A Ruelle Operator for continuous time Markov Chains

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Abstract. We consider a generalization of the Ruelle theorem for the case of continuous time problems. We present a result which we believe is important for future use in problems in Mathematical Physics related to C*-Algebras.

We consider a finite state set $S$ and a stationary continuous time Markov Chain $X_t$, $t \geq 0$, taking values on $S$. We denote by $\Omega$ the set of paths $w$ taking values on $S$ (the elements $w$ are locally constant with left and right limits and are also right continuous on $t$). We consider an infinitesimal generator $L$ and a stationary vector $p_0$. We denote by $P$ the associated probability on $(\Omega, \mathcal{B})$. All functions $f$ we consider below are in the set $L^\infty(P)$.

From the probability $P$ we define a Ruelle operator $L_t^t, t \geq 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $L^\infty(P)$. Given $V : \Omega \to \mathbb{R}$, such that is constant in sets of the form $\{X_0 = c\}$, we define a modified Ruelle operator $L_V^t, t \geq 0$, in the following way: there exist a certain $f_V$ such that for each $t$ we consider the operator acting on $g$ given by

$$L_V^t(g)(w) = \left[ \frac{1}{f_V} L^t_t(e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} g f_V) \right](w)$$

We are able to show the existence of an eigenfunction $u$ and an eigen-probability $\nu_V$ on $\Omega$ associated to $L_V^t, t \geq 0$.

We also show the following property for the probability $\nu_V$: for any integrable $g \in L^\infty(P)$ and any real and positive $t$

$$\int e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} \left[ (L_V^t(g)) \circ \theta_t \right] d\nu_V = \int g d\nu_V$$

This equation generalize, for the continuous time Markov Chain, a similar one for discrete time systems (and which is quite important for understanding the KMS states of certain C*-algebras).

* Beneficiary of grant given by CAPES - PROCAD.
† Partially supported by CNPq, Instituto do Milenio, PRONEX - Sistemas Dinâmicos, INCT em Matemática - CNPq, Beneficiary of grant given by CAPES - PROCAD.
‡ Partially supported by CNPq, Instituto do Milenio, PRONEX - Sistemas Dinâmicos.
1. Introduction

We want to extend the concept of Ruelle operator to continuous time Markov Chains. In order to do that we need a probability a priori on paths. This fact is not explicit in the discrete time case (thermodynamic formalism) but it is necessary here.

We consider a continuous time stochastic process: the sample paths are functions of the positive real line $\mathbb{R}^+ = \{ t \in \mathbb{R} : t \geq 0 \}$ taking values in a finite set $S$ with $n$ elements, that we denote by $S = \{1, 2, \ldots, n\}$. Now, consider a $n$ by $n$ real matrix $L$ such that:

1) $0 < -L_{ii}$, for all $i \in S$,
2) $L_{ij} \geq 0$, for all $i \neq j$, $i \in S$,
3) $\sum_{i=1}^{n} L_{ij} = 0$ for all fixed $j \in S$.

We point out that, by convention, we are considering column stochastic matrices and not line stochastic matrices (see [N] section 2 and 3 for general references).

We denote by $P_t = e^{tL}$ the semigroup generated by $L$. The left action of the semigroup can be identified with an action over functions from $S$ to $\mathbb{R}$ (vectors in $\mathbb{R}^n$) and the right action can be identified with action on measures on $S$ (also vectors in $\mathbb{R}^n$).

The matrix $e^{tL}$ is column stochastic, since from the assumptions on $L$ it follows that

$$(1, \ldots, 1)e^{tL} = (1, \ldots, 1)(I + tL + \frac{1}{2}t^2L^2 + \cdots) = (1, \ldots, 1).$$

It is well known that there exist a vector of probability $p_0 = (p_0^1, p_0^2, \ldots, p_0^n) \in \mathbb{R}^n$ such that $e^{tL}(p_0) = P^tp_0 = p^t$ for all $t > 0$. The vector $p_0$ is a right eigenvector of $e^{tL}$. All entries $p_0^i$ are strictly positive, as a consequence of hypothesis 1.

