and phases \( (j+n)(\theta_0 - \theta_j) \); this will give for \( -\theta_0 / \theta \):

\[ c_n = \frac{1}{2\pi\theta} \int_{-\theta_0}^{\theta_0} K(0) e^{-ink\theta_0} d\theta \] (31)

where

\[ c_n = \frac{2\pi}{2\pi} = \frac{\pi}{\theta} \] (3m)

But according to (3j) a good approximation is given by
\[ c''_n = \frac{1}{2\pi} \sum_{m=-N/2+1}^{N/2-1} \gamma_m e^{-i(q_m - q_0)m} \]

or
\[ c''_n = \frac{1}{2\pi} \sum_{m=-N/2+1}^{N/2-1} \gamma_m \sin \left( \frac{(q_m - q_0)\pi}{N/2} \right) \]

However, the Fourier analysis of \( x(t) \) and \( y(t) \) give, respectively,
\[
\begin{align*}
\epsilon_n &= R_n \cos r_n \\
\eta_n &= R_n \sin r_n
\end{align*}
\]
and, if we designate the complex conjugate of \( c''_n \) by \( c''_n^* \), it follows that
\[
\begin{align*}
c_n &= \frac{1}{2} \left( \epsilon_n + \epsilon_{n}^{*} \right) e^{-i\phi_n} \\
c''_n &= \frac{1}{2} \left( R_n \cos r_n + R_{n}^{*} \cos r_{n}^{*} \right) e^{-i\phi_n}
\end{align*}
\]

Thus,
\[ c_n = (\epsilon_n + \epsilon_n^{*})/2 = \frac{1}{2} R_n e^{-i\phi_n} \]
\[ c''_n = (R_n \cos r_n + R_{n}^{*} \cos r_{n}^{*})/2 = \frac{1}{2} R_{n} e^{-i\phi_{n}} \]

and, according to (3k)
\[ c_n^* c_{n}^{*'} = \gamma_n \]

Thus
\[ c''_n = \frac{1}{4\pi} \sum_{m=-N/2+1}^{N/2-1} \gamma_m \sin \left( \frac{(q_m - q_0)\pi}{N/2} \right) \]

Since \( c_{n}^* = c_{n}^{*'} \) we can avoid the summation through negative values of \( n \) by changing (3.0) into
\[ c''_n = \frac{1}{4\pi} \sum_{m=0}^{N/2-1} \gamma_m \sin \left( \frac{(q_m - q_0)\pi}{N/2} \right) \]

Hence, the cross spectrum $c''_{xy}$ is a weighted sum of the values of $c_n c'^*_{n'}$ and $c_{n'} c'^*_{n}$, the maximum weight corresponding to $q_n = q_{n'}$.

Since $c_n$ and $c_{n'}$ are the complex Fourier coefficients resulting from the analysis of $N$ values of $x(t)$ and $y(t)$, respectively, and with sampling interval equal to $\tau$ the angular frequency will be $q_n = 2\pi n/N\tau$. Thus, according to (30) we have:

$$q_n \pm q_{n'} = 2\pi (mN/2^m \pm s)/N\tau$$

or, if we call

$$s N\tau/2^m = p$$

so that $p$ is an integer, then

$$q_n q_{n'} = 2\pi (pn)/N\tau$$

Consequently, if we replace this value of $q_n q_{n'}$ in (30) and put $c''_{xy}(p)$, this will be the cross spectrum estimate centered at $p$:

$$c''_{xy}(p) = \frac{N/2-1}{n=0} \left[ c_n c'^*_{n'} \sin \left[ 2\pi (p-n)\mu N\tau \right] + c_{n'} c'^*_{n} \sin \left[ 2\pi (p+n)\mu N\tau \right] \right]$$

Function $(\sin(m)/mx)$ decays very quickly when $x>2$. Thus the limits of summation can be conveniently reduced according to the condition

$$(p-n)2m/N\tau = 2$$

and

$$(p+n)2m/N\tau = 2$$

In the first case we derive

$$n = \begin{cases} 
\mu & p = p^{\text{NT}}/m \\
\mu' & p = p^{\text{NT}}/m 
\end{cases}$$

and in the second

\[ n = n_1 = \frac{Nt}{m-p}\quad (3r) \]

thus

\[ S_{xy}(p) = \sum_{n=0}^{\infty} e^{-t} \sin\left[\frac{\pi(p-n)2\pi}{NT}\right] \]

The term in (p+n) of this formula can be neglected for values of p greater than NT/m. Since the practical values of NT/m are not usually high, then the second term must only be used for very low frequencies. Consequently it is possible to discuss the behaviour of only the (p-n) term. The graph of Fig. 3 represented by the dashed line is the curve of equation:

\[ \theta(p-n) = \sin\left[\frac{\pi(p-n)2\pi}{NT}\right] \]

for 2m/NT = 1.

The figure shows two undesirable side lobes. If we remember that \( S_{xy}(p) \) is only an estimate these side lobes can be eliminated by multiplying the cross correlation function \( R(t) \) by the fading
function \((1 + \cos n\theta/m)\). Thus according to (3j) we have

\[
E(0) (1 + \cos n\theta/m) = \frac{N}{2} \sum_{-N/2}^{N/2-1} \frac{1}{2} (2\cos\theta/m + e^{-i\theta/m}) e^{i\theta} 0
\]

Now we replace the value of \(E(0)\) in (3j) by the second member of the above expression and integrate, in the same way we obtained (3h). Then we use the same arguments with which we arrived at (3s) and find

\[
\eta(p) = \frac{n}{\pi} \sin \left( \frac{n(p+n\pi)}{2m/NT} \right) \right]
\]

where

\[
\psi(p) = \frac{n}{\pi} \sin \left( \frac{n(p+n\pi)}{2m/NT} \right) \right]
\]

This function is represented in Fig. 1 by the solid line. It is seen that this function is negligible for arguments greater than 2.

According to (3q) the "filter" \(\psi(p)\) covers \(2m/\pi\) harmonics. It will be shown later that this quantity has an important statistical meaning.

The power spectrum density is obtained as a particular case of the cross spectrum. In fact if we establish the correlation for the same function, with a lag \(\theta\) we have from (3h) for \(\gamma_{1J}\) and \(\gamma'_{1J}\)

\[
\gamma_{1J} = \frac{1}{N} \sum_{n=0}^{N-1} e^{-in\pi/4m} \gamma_{1J} e^{i\theta}
\]

and expression (3l) takes the form

\[
A(\theta) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{i\theta} e^{-in\theta} e^{i\theta}
\]

or, in trigonometric form,

\[
A(\theta) = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} e^{i\theta} e^{-in\theta} e^{i\theta}
\]

which is the autocorrelation function. Its Fourier analysis will give an estimate of the power spectral density \(\frac{1}{2} \tilde{R}^2(p)\) centered.
in the harmonic of order \( p \). Such an estimate can be derived immediately from (36) by making \( c_n = c'_n \), which gives

\[
\mathbb{E}_{yy}(p) = \sum_{n=0}^{\infty} |c_n|^2 \phi(p-n) + \sum_{n=0}^{\infty} |c_n|^2 \phi(p+n)
\]  

(36)

where

\[
|c_n|^2 = (\alpha_n^2 + \beta_n^2)/4
\]  

(36)

A computer program for finding \( \mathbb{E}_{yy}(p) \) is given in Appendix IV.