Now, let’s consider the space $\tilde{\Omega} = \{1, 2, \ldots, n\}^{\mathbb{R}^+}$ of all functions from $\mathbb{R}^+$ to $S$. In principle this seems to be enough for our purposes, but technical details in the construction of probability measures on such a space force us to use a restriction: we consider the space $\Omega \subset \tilde{\Omega}$ as the set of right-continuous functions from $\mathbb{R}^+$ to $S$, which also have left limits in every $t > 0$. These functions are constant in intervals (closed in the left and open in the right). In this set we consider the sigma algebra $\mathcal{B}$ generated by the cylinders of the form

$$\{w_0 = a_0, w_{t_1} = a_1, w_{t_2} = a_2, \ldots, w_{t_r} = a_r\},$$
where $t_i \in \mathbb{R}_+, \ r \in \mathbb{Z}^+, a_i \in S$ and $0 < t_1 < t_2 < \ldots < t_r$. It is possible to endow $\Omega$ with a metric, the Skorohod-Stone metric $d$, which makes $\Omega$ complete and separable ([EK] section 3.5), but the space is not compact.

Now we can introduce a continuous time version of the shift map as follows: we define for each fixed $s \in \mathbb{R}^+$ the $\mathcal{B}$-measurable transformation $\Theta_s : \Omega \to \Omega$ given by $\Theta_s(w_t) = w_{t+s}$ (we remark that $\Theta_s$ is also a continuous transformation with respect to the Skorohod-Stone metric $d$).

For $L$ and $p_0$ fixed as above we denote by $P$ the probability on the sigma-algebra $\mathcal{B}$ defined for cylinders by

$$P(\{w_0 = a_0, w_{t_1} = a_1, \ldots, w_{t_r} = a_r\}) = P_{t_r}^{t_r-t_{r-1}} \ldots P_{t_2}^{t_2-t_1} P_{t_1}^{t_1-t_0} p_0.$$

For further details of the construction of this measure we refer the reader to [B].

The probability $P$ on $(\Omega, \mathcal{B})$ is stationary in the sense that for any integrable function $f$, and, any $s \geq 0$

$$\int f(w) dP(w) = \int (f \circ \Theta_s) dP(w).$$

From now on the Stationary Process defined by $P$ is denoted by $X_t$ and all functions $f$ we consider are in the set $L_\infty(P)$.

There exist a version of $P$ such that for a set of full measure we have that all sample elements $w$ are locally constant on $t$, with left and right limits, and $w$ is right continuous on $t$. We consider from now on such probability $P$ acting on this space (see [EK] chapter 3).

From $P$ we are able to define a continuous time Ruelle operator $L^t$, $t > 0$, acting on functions $f : \Omega \to \mathbb{R}$ of $L_\infty(P)$. It’s also possible to introduce the endomorphism $\alpha_t : L_\infty(P) \to L_\infty(P)$ defined as

$$\alpha_t(\varphi) = \varphi \circ \Theta_t, \quad \forall \varphi \in L_\infty(P).$$

We relate in the next section the conditional expectation with respect to the $\sigma$-algebras $\mathcal{F}_t^+$ with the operators $L^t$ and $\alpha_t$, as follows:

$$[L^t(f)](\Theta_t) = E(f|\mathcal{F}_t^+).$$

Given $V : \Omega \to \mathbb{R}$, such that it is constant in sets of the form $\{X_0 = c\}$ (i.e., $V$ depends only on the value of $w(0)$), we consider a Ruelle operator family $\tilde{L}^t_V$, for all $t > 0$, given by

$$\tilde{L}^t_V(g)(w) = \left[ \frac{1}{f_V} \mathcal{L}^t(e^\int_0^t (V \circ \Theta_s)(.) ds \ f_V) \right](w),$$

for any given $g$, where $f_V$ is a fixed function.
We are able to show the existence of an eigen-probability $\nu_V$ on $\Omega$, for the family $\hat{\mathcal{L}}_t$, for all $t > 0$, such that satisfies:

**Theorem A.** For any integrable $g \in \mathcal{L}^\infty(P)$ and any positive $t$

$$\int \left[ \frac{1}{e^{\int_0^t (V \circ \Theta_s)(\cdot) ds}_f V} \right] \mathcal{L}^t \left( \left[ e^{\int_0^t (V \circ \Theta_s)(\cdot) ds}_f V \right] \circ \Theta_t \right) \, d\nu_V = \int g \, d\nu_V.$$

The above functional equation is a natural generalization (for continuous time) of the similar one presented in Theorem 7.4 in [EL1] and proposition 2.1 in [EL2].

In [EL1] and [EL2] the important probability in the Bernoulli space is an eigen-probability $\nu$ for the Ruelle operator associated to a certain potential $V = \log H : \{1, 2, \ldots, d\}^\mathbb{N} \rightarrow \mathbb{R}$. This probability $\nu$ satisfies: for any $m \in \mathbb{N}$, $g \in C(X)$,

$$\int \lambda_m E_m(\Lambda^{-1}_m g) d\nu = \int g d\nu,$$

where

$$\lambda_m(x) = \frac{H^\beta[m](x)}{\Lambda[m](x)} = \frac{H^\beta(x)}{\Lambda[m](x)} \frac{H^\beta(\sigma(x)) \cdots H^\beta(\sigma^{n-1}(x))}{e^\log(H^\beta(\sigma(x))) + \log(H^\beta(\sigma(x))) + \cdots + \log(H^\beta(\sigma^{n-1}(x))) - \Lambda[m](x)},$$

and $\sigma$ is the shift on the Bernoulli space $\{1, 2, \ldots, d\}^\mathbb{N}$. Here $E_m(f) = E(f|\sigma^{-m}(B))$ denotes the expected projection (with respect to a initial probability $P$ on the Bernoulli space) on the sigma-algebra $\sigma^{-m}(B)$, where $B$ is the Borel sigma-algebra and $\Lambda[m]$ is associated to the Jones index.

We refer the reader to [CL] for a Thermodynamic view of C*-Algebras which include concepts like pressure, entropy, etc..

We believe it will be important in the analysis of certain C* algebras associated to continuous time dynamical systems a characterization of KMS states by means of the above theorem. We point out, however, that we are able to show this theorem for a certain $\rho_V$ just for a quite simple function $V$ as above. In a forthcoming paper we will consider more general potentials $V$.

One could consider a continuous time version of the C*-algebra considered in [EL1]. We just give an idea of what we are talking about. Given the above defined $P$ for each $t > 0$, denote by $s_t : \mathcal{L}^2(P) \rightarrow \mathcal{L}^2(P)$, the Koopman operator, where for $\eta \in \mathcal{L}^2(P)$ we define $(s_t \eta)(x) = \eta(\theta_t(x))$.

Another important class of linear operators is $M_f : \mathcal{L}^2(P) \rightarrow \mathcal{L}^2(P)$, for a given fixed $f \in C(\Omega)$, and defined by $M_f(\eta)(x) = f(x)\eta(x)$, for any $\eta$ in

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We assume that $f$ is such defines an operator on $L^2(P)$ (remember that $\Omega$ is not compact).

In this way we can consider a $C^*$-algebra generated by the above defined operators (for all different values of $t > 0$), then the concept of state, and finally given $V$ and $\beta$ we can ask about KMS states. There are several technical difficulties in the definition of the above $C^*$-algebra, etc... Anyway, at least formally, there is a need for finding $\nu$ which is a solution of an equation of the kind we describe here. We need this in order to be able to obtain a characterization of KMS states by means of an eigen-probability for the continuous time Ruelle operator. This setting will be the subject of a future work. This was the motivation for our result.

With the operators $\alpha$ and $\mathcal{L}$ we can rewrite the theorem above as

$$\rho_V (G_T^{-1} E_T (G_T \varphi)) = \rho_V (\varphi),$$

for all $\varphi \in L^\infty$ and all $T > 0$, where, as usual, $\rho_V (\varphi) = \int \varphi \rho_V$, $E_T = \alpha_T \mathcal{L}^T$ is in fact a projection on a subalgebra of $\mathcal{B}$, and $G_T : \Omega \to \mathbb{R}$ is given by

$$G_T(x) = \exp \left( \int_0^T V(x(s)) ds \right).$$

For the map $V : \Omega \to \mathbb{R}$, which is constant in cylinders of the form $\{w_0 = i\}$, $i \in \{1, 2, ..., n\}$, we denote by $V_i$ the corresponding value. We also denote by $V$ the diagonal matrix with the $i$-diagonal element equal to $V_i$.