In order to establish the filter's maximum width and still avoid mixing different species of tides, the following method is suggested. Let \( q_s \) and \( q_{s+1} \), respectively, be the largest angular frequency of species \( s \) and the smallest angular frequency of species \( s+1 \). Thus, if \( n_s \) and \( n_{s+1} \) are the respective frequencies in units of the fundamental frequency \( 1/NT \), then we can write

\[
q_s = 360^n_s \text{N} \quad \text{and} \quad q_{s+1} = 360^n_{s+1} \text{N},
\]

consequently

\[
n_{s+1} = n_s = 360^n_s (n_{s+1} - n_s) \text{N},
\]

and

\[
n_{s+1} = n_s = (q_{s+1} - q_s) \text{N}/3600^N.
\]

From an extended table of tidal constituents as derived by Zetler-Cummings or Lennon-Rossiter it can be concluded that \( q_{s+1} - q_s \) is about \( 11^N \) which gives for the maximum filter width

\[
n_{s+1} = n_s = 0.015^N.
\]  

(3y)

4 - THE NOISE SPECTRUM

It was shown in section 2 that \( c_0 \) is the Fourier complex coefficient of the noise analysis. Thus, according to (36), the noise power spectrum is given by

\[
\mathbb{E}_{uv}(p) = \sum_{n=0}^{\infty} |c_n|^2 \phi(p+n)
\]  

(4x)

where, according to (3w) and (21)
\[
|\tau_n|^2 = (\xi_n^2 + \eta_n^2)/4
\] (4b)
Values of $\xi_n$ and $\eta_n$ can be found for the tidal frequency bands through (2r) and (2e). Outside these bands it can be assumed that coefficients $\xi_n$ and $\eta_n$ given the FFT do not have tidal contributions. Thus the values of $|\xi_n|^2$ to be introduced into (4a) are those found through (2r) and (2e) for the tidal frequency bands and $|\eta_n|^2$ given by the FFT, outside these bands. This procedure corresponds to the usual "prewhitening" which consists in obtaining the power spectrum of the residuals equal to the difference between the actual and predicted tides.

Since the known tidal effect has been eliminated before the energy density power spectrum has been determined, any spike of such a spectrum may be understood as the effect of a tidal constituent not included in the matrices. However, if no spike appears in the spectrum we can admit that only gaussian noise is present.

In order to establish the "confidence interval" of $\hat{\xi}_{\nu\nu}(p)$ when $v(t)$ is a gaussian noise, we begin by simplifying (4a). In fact, the term in $\eta_n$ only applies to very low frequencies and thus expressions (4a) and (4b) give

\[
\hat{\xi}_{\nu\nu}(p) = \sum_{n=m}^{n'} (\xi_n^2 + \eta_n^2) \phi(p-n)
\] (4c)

Now, since $v(t)$ is gaussian and it is linked to $\xi_n$ and $\eta_n$ through a linear equation, then $\xi_n$ and $\eta_n$ are also gaussian. Thus, it is reasonable to assume that the mean values of $\xi_n^2$ and $\eta_n^2$ are nearly equal to the same quantity, say, $\bar{\nu}^2$. Consequently, we have from (4c) the following approximation:

\[
4\hat{\xi}_{\nu\nu}(p) = \bar{\nu}^2 \sum_{n=m}^{n'} 2\phi(p-n)
\]

or, if we call
\[
2 \sum_{n=m}^{n'} \phi(p-n) = \psi
\] (4d)

then

$$\Delta \theta_v^2(p) = \theta_v^2$$  (4a)

But a better approximation will be reached if we return from \(\theta_v^2\) to
the individual values of \(\theta_v^2\) and \(n_v^2\). In this case we must have \(v/2\)
values of \(\theta_v^2\) and \(v/2\) values of \(n_v^2\), hence

$$\Delta \theta_v^2(p) = \frac{p+\nu/2}{2} \left( \theta_v^2 + \frac{n_v^2}{2} \right)$$

which is a chi-squared distribution with \(v\) degrees of freedom. In
order to find \(v\) let us recall that the approximate area of the
curve \(\theta(p-n)\) is given by

$$\text{Area} = \frac{\pi}{2}, \theta(p-n) \Delta n$$  (4f)

where \(\Delta n=1\). But a close approximation of the curve of \(\theta(p-n)\)
between the limits shown in Fig.4

$$\sigma = \frac{n}{\sqrt{\pi}}$$  (4g)

Fig.4 - Function \(\theta(p-n)\)

Hence, since we assume that \(\theta(p-n) = 0\) for \(|x|>\pi\), the area of the
curve is

$$\text{Area} = \int_{-1/2}^{1/2} \left[ (1-n^2x^2)^2 + (1-n^2x^2)^2 \right] dx = \frac{2n}{\pi} = 1/n$$

or, according to (4d), (4f) and (4g)

$$\nu = 2n/\pi$$  (4h)

This is the result which is found in classical books by following
a much more complicated procedure.

Since \(\nu\) is known it is possible to determine the "confidence"
interval of \(\Delta \theta_v^2(p)\). This is an interval where the value of
\(\Delta \theta_v^2(p)\) will have 95% probability to be included. The extreme
Values of $\Delta \nu_n(p)$ are found with the aid of the coefficients taken from Table 4-1 (Munk, Snodgrass & Tucker, 1959) which must be multiplied by $4\nu_n(p)$ to give the extreme values. Thus it is evident that the confidence limits of $\delta \nu_n(p)$ are $1/4$ of those of $\Delta \nu_n(p)$. Consequently, these limits are the products of $\delta \nu_n(p)$ multiplied by the coefficients.

**TABLE 4-1**

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>Conf.</th>
<th>$\nu$</th>
<th>Conf.</th>
<th>$\nu$</th>
<th>Conf.</th>
<th>$\nu$</th>
<th>Conf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>1</td>
<td>0.2</td>
<td>1</td>
<td>0.2</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>0.31</td>
<td>4</td>
<td>0.30</td>
<td>8</td>
<td>0.43</td>
<td>15</td>
<td>0.63</td>
</tr>
<tr>
<td>1</td>
<td>0.32</td>
<td>16</td>
<td>0.42</td>
<td>4.8</td>
<td>0.55</td>
<td>2.4</td>
<td>0.78</td>
</tr>
</tbody>
</table>

Expression (4b) shows that if $m$ is increased the value of $\nu$ is reduced for a constant value of the parameter $2\nu T$. Consequently, the accuracy of the analysis, from the statistical point of view, is greater for large values of $\nu$. However, expression (3b) shows that the larger $m$ is, the larger is the filter width and the less is the resolution. Hence $m$ must be fixed according to the purpose of the analysis.