Now, consider $P^t_V = e^{t(L+V)}$. The classical Perron-Frobenius Theorem for such semigroup will be one of the main ingredients of our main proof.

As usual, we denote by $\mathcal{F}_s$ the sigma-algebra generated by $X_s$. We also denote by $\mathcal{F}^+_s$ the sigma-algebra generated $\sigma(\{X_s, s \leq u\})$. Note that a $\mathcal{F}^+_s$-measurable function $f(w)$ on $\Omega$ does depend of the value $w_s$.

We also denote by $I_A$ the indicator function of a measurable set $A$ in $\Omega$.

In a forthcoming paper we will analyze the case where the potential $V$ is of a more general type (not just $V : \Omega \to \mathbb{R}$, such that it’s constant in sets of the form $\{X_0 = c\}$)

2. A continuous time Ruelle Operator

We present a quite general definition: let $\mathbb{X}$ and $X$ be a separable metric Radon spaces, $\hat{\mu}$ probability on $\mathbb{X}$, $\pi : \mathbb{X} \to X$ Borel measurable and $\mu = \pi_* \hat{\mu}$. Then there exists a Borel family of probabilities $\{\hat{\mu}_x\}_{x \in X}$ on $\mathbb{X}$, uniquely determined $\mu$-a.e, such that,

1) $\hat{\mu}_x(\mathbb{X} \setminus \pi^{-1}(x)) = 0$, $\mu$-a.e;
2) $\int g(z) d\hat{\mu}_x(z) = \int_X \int_{\pi^{-1}(x)} g(z) d\hat{\mu}_x(z) d\mu(x)$. 

We refer the reader to [AGS] for the proof. Here we will need a more simple version of this general result that can be obtained in an explicit form.

We consider the disintegration of $P$ given by the family of measures, indexed by the elements of $\Omega$ and $t > 0$ defined as follows: first, consider a sequence $0 = t_0 < t_1 < \cdots < t_{j-1} < t \leq t_j < \cdots < t_r$. Then for $w \in \Omega$ and $t > 0$ we have on cylinders:

$$\mu_t^w([X_0 = a_0, \ldots, X_{t_r} = a_{t_r}]) = \begin{cases} \frac{1}{p_0(t)}\prod_{a_{t_j}a_j=1}^{t_{j-1}} P_{w(t)a_{j-1}}^{t_{j-1}} \prod_{a_{t_j}a_j=1}^{a_0} P_{a_{t_j}a_j}^{a_0} & \text{if } a_j = w(t_j), \ldots, a_{t_r} = w(t_r) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1. Under the above conditions on the sequence $0 = t_0 < t_1 < \cdots < t_{j-1} < t \leq t_j < \cdots < t_r$, we have that $\mu_t^w$ is the disintegration of $P$ along the fibers $\Theta_t^{-1}(\cdot)$.

Proof: It is enough to show that for any integrable $f$

$$\int_{\Omega} f dP = \int_{\Omega} \int_{\Theta_t^{-1}(w)} f(x) d\mu_t^w(x) dP(w).$$

For doing that we can assume that $f$ is in fact the indicator of the cylinder $[X_0 = a_0, \ldots, X_{t_r} = a_{t_r}]$; then the right hand side becomes

$$\int \int f d\mu_t^w(x) dP(w) = \sum_{a=1}^n \int_{[w(t_j)=a_j, \ldots, w(t_r)=a_{t_r}]} \frac{1}{p_0(t)} \prod_{a_{t_j}a_j=1}^{t_{j-1}} P_{w(t)a_{j-1}}^{t_{j-1}} \prod_{a_{t_j}a_j=1}^{a_0} P_{a_{t_j}a_j}^{a_0} dP(w) = \sum_{a=1}^n P_{t_r-t_{r-1}}^{t_{r-1}} \prod_{a_{t_j}a_j=1}^{t_{j-1}} P_{a_{t_j}a_j}^{t_{j-1}} \prod_{a_{t_j}a_j=1}^{a_0} P_{a_{t_j}a_j}^{a_0} = P([X_0 = a_0, \ldots, X_{t_r} = a_{t_r}]) = \int f dP.$$

In the second inequality we use the fact that $P$ is stationary.

The proof for a general $f$ follows from standard arguments. $\square$
Definition 2.2. For $t$ fixed we define the operator $L^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P)$ as follows:

$$L^t(\varphi)(x) = \int_{y \in \Theta^{-1}_t(x)} \varphi(y) d\mu^x_t(y).$$

Remark 2.3. The definition above can be rewritten as

$$L^t(\varphi)(x) = \int_{y \in D[0,t)} \varphi(yx) d\mu^x_t(yx),$$

where the symbol $yx$ means the concatenation of the path $y$ with the translation of $x$:

$$xy(s) = \begin{cases} y(s) & \text{if } s \in [0,t) \\ x(s-t) & \text{if } s \geq t, \end{cases}$$

and, $D[0,t)$ is the set of right-continuous functions from $[0,t)$ to $S$. This follows simply from the fact that, in this notation, $\Theta^{-1}_t(x) = \{yx: y \in D[0,t)\}$.

Note that the value $\lim_{s \to t} y(s)$ do not have to be necessarily equal to $x(0)$.

In order to understand better the definitions above we apply the operator to some simple functions. For example, we can see the effect of $L^t$ on some indicator function of a given cylinder: consider the sequence $0 = t_0 < t_1 < \ldots < t_j < \ldots < t_r$ and then take $f = I_{\{X_0=a_0, X_{t_1}=a_1, \ldots, X_{t_r}=a_r\}}$. Then, for a path $z \in \Omega$ such that $z_{t_j-t} = a_j, \ldots, z_{t_{r}-t} = a_r$ (the future condition) we have

$$L^t(f)(z) = \frac{1}{p_0} p_{t_0}^{a_{j-1}} \ldots p_{a_{t_1}}^{a_1} p_{a_0}^{a_0} p_{00},$$

otherwise (i.e., if the path $z$ does not satisfy the condition above) we get $L^t(f)(z) = 0$.

Note that if $t_r < t$, then $L^t(f)(z)$ depends only on $z_0$. For example, if $f = I_{\{X_0=i_0\}}$ then

$$L^t(f)(z) = \int_{y \in D[0,t)} I_{\{X_0=i_0\}}(yx) d\mu^x_t(yx) = \mu^x_t([X_0=i_0]) = \frac{1}{p_0} p_{t_0}^{a_{j-1}} p_{00}^0.$$

In the case $f = I_{\{X_0=i_0, X_t=j_0\}}$, then $L^t(f)(z) = P_{z_0,i_0}^{j_0} z_0$, if $z_0 = j_0$, and $L^t(f)(z) = 0$ otherwise.

We describe bellow some properties of $L^t$.

Proposition 2.4. $L^t(1) = 1$, where $1$ is the function that maps every point in $\Omega$ to 1.
Proposition 2.6. Given the functions $\varphi, \psi \in L^\infty(P)$, then

\[ L^\ell(\varphi \times (\psi \circ \Theta_t))(z) = \psi(z) \times L^\ell(\varphi)(z). \]

Proof: Using the formula of $L^\ell$ given by Remark 2.3 we get

\[ L^\ell(\varphi \circ \Theta_t)(x) = \int_{y \in D[0,t]} \varphi(iy)(\psi \circ \Theta_t)(iy) \, d\mu^\ell(i) = \]

\[ \sum_{a=1}^n \mu^\ell([X_0 = a, X_t = x(0)]) = \frac{1}{p_{x(0)}} \sum_{a=1}^n P^t_{x(0)a} p_0^a = 1. \]

We can also define the dual of $L^\ell$, denoted by $(L^\ell)^*$, acting on the measures. Then we get:

Proposition 2.5. For any positive $t$ we have that $(L^\ell)^*(P) = (P)$.