It was seen that formula (4g) gives only an approximate value of $\nu$. But a more accurate value can be obtained from the analysis itself. In fact, if we know $4\nu_n(p)$, expression (4e) can be considered an actual equality where $\nu$ is not known and is given by

$$\nu = \frac{4\nu_n(p)}{\nu_n^2} \quad (4i)$$

where

$$\nu_n^2 = \frac{N^{1/2}}{s^2} \left( \frac{1}{n_1} + \frac{n_2}{n} \right) \quad (4j)$$

is the mean value of the noise energy.

Bol. Inst. oceanogr. S. Paulo, 20(145-199), 1971
5 - CONCLUSION

It is interesting to quote the following statement made by Franco (1970): "

"We are now in a position to foresee a new development of this subject so far as tidal analysis is concerned. In actual fact, it is not usual to take advantage of the Fourier analysis used to obtain the power spectrum to compute tidal harmonic constants. However, we believe that either the Myazaki or the Cartwright-Catton method may be used to "adjust" the Fourier analysis to the angular frequency of the astronomical constituents in order to find these constants. If so, we will be able to "fish" the needed harmonic terms from among those given by the Fourier analysis. The only objection is that the span is tied to a power of 2 and not to a classical multiple of one lunation. We know, however, that some least square analyses have been effected with no regard paid to the conventional spans and that the results were shown to be correct. Hence we hope to find an economical solution for avoiding heavy supplementary computations in order to arrive at the harmonic constants from the Fourier analysis itself, such as it is used to obtain the power spectrum."

Thus the present work confirms the above statement. The Cartwright-Catton method in fact has been extended by the addition of a number of redundant equations which adjust the Fourier coefficients to the known constituent terms. Consequently, the method is formally similar to the least square method and table 2-1 shows how the inverse matrices to obtain $R \cos r$ and $R \sin r$ are well conditioned. Each matrix is inverted by species, the band of Fourier frequencies slightly exceeding the known tidal frequency band.

The central time used in the Cartwright-Catton method does not permit the use of the FFT algorithm; but the difficulty can be overcome by adding a small correction to the Fourier coefficient of the cosine term.

The fact that tidal analysis is not necessarily tied to a whole number of semi-lunations was demonstrated by Munk and Hasselman (1964). They made it clear that a good separation of the constituents with frequencies $f_1$, $f_2$ depends only on the length.
record T and the signal/noise ratio. They proved, that for the usual noise level, resolution can be better than 1/T if

$$|\hat{\epsilon}_T - \tilde{\epsilon}_n| > 1/\text{Signal/Noise}$$

Godin (1970) studied very recently the effect of background noise on the resolution of the tidal constituents. His conclusion is that for constituents with very different frequencies such an effect can be disregarded even for very short spans. "However the noise does disturb drastically the resolution of close constituents and actually prohibits the attempt of resolving constituents whose relative phase difference is less than a given minimum value". In addition he says that in his personal experience of tidal analysis, components with close frequencies can be resolved if the phase shift is about 288°. Since his approach is through the least square analysis we believe that the same results can be reached with the procedure here described.

Godin's criterion is adopted to select new constituents we are able to search on the line spectrum of the Fourier analysis, for the new constituents which can be considered in the analysis of the residuals. In order to give an idea of the work involved in such a selection, let us take the example of the semidiurnal tide. Table 5-1 shows amplitudes \( \Delta m = \sqrt{\tilde{m}_n^2 + \tilde{\epsilon}_n^2} \) of the residuals corresponding to the angular frequencies \( \tilde{\epsilon}_n \). Resolution of the Fourier analysis is about 0.04 degrees per hour, which corresponds to 328° in 8192 hours. However, if Godin's criterion is adopted, the difference between the hourly speeds of the old and new constituents must be \( \Delta \tilde{\epsilon}_n = 288/8192 = 0.035 \). The hourly speeds of the new constituents are obtained, as usually, through the combinations of the hourly speeds of the main constituents.

By finding the residual amplitudes

$$\Delta m = \sqrt{\tilde{m}_n^2 + \tilde{\epsilon}_n^2}$$

and the hourly speeds

$$\tilde{\epsilon}_n = 360n/N$$


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for the values of $n$ (cycles per period $N$), it is possible to
organize tables for the frequency bands of the tidal oscillation,
e.g. Table 5-1 where one can see the $q$ values corresponding to
important values of $n$. Thus, it is possible to search for new
shallow-water constituents having hourly speeds near the tabu-
lated values of $q$. Such a selection is based upon the same oper-
ations indicated by Doodson (1928). These operations are very
delicate, and difficulties appear in the selection of the consti-
tuents. For instance, there exists the possibility of obtaining
the same hourly speed with different combinations. In such case,
one should select the combination, the amplitude constituents of which give the largest product. Since the nodal factors of the compound constituents are given by the product of the individual node factors, only a very long period analysis (18.67 years) will show the appropriate combination.

Table 5-1 shows that residual amplitudes resulting from cleaning the tidal spectrum from the 18 classical (Doodson) semi-diurnal constituents are very small indeed. According to Zetler - Cummings (1967) these residuals in the semi-diurnal band do not justify the extra work of searching for new constituents.

The above mentioned choice of new constituents is sufficient inside the frequency bands. However, only a more elaborate spectral analysis will show all the frequency bands to which the research must be extended. In addition, such an analysis will show the statistical accuracy of the results.

Appendices II and III are the programs to compute the values for $s_i$ and $b_j$ for any number of tidal constituents.

It remains to draw some most important conclusions about the search for new constituents. Fig. 5 shows that some energy is due to the fifth diurnal tide which is not represented by the classical 61 constituents. It is obvious that such a peak would persist in the residual spectrum resulting from the removal of the 61 constituents. Fig. 6 shows that peak (solid line). In the figure the interrupted line represents the power spectrum of the residuals resulting from a 167 constituents tidal analysis. These constituents, extended up to the 12$^{th}$ diurnal species, except for the 3$^{rd}$ and 5$^{th}$ diurnal species, did not show any improvement on the spectrum of the residuals. In fact the use of 29 semi-diurnal constituents, instead of the classical 18 constituents, increased the residual energy of the power spectrum. The same can be repeated for the 4$^{th}$ diurnal species. Thus, it was decided to use the classical constituents only for the 1$^{st}$, 2$^{nd}$, 3$^{rd}$ and 6$^{th}$ diurnal species and new shallow water constituents to represent the 3$^{rd}$, 4$^{th}$ and 5$^{th}$ diurnal species. The final result can be seen in Fig. 7.

*Existence of tidal cusps according to Munk, Zetler & Groves (1965) were considered.

The total number of constituents corresponding to the final result is 82.

One of the most important steps in the analysis, that of generating new shallow water constituents to explain observed peaks, is shown in the flow diagram as a manual process. A computer program has been written to perform this function, but it is not included herein for the sake of simplicity.

The implications of the method in the field of tidal analysis are many and varied, but perhaps the most important is that an increase in the number of points analysed (in this case 8192 hourly readings) in order to increase the resolution of the analysis does not occasion a disproportionate increase in computer processing time.

Although the entire analysis was carried out on an IBM/360/44, a high powered computer is not essential, since the FFT and the matrix operations can be carried out in a series of separate stages. Thus with certain program modifications a 16K word memory with suitable high-speed input/output facilities should be sufficient.

RESUMO

Este trabalho propõe um novo caminho para a análise espectral da maré baseada no algoritmo de Cooley-Tukey. A análise através da "Transformação Rápida de Fourier" (Fast Fourier Transform - FFT) é empregada tanto para calcular as constantes harmônicas da maré quanto para obter o espectro de energia. Este é calculado por meio de uma soma ponderada. Também é dada uma nova dedução da fórmula que exprime o número de graus de liberdade em que se baseia o intervalo de confiança correspondente ao espectro do ruído.

O trabalho foi redigido em inglês a fim de facilitar o intercâmbio de informações.
REFERENCES

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COOLEY, J.W. & TUCKER, J.W.

DODDSON, A.T.

FRANCO, A.S.

HORN, W.

HORN, W.V.

MIYASAKI, M.

MUNK, W.H. & CARTWRIGHT, D.E.

MUNK, W.H. & HASSELMANN, K.


**LIST OF SYMBOLS USED**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a,b</td>
<td>cosine/sine component</td>
</tr>
<tr>
<td>A,B</td>
<td>cosine/sine matrices</td>
</tr>
<tr>
<td>A(k)</td>
<td>auto-correlation function for time lag k</td>
</tr>
<tr>
<td>c</td>
<td>vector denoting phase and amplitude of oscillation</td>
</tr>
<tr>
<td>C_s</td>
<td>spectral estimate at frequency clf (lf = fundamental frequency)</td>
</tr>
<tr>
<td>F_i</td>
<td>nearest Fourier frequency to i th tidal constituent</td>
</tr>
<tr>
<td>F_i</td>
<td>subscript denoting tidal constituent</td>
</tr>
<tr>
<td>K(k)</td>
<td>cross-correlation function for time lag k</td>
</tr>
<tr>
<td>n</td>
<td>maximum number of lags</td>
</tr>
<tr>
<td>n</td>
<td>subscript denoting Fourier number</td>
</tr>
<tr>
<td>N</td>
<td>number of values in Fourier series</td>
</tr>
<tr>
<td>N</td>
<td>subscript denoting Fourier number</td>
</tr>
<tr>
<td>Q</td>
<td>number of tidal constituents</td>
</tr>
<tr>
<td>c</td>
<td>phase lag reckoned from the time origin</td>
</tr>
<tr>
<td>s</td>
<td>amplitude of tidal constituent</td>
</tr>
<tr>
<td>s</td>
<td>index denoting discrete frequency of spectral estimate</td>
</tr>
<tr>
<td>S_y</td>
<td>cross-spectral estimate between series x and y</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>v(t)</td>
<td>gaussian noise as a function of time</td>
</tr>
<tr>
<td>y(t)</td>
<td>time series</td>
</tr>
<tr>
<td>y(t)</td>
<td>tidal heights as a function of time</td>
</tr>
<tr>
<td>s</td>
<td>index for values of discrete weighting function</td>
</tr>
<tr>
<td>n</td>
<td>damping coefficient for weighting function</td>
</tr>
<tr>
<td>F</td>
<td>raw Fourier spectral estimate</td>
</tr>
<tr>
<td>c</td>
<td>vector denoting phase and amplitude of random oscillation</td>
</tr>
<tr>
<td>n</td>
<td>time lag</td>
</tr>
<tr>
<td>D,γ</td>
<td>weighting functions</td>
</tr>
<tr>
<td>H_n</td>
<td>r.m.s. amplitude of white noise</td>
</tr>
<tr>
<td>v</td>
<td>degrees of freedom</td>
</tr>
<tr>
<td>u</td>
<td>sampling interval</td>
</tr>
<tr>
<td>w</td>
<td>angular speed difference/sum</td>
</tr>
<tr>
<td>ζ,η</td>
<td>cosine/sine of residual noise</td>
</tr>
</tbody>
</table>

**ACKNOWLEDGEMENTS**

Grateful appreciation is extended to the "Instituto de Física", USP, for allowing unrestricted use of the IBM 360 computer installation.

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A special thank you to Mr. Sylvio José Correia for his valuable collaboration in the preparation of computer programs in Appendices V, VI.

APPENDICES

SET OF PROGRAMS FOR TIDAL ANALYSIS

by the

"Instituto Oceanográfico" Method

N.B. The programs are presented separately to provide greater flexibility to the user. In practice, however, programs in Appendices I, II, III, IV, are interconnected and executed sequentially by the computer.

INPUT OF DATA

I - F.I.T. PROGRAM -

HEADER CARD
GAMA = Power of two \(2^{\text{GAMA}}\)

MAIN DECK
Y(I) = Tidal heights every hour

II - MATRIX GENERATION -

HEADER CARD
N = No. of values in series \(2^{\text{GAMA}}\)
NBLOC = No. of tidal species present

TIDAL CARDS
SPEED = Angular speed of constituent in degrees per solar hour

SHALLOW WATER
COMPOSITION FACTORS = ICOM(I), Positive or negative integers indicating the composition of the shallow water constituent in terms of 30 fundamental constituents. (Sa, Ssa excepted)

PRINCIPAL CONSTITUENT NO. = Number denoting one of thirty-two principal constituents (see list); if not then =0

CON = Symbolic name of constituent

III - TITLE CARD -

TITLE CARD
YEAR CARD
YEAR = Year of the start of the series
N = No. of values in tidal series
NDAY = First day of the series according to the Julian calendar

HEADER CARD
N = No. of values in tidal series 14
NBLOC = No. of species to be resolved 14
YN = Last value (n^th) of tidal series F4.0
NTAPE = 1 if matrices are written on same tape as Fourier series, else = 2 14
NTAPE = Symbolic tape unit for output of residual Fourier series 14

IV - AUTO-SPECTRUM
HEADER CARD
N = No. of values in tidal series
NO = No. of series to be processed
MDIV = Arbitrary divider to increase the resolution from a predetermined minimum (>5)
L = When MDIV is not specified the resolution can be controlled by specifying the half-filter width L
NTAPE = If non-zero then auto-spectra are written on Magnetic Tape (2), else print-out and punch-out of auto-spectra are effected.

LIST OF 32 PRINCIPAL CONSTITUENTS -
Sa, Ssa, Ma, MF, M2, N2, Q1, O1, G1, M1, X1, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100.
APPENDIX I

FAST FOURIER TRANSFORM
COMPLEX YA(0192), MK, YF2, AUX
INTEGER*2 BETA(0192), GAM, COEF, CO2
REAL BY(0192)
DIMENSION S(2049)
EQUIVALENCE (Y(1), YA(1))
READ(5, 1000) GAM
REWIND 4
N = 2**GAM
XN = N
FACT = 2.0/318/XN
FCT = XN/360.
ND0 = FCT
ND4 = ND8 = ND4 & 2
ND41 = ND4 & 1
ND2 = ND4 & ND4
ND22 = ND2 & 2
ND8 = ND4 & 2
C generating the sine series
ANG = FACT * ND8
S(ND4) = SIN(ANG)
S(1) = 0.
S(ND41) = 1.
DO 100 I = ND42, ND4
JJ = ND42 - 1
ANG = ANG * FACT
S(I) = SIN(ANG)
100 S(JJ) = COS(ANG)
C forming the bit-reversed coefficients
IE = GAM - 2
NL = 1
BETA = 0
DO 200 M = 1, IE
LL = NL
NL = NL * NL
DO 200 JJ = 1, LL
BETA(JJ) = BETA(JJ) * 2
200 BETA(JJ) = BETA(JJ) + BETA(JJ)
READ(5, 1200) (Y(I), I = 1, N)
YSUM = 0.
DO 105 I = 1, N
105 YSUM = YSUM + Y(I)
YSUM = YSUM/N
DO 106 I = 1, N
Y(I) = YSUM - Y(I)
106 YA(I) = CMPLX(Y(I), 0.)
YN = Y(I)
MK = 1
KK = 1
C the hard core
DO 520 I = 1, GAM
MK = MK * 2
MK = MK + 1
IB = 1
C selecting the complex multipliers
J = 1
K = 1
KEN = KK
380 Bolm Inst. oceanogr. 8 Paulo, 20:145-199, 1971
220 MJ=BET-A(K)&J
IF(MJ>NO41)GOTO 403
NC=NO42-MJ
WBK=CMPLX(S(NC),S(MJ))
GOTO 410
403 NS=NO22-MJ
NC=NO4Z-NS
WBK=CMPLX(-S(NC),S(NS))
410 IEND=IB&MMM
DO 500 COEF1=IB,IEND
COEF2=COEF1&WK
Y2=YA(COEF2)*WBK
YA(COEF2)=YA(COEF1)-Y2
500 YA(COEF1)=YA(COEF1)*YF2
C DECIMATION IN TIME FOR THE FAST FOURIER TRANSFORM
C THE DATA IS NATURALLY ORDERED ON ENTRY AND BIT REVERSED ON EXIT
IB=IB&MM2
K=61
IFK.LE.NO4) GOTO 515
K=2
KEND=KEND-NO4
515 IFK.LE.KEND GOTO 220
K=K&KRK
52: MM2=MM
C BIT-REVERSING THE SERIES IN TWO PARTS:
KK=NO4
DO 600 K=1,NO4
MM=KETAIK1
MK=MK&MKL1
KK=KK&1
IFK.LE.KK GOTO 590
AUX=YA(KK)
YA(KK)=YA(MK)
YA(MK)=AUX
590 IFK.LE.KK DO 600
YA(MK)=AUX
600 CONTINUE
RI=1./NO2
ND21=NDZ&1
DO 610 K=1,ND21
AUX=YA(K)*RI
YA(K)=YA(MK)*RI
610 CONTINUE
C NOTE THAT ONLY THE THE N/2 COEFFICIENTS DESIRED ARE FULLY BIT-REVERSED
WRITE(4)(YA(I),I=1,NO21)
REWIN 4
100 FORMAT(24F3.0)
CALL EXIT
END
APPENDIX II
MATRIX GENERATION
EXTERNAl

MFSD

DOUBlE PRECISION (CONS(3),CONS(2))
DIMENSION SPEED(1),NFOUR(1),EPS(1),CONS(1),NSPEC(1),NMAT(1),NFOUR(1)
FORMAT(16H14,4F4.0)

READ(21,184)

N = the number of values per series.
YN = the last value of the series

READ(21,184)

C N IS THE NUMBER OF VALUES PER SERIES, NBL0C THE NUMBER OF TIDAL

DIMENSION ISP(1),JOUM(1),CONS(1),NFOUR(1)

READ(21,184)

C USE OF EXTRA VARIABLES IS AN ARTIFICE TO WRITE FOUR WHOLE

C IN MAGNETIC TAPE

C THE PROGRAM USER SHOULD ASCERTAIN THAT HIS COMPUTER CAN HANDLE AN

C INTEGER NUMBER OF 10 DIGITS. IF NOT THEN MAKE THE FORMAT 216,A8

C AND COMBINE (ISP,JOUM,CONS) TO OBTAIN A REGULAR SPEED

C THE USE OF EXTRA VARIABLES IS AN ARTIFICE TO WRITE FOUR WHOLE CASES

C ON MAGNETIC TAPE

C THE PROGRAM USER SHOULD ASCERTAIN THAT HIS COMPUTER CAN HANDLE AN

C INTEGER NUMBER OF 10 DIGITS. IF NOT THEN MAKE THE FORMAT 216,A8

C AND COMBINE (ISP,JOUM,CONS) TO OBTAIN A REGULAR SPEED

C THE USE OF EXTRA VARIABLES IS AN ARTIFICE TO WRITE FOUR WHOLE CASES

C ON MAGNETIC TAPE

C THE PROGRAM USER SHOULD ASCERTAIN THAT HIS COMPUTER CAN HANDLE AN

C INTEGER NUMBER OF 10 DIGITS. IF NOT THEN MAKE THE FORMAT 216,A8

C AND COMBINE (ISP,JOUM,CONS) TO OBTAIN A REGULAR SPEED
SPEEO=SPEO IM)
SIGN=1
IFIHOO=NST,NFIN
SIGN=-SIGN
KK=KK&1
ARG=I-SPEED&IFOUR)*FCT2
ANG=ISPEEO&IFOUR)*FCT2
PART=SIN(ANG)*XNI)*RN/SINIAKGI
SENQ=XNI*RN
IFIARG.NE.O.)SENQ=SINIARC*XNI)*RN/SINIARG
ARAIKK)=ISENQ&PART)*SIGN
15 0 ARBIKK)=ISENQ-PART)*SICN
200 CONTINUE
C ALGIA TO FORM THE ORIGINAL MATRIX
CALL MATSYIARA,RINV,NCON,NOFOR)
OELTA=O.OOO1
C NORMALISING THE MATRIX TO SYMETRIC "CRM VIA A SUBROUTINE
C INVERTING THE MATRIX VIA A SUBROUTINE
IFIIER)699,205,202
20Z WRITEI6,620Z)
620 2 FORMAT(1I10X,'IN S TABILITY AT STAG : NO. ',14,')
659 FORMAT(1IIIIIlOX,'INVERTED
M ATRIX')
I E=O
00
355
K=l,NCON
IB=I E &l
IE=IE&K
WRITEI6,646)(RINVIKK),KK=IB,IE)
646 FORMAT(1/13F8.5)
IFILSW.EQ.2) GaTO 465
C ALL SYMTPRIRINV,ARB,RONE,NCON,NOFOR,RFIM)
C MULTIPLYING THE INVERSE BY THE TRANSPOSE MATRIX
KSW=2
GOTO 410
465 CALL SYMTPRIRINV,ARB,RONE,NCON,NOFOR,RFIM)
KSW=1
470 WRITEI6,6651
665 FORMAT(1IIIIIlOX,'FINAL
NORMALISED
M ATRIX')
WRITEI6,630(ICONSTIKKI , KK=1,NCON)
WRITEI6,631)(INFOURIKK),KK=1,NCON)
WRITEI6,632)'EPSIKK),KK=1,NCON)
CALl MATSYM(ARB,RINV,NCON,NOFOR)
SUBROUTINE MATSYM(A,R,N,M)
OIMENSION All',R(L'
C SUBROUTINE TO MULTIPLY A GENERAL MATRIX BY THE TRANSPOSE OF ITSELF
C RETURNING THE RESULT AS AN UPPER TRIANGULAR SYMMETRICAL MATRIX
C A IS THE INPUT MATRIX STORED AS A TALL THIN MATRIX IN ONE DIMENSION
C M IS THE NUMBER OF ROWS OF A
C N IS THE NUMBER OF COLUMNS OF A (LESS THAN M)
C NENO = 0
IR = 0
DO 5 JJ = 1,M
NEEG = NENO + 1
MENG = MENG + 1 &
JJ = 0
DO 5 JJ = 1,N
IR = IR + 1
YSUM = 0.
DO 5 JJ = 1,N
YSUM = A(LM)*AlJJ)&YSUM
5 RIR) = YSUM
DO CONTINUE
RETURN
END
SUBROUTINE SINV(A,N,EPS,IER)
OIMENSION All)
DOBLE PRECISION SIN,WORK
C FACTORIZE GIVEN MATRIX BY MEAN OF SUBROUTINE MFSO
C A = TRANSPOSE(A)* T
C CALL MFSO(A,N,EPS,IER)
IF(lER) 9,1,1
C INVERT UPPER TRIANGULAR MATRIX T
C PREPARE INVERSION-LOOP
1 IPIVMNT=I/2
2 IPIV=IPIV
C INITIALIZE INVERSION-LOOP
DO 2 I=1,N
MIN = I
DO 2 K=I+1,N
IF(KEND(l)) 9,9,2
IF(KEND) 9,9,2
2 J=END
C INITIALIZE RNW-LOOP
DO 2 K=1,N
WORK=0.DO
DO 2 K=1,N
MIN=MIN-1
GOTO 485
FORMAT(10(2X,F8.5))
WRITE(6,6099)
FORMAT(1//40X,'MATRIX INVERSION UNSUCCESSFUL')
END
LHOR=IPIV
LVER=J

CALCULATE INVERSE(A) BY MEANS OF INVERSE(T) = INVERSE(T) * TRANSPOSE(INVERSE(T))
TOI = EPS*A(KPIV)

START FACTORIZATION-LOOP OVER I-TH ROW

DO 11 I = K,N

OSUM = O.OO

IF (IENO I 2, 4, 2)

C

START INNER LOOP

DO 3 L = I, IENO

LANF = KPIV - L

LINO = LINO - L

OSUM = OSUM + A(LANF)*A(LINO)

END IF

END IF

TRANSFORM ELEMENT A(LANO)

OSUM = OPIV*LANO

IF (I - K, 10, 5, 10)

C

TEST FOR NEGATIVE PIVOT ELEMENT AND FOR LOSS OF SIGNIFICANCE

IF (OSUM - TOL)

C

IF (OSUM)

C

ifer = K - 1

C

COMPUTE PIVOT ELEMENT

OPIV = OSQRT(OSUM)

A(KPIV) = OPIV

IOPIV = I.OO/OPIV

GO TO 11

END DF

END DF

END DF

RETURN

MULTIPLICATION OF A SYMMETRIC UPPER TRIANGULAR MATRIX BY THE TRANSPOSE

OF A GENERAL MATRIX

A IS A SYMMETRIC UPPER TRIANGULAR MATRIX WITH N ROWS

B IS A WORK VECTOR OF SIZE N

C IS THE OUTPUT MATRIX OF SIZE N BY M AND CANNOT OCCUPY THE SAME

POSITION AS THE INPUT MATRIX

B IS THE OUTPUT MATRIX OF SIZE N BY M AND CANNOT OCCUPY THE SAME

POSISION AS THE INPUT MATRIX

N IS THE NUMBERS OF ROWS

M IS THE NUMBER OF COLUMNS

DIMENSION A(I1,I2,I3,I4,I5)

JST = 0

TH = 0

DO 10 L = 1,N

JJ = JST

JSTEP = 1

10 JJ = JJ + JSTEP

BM = BM + 1

DIMM = 131, 132, 133, 174, 175
APPENDIX III

CALCULATION OF TIDAL COMPONENTS $a_j$ AND $b_j$, HARMONIC CONSTANTS $H_{ij}$ AND CORRECTION OF FOURIER COEFFICIENTS FOR TIDAL EFFECTS TO OBTAIN RESIDUALS $r_n$ AND $n_n$. 

ANALYSIS AND CALCULATION OF THE RESIDUAL FOURIER SPECTRUM

DATA Z/4*0.,5*Z70.,Z*90.,270.,2*90.,3*90.,4*90.,5*0.,Z*180.

DATA AU/9*0.,-23.74,10.80,-8.86,-12.94,-36.68,-2.14,-17.74,

DATA AF/l.0000,1.0429,1.0089,1.00,0,1.0129,1.1021,1.0004,1.0241,

DATA IFU/2*1,2,3,5*4,11,6,3*1,5,11*1,2*6,1,6*8,10,3*1,9,12

DATA SPEEO/0.0410686,0.0821373,0.04443147,1.0980331,12.8542862,

READ(5,570)

READ(5,502) YEAR,N

WRITE(6,630) YEAR,N

570 FORMAT(1)

502 FORMAT(F5.0,255)

630 FORMAT(20X,'NODE FACTORS ANO EPSIC ANGLES FOR ',F6.0, ' CENTRED',

190

XSUM=XKK
DO J=1,3
X=KK&32
19
XSUM=AAM'J'*Z'KK & XSUM
20 V'I'=XSUM
C CALCULATION OF THE BASIC ANGLES
PIF=6.28318/360.
ANOD=AN*PIF
ANG=ANOD
DO 25 I=1,3
BCOS'1'=COS'ANG
BSEN'I'=SIN,ANG
25 ANG=ANG&ANOD
C CALCULATION OF THE NODE FACTORS F AND THE LUNAR ANGLES U
F(J)=1.000
U(J)=0.
F(J)=XSUM
21: U(J)=XSUM
30 U(J)=YSUM
P=AAM'3'*PIF
PP=P&P
PN=P-ANOD
PPNN=PN&PN
PPN=PP-ANO
FCOSUL=I.-COS'PP'*O.2505-COS'PPN'*.1102-COS'PPNN'*O.0156-BCOS'1'*O.0370
FSENUL=I.-SIN'PP'*O.2505)-SIN'PPN)*.1102-SIN'PPNN)*.0156-BSEN'1'*.0370
FCOSUM=COS'P)*2. &COS'PN)*.4
FSENUM=SIN'P)*.2 &SIN'P)
F(10)=SQRT'FCOSUL*FCOSUL &FSENUL*FSENUL,
U(10)'=ATAN(FSENUL/FCOSUL)
F(11)=SQRT(FCOSUM*FCOSUM &FSENUM,=SENUM
U(11)'=ATAN(FSENUM/FCOSUM,
F(12)=F(S'**1.5
U(12)=U(S'*1.5
C SEPARATE CALCULATIONS OF F AND U FOR L2,M1,M3 RESPECTIVELY
DO 40 J=1,32
C DUMMY SUBSTITUTIONS
VMU(J)=REOUZ(SPEEO(J)*RESTO &V(J, &U(KK')
40 FF(J)=F(KK')
C FACTD=IS0 . /3.14159
1 READ 5,500,N,N8LDC,YN,NTAPE,NPUN,NTAPE
500 FORMAT(214,F4.0,3I4)
C N IS THE EXTENT OF ORIGINAL SERIES
C NBLOC IS THE NUMBER OF SPECIES FOR PROCESSING
C YN IS THE LAST VALUE OF ORIGINAL SERIES
C H = 1 IF THE MATRICES ARE WRITTEN ON SAME TAPE AS FOURIER SERIES,
C ELSE H = 2
C NPUN = 0 IF SUPPRESSION OF PUNCHING H & G FOR TIDAL CONSTITUENTS IS
C REQUIRED, ELSE ANY NON-ZERO VALUE
C MTAPe IS THE SYMBOLIC TAPE UNIT FOR THE OUTPUT OF THE RESIDUAL SERIES
C IF MTAPE = 0 THEN NO OUTPUT IS MADE
IF(MTAPE.NE.0)
REWIND MTAPe
REWIND TAPE
            IF(MTAPE.NE.0)REWIND MTAPE
C NOTE IT IS POSSIBLE FOR MTAPe TO BE EITHER TAPE 1 OR 2
FCT=XN/360.
REWIND 1
REWIND NTAPe
            IF(MTAPE.NE.0)REWIND MTAPE
C NOTE IT IS POSSIBLE FOR MTAPe TO BE EITHER TAPE 1 OR 2
READ(1)(X(I),YII),I=1,ND21)
FCT=XN/360.
SPINC=360./XN
CORR=YII-X(1)/ND21)
X(I)=O.
KOUNT=O
2 J=J+1
KOUNT=KOUNT&1
WRITE(6,6001)
6001 FORMAT(/IOX,'SPEED READ',IOX,'SPEED CALCD',IOX,'NODE FACTOR',IOX
C BLOCK TO READ NAME/SPEEDS AND COMPOSITION FACTORS FROM TAPE
5 READMTAPe,2100)(ISP(JK),(ICOM(I,JK),I=3,33),CON{JK),JK=1,4)
2100 FORMAT(/IO,3112,A8)
00
8 JK=1,4
J = J & 1
IFIISP(JK).EQ.O)
GOTO 120
SPED{J)=ISPIJK)*I.E-01
CONSIJ)=CONIJK)
IFI{ICOMI33,JK).EQ.0)
GOTO 6
JS=ICOMI33,JK)
FNIJ)=FFIJS)
VZUIJ)=
VElOC = SPEED{JK)*IC & VElOC
GOTO 8
6 ANG=O.
FNO = 1.
VElOC = O.
DO 7 KS=3,32
IC=ICOMIKS,JK)
IFIIC.EQ.O)
GOTO 7
ANG = VMUIKS)*IC & ANG
FNO = FFIJS)**IABSIIC)*FNO
VElOC = SPEED{KS)*IC & VElOC
7 CONTINUE
FIN (J) = FNO
VII (J) = BDEVJ (ANG)
5 READMTAPe,6002)(SPED{J),VElOC,FN(J),VZU{J,CONS(J)
6002 FORMAT(/IO,3112,F11.1,9X,F11.7,9X,F10.2,10X,A8)
GOTO 5
120 NCON = J - 1
NST=SPED{J)*FCT-O.5
IF(KOUNT.GT.NST)
SPED{J)=LSPED{J)+0.5
KOUNT=KOUNT-1
C KOUNT INDICATES THE SPECIES NUMBER
C INCREASED BY ONE -- THE SPAN OF THE MATRIX IS
C INCREASED BY ONE -- THE SPAN OF THE MATRIX IS
C AUGMENTED BY 2 AND INCREASING THE SPAN BY 2 AUGMENTING
C AUGMENT THE SPAN BY 2 AUGMENTING
C AUGMENT THE SPAN BY 2 AUGMENTING
C AUGMENT THE SPAN BY 2 AUGMENTING
C AUGMENT THE SPAN BY 2 AUGMENTING

NMAT=NDFORM*CON
DO 36 K=NST,NFIM
C CORRECTION OF THE COSINE TERM
READINTAPE(VECTOR(K),K=1,NMAT)
C THE INVERSE COSINE MATRIX IS READ AND THE COSINE CONSTITUENT OF
C THE FOURIER CALCULATED
IP=0
DO 50 L=1,LWCON
RSUM=0.
IM=IP
50 CONTINUE
DO 58 K=NST,NFIM
VECTOR=VECTOR(K)*CORRECTION
DO 58 XIK=XIK&CORR
58 CONTINUE
READINTAPE(VECTOR(K),K=1,NMAT)
C THE NORMAL COSINE MATRIX IS READ AND THE COSINE
C COMPONENT OF THE TIDAL FOURIER SERIES CALCULATED.
KX=0
DO 50 K=NST,NFIM
RSUM=0.
50 CONTINUE
DO 38 K=NST,NFIM
IP=KX
DO 55 L=1,LWCON
VECTOR=VECTOR(L)*XIK
55 CONTINUE
READ(NTAPE)VECTOR(K),K=1,NMAT
RSUM=0.
58 CONTINUE
DO 38 K=NST,NFIM
RSUM=VECTOR(K)*XIK
58 CONTINUE
XIK=XIK-VECTOR(K)
58 CONTINUE
IP=XK
DO 55 L=1,LWCON
VECTOR=VECTOR(L)*XIK
55 CONTINUE
READ(NTAPE)VECTOR(K),K=1,NMAT
RSUM=0.
60 CONTINUE
DO 38 K=NST,NFIM
RSUM=VECTOR(K)*XIK
60 CONTINUE
KX=KX+1
DO 55 L=1,LWCON
VECTOR=VECTOR(L)*XIK
55 CONTINUE
READ(NTAPE)VECTOR(K),K=1,NMAT
RSUM=0.
60 CONTINUE
DO 38 K=NST,NFIM
RSUM=VECTOR(K)*XIK
60 CONTINUE
XIK=XIK-VECTOR(K)
58 CONTINUE
IP=XK
DO 55 L=1,LWCON
VECTOR=VECTOR(L)*XIK
55 CONTINUE
READ(NTAPE)VECTOR(K),K=1,NMAT
RSUM=0.
60 CONTINUE
DO 38 K=NST,NFIM
RSUM=VECTOR(K)*XIK
60 CONTINUE
BMAR(NF1)=BMAR(NF1)-VECTOR(K)
BMAR(NF1)=NF1-0.5
BMAR(NF2)=NF2-1
DO 85 MA=NF2-200
BMAR(N1)=0.
85 CONTINUE
THE TIDAL FOURIER COEFFICIENTS ARE NOW
C CONTAINED IN TWO BLOCKS OF 200 MAXIMUM NUMBERS
CFields & MAN FOR FUTURE USE IF DESIRED
C THE LAST TWO NUMBERS CONTAIN THE EXTENT
C OF THE ARRAY FOR FUTURE MANIPULATION
WRITE(NOS5001)
STOP
END
C
CALCULATION OF PHASE ANGLE

90 IF(L11>0.190.92,91)
91 FN(L) = SQRT(A(L)*A(L)+8(L)*8(L))

10 IFN(L) = IF4 O.

11 IFOU= SPEO(L)*FCT&0.5

12 CALCULATION OF PHASE ANGLE
FI=O.

13 IF(A(L'-O.,90,92,91

14 IF(A(L'-90) FI=180.

15 THETA=ATAN(8(L)/A(L))*FACT&FI

16 IF(THETA.LT.O.)THETA=THETA&360.

17 GO TO 119

18 VZU(L) = REOUZ(VZU(L) & 

19 FORMAT(FII.7,IOX,F9.3,4X,F6.2,32~A8)

20 FORMAT(4(4X,F7.2),4X,A8,5X,14)

21 WRITE(6,640)A(L),8(L),HT,THETA,CO~S(L),IFOU

22 640 FORMAT(4(4X,F7.2),4X,A8,5X,14)

23 WRITE(6,6010)

24 6010 FORMAT(1140X,'VALUES OF H & G'/)

25 DUM = O.

26 WRITE(6,6002)(SPED(L),DUM,FNL11,WEFL11,CONS(L),L=1,NCON

27 WRITE(6,6003)

28 6003 FORMAT(/DUM,6(4X,F7.2),4X,A8,70u)

29 FORMAT(1120X,'SPEED NUMBERS AND RESIDUAL FOURIER ENERGY IN '/,

30 STATEMENT TO WRITE END OF BLOCK
31 IF(JK.NE.IO'GOTO 190

32 WRITE(6,659'(IQ(J),J=I,lO),(SPED(J),J=I,10),(SPED(J),J=11,20)

33 659 FORMAT(/IO(4X,I4,4X'/IO(IX,FI1.7)110(IX,EI1.5')

34 JK=O

35 190 VARIA=VARIA~RESEN

36 JK1=JK&l

37 JKIO=JK&lO

38 DO 195 J=JK1,lO

39 IQ(J)=O.

40 SPED(J)=O.

41 WRITE(6,659)(IQ(J),J=I,lO),(SPED(J),J=I,10),(SPED(J),J=11,JKIO

42 C STATEMENT TO WRITE END OF BLOCK
43 VARIA = SQRT(VARIA/NOFOR1

44 WRITE(6,645)VARIA

45 645 FORMAT(' RESIDUAL ENERGY IN TIDAL BAND =',E12.4)

46 IF(NBLOC.NE.KOUNT1GOTO 2

47 XSX = O.

48 DO 205 J = 1,N21

49 XSF = X(J)*X(J)+Y(J)*Y(J) XSX = XSX +XSF

50 205 XSX = SQRT(XSX/N21)

WRITE(6,6005)X$ X$ (IVXL)X$ (XL)X$ (I)X$ (X)
6005 FORMAT((X',TOTAL NOISE LEVEL =',E12.4)),(X',NOISE LEVEL IN',E12.4))
WRITE(6,6005)X$ X$ (IVXL)X$ (XL)X$ (I)X$ (X)
6005 FORMAT((X',TOTAL NOISE LEVEL =',E12.4)),(X',NOISE LEVEL IN',E12.4))
WRITE(IMTAPE,IMTAPE)X$ X$ (IVXL)X$ (XL)X$ (I)X$ (X)
CALL EXIT
END
FUNCTION REDUZ(ANG)
ANG=ANG
1 IF(ANG.GE.0.)GOTO 2
ANG=ANG-360.
GOTO 1
2 IF(ANG.LT.360.)GOTO 3
ANG=ANG+360.
GOTO 2
3 REDUZ=ANG
RETURN
END
APPENDIX IV
POWER SPECTRAL ANALYSIS

Bole Inst. oceanogr. 8 Paulo, 20:145-199,1971
COMPLEX C(4097), CZ
DIMENSION Z(4097),PEPSI(30),P30(30),V(30),X(30)
COMMON RAYF
COMMON CV(12),XV(12),Y(12)
READ (5) Y(12),XV(12),RAYF
501 FORMAT (15)
C N IS THE NO. OF VALUES IN THE ORIGINAL SERIES (N=2**M)
C NOV IS THE NO. OF AUTO-SPECTRA TO BE FORMED
C PREDETERMINED MDIV, IF MDIV>0 THEN THE HALF WIDTH OF THE FILTER IS
C TAPS. M+1 IS A PARAMETER THAT SPECIFIES AUTO-SPECTRA OUTPUT IS TO
C ON MAGNETIC TAPE, OUTPUT AND PUNCH-OUT ARE EXPECTED
N=2**21
IF(L,EQ.O)0.0162829
NP=MOD(NE.O,1.)/MDIV
SCL=01.1/SCAL*0.5
EDF=0.1
(EK(Y),ME.0,12)MM1
9 PEPSI=(SINC(U,1,1593)+SCAL(0.1383/E-U)*ME.1,141)
1 CONTINUE
KOUNT=0
WRITE(6,101)PEPSI(1,1,41)
10 KOUNT=KOUNT+1
WRITE(6,110)KOUNT
11 FORMAT(10X,'AUTO-SPECTRUM NO.: ',141)
REAO(1)
CCV(1,1,2)
VSUM=0.
C INITIALLSATION AND LOW FREQUENCY CORRECTION
DO 12 J=1,1
C=CIJ
U=RAYF
Y(1,1)=2.
Y(1,1)=Y(1,1)+VSUM
12 CONTINUE
M=O
LL=O.
JSW=2
IF(L,GT.NIIGOTO
45
14 VA=XU
   XU=V(X)
   DO 15 IM=1,L
   XU(II,IM)=Y(II,IM)
   15 IM=IM+1
   VA=XU
   XU=V(X)
   V(II,IM)=XU
   XIIM=XU
   XY=XIIM
   XIIM=XY
   XU=XY
   VA=IYIIM)&XYI*PEPSIIIM)&VA
   YIIM=Y(IM&II
   Y{L)=lZ
   ZIIM)=VA
   GOTO(30,31),JSW
45 IF(KTAPE.NE.O)GOTO 60
   WRITE(*,620)(Z(II,I,II=I,NII
620 FORMAT(I8(1X,E9.3,I4))
   JB=0
   KCARD=0
   DO 13 I=1,N
   JB=JB+1
   KCARD=KCARD+1
131 PUNCH 1000,(Z(II,I=JB,JEI,KOUNT,K:ARO
1000 FORMAT(8E9.3,lX,I2,lX,I4)
   GOTO 41
60 WRITE(*,620)(Z(II,I=I,N)
61 WRITE(*,640)NCARD
   CALL EXIT
   END