Proof: For a fixed $t$ we have that $(L^\ell)^*(P) = (P)$, because for any $f$ of the form $f = I\{X_0=a_0,X_{t_1}=a_1,...,X_{t_r}=a_r\}$, $0 = t_0 < t_1 < .. < t_{j-1} < t \leq t_j < ... < t_r$, we get

\[ \int L^\ell(f)(z) \, dP(z) = \sum_{b=1}^n \int_{\{X_0=b\}} L^\ell(f)(z) \, dP(z) = \]

\[ \sum_{b=1}^n \int I\{X_0=b,X_{t_j-t}=a_j,...,X_{t_r-t}=a_r\}(z) dP(z) \frac{1}{p_0} P^{t_j-t_{j-1}}_{ba_{j-1}} P^{t_{j+1}-t_j}_{a_j a_{j+1}} P^{t_{j+2}-t_{j+1}}_{a_{j+1} a_{j+2}} P_0^a = \]

\[ \sum_{b=1}^n P\{X_0 = b, X_{t_{j+1}-t_j} = a_{j+1}, ..., X_{t_r-t} = a_r\} \frac{1}{p_0} P^{t_{j+1}-t_{j+2}}_{a_{j+1} a_{j+2}} P^{t_{j+2}-t_{j+1}}_{a_{j+1} a_{j+2}} P_0^a = \]

\[ \int f(w) \, dP(w). \]
ψ(x)∫_{i∈D[0,t]}φ(ix)dµ{\tilde{t}}(i) = (ψL^t(φ))(x) = ψ(x)L^t(φ)(x),

since ψ ◦ Θ_t(ix) = ψ(x), independently of i. □

We just recall that the last proposition can be restated as

\[ L^t(φ_α(x)) = ψL^t(φ). \]

Then we get:

**Proposition 2.7.** \( α_t \) is the adjoint of \( L^t \) on \( L^2(P) \).

**Proof:** From last two propositions

\[ ∫ L^t(f)g\,dP = ∫ L^t(f×(g ◦ Θ_t))\,dP = ∫ f×(g ◦ Θ_t)\,dP = ∫ fα_t(g)\,dP, \]

as claimed. □

We want to obtain conditional expectations in a more explicit form. For a given \( f \), recall that the function \( Z(w) = E(f|F_{i+t}) \) is the \( Z \) (almost everywhere defined) \( F_{i+t} \)-measurable function such that for any \( F_{i+t} \)-measurable set \( B \) we have \( ∫_B Z(w)\,dP(w) = ∫_B f(w)\,dP(w). \)

**Proposition 2.8.** The conditional expectation is given by

\[ E(f|F_{i+t})(x) = ∫_B f\,dμ^t_i. \]

**Proof:** For \( t \) fixed, consider a \( F_{i+t} \)-measurable set \( B \). Then we have

\[ ∫_B (fI_B)\,dμ^w_i\,dP(w) = ∫ (fI_B)\,dP(w) = ∫_B f\,dP. \] □

Now we can relate the conditional expectation with respect to the \( σ \)-algebras \( F_{i+t} \) with the operators \( L^t \) and \( α_t \) as follows:

**Proposition 2.9.** \( [L^t(f)](Θ_t) = E(f|F^t_{i+t}). \)
Proof: This follows from the fact that for any \( B = \{ X_{s_1} = b_1, X_{s_2} = b_2, \ldots, X_{s_u} = b_u \} \), with \( t < s_1 < \ldots < s_u \), we have \( I_B = I_A \circ \Theta_t \) for some measurable \( A \) and

\[
\int_B \mathcal{L}_t^t(f)(\Theta_t(w))dP(w) = \int I_B(w)\mathcal{L}_t^t(f)(\Theta_t(w))dP(w) = \\
\int (I_A \circ \Theta_t)(w) \mathcal{L}_t^t(f)(\Theta_t(w)) dP(w) = \int I_A(w)\mathcal{L}_t^t(f)(w)dP(w) = \\
\int I_A(\Theta_t(w)) f(w) dP(w) = \int_B f(w)dP(w).
\]

\[ \square \]

3. The modified operator Ruelle Operator associated to \( V \)

We consider \( V : \Omega \to \mathbb{R} \), such that is constant in sets of the form \( \{ X_0 = c \} \). We are interested in the operator obtained by the perturbation of the operator \( \mathcal{L}^t \) by \( V \).

**Definition 3.1.** We define \( G_t : \Omega \to \mathbb{R} \) as

\[
G_t(x) = \exp \left( \int_0^t V(x(s))ds \right)
\]

**Definition 3.2.** We define the \( G \)-weighted transfer operator \( \mathcal{L}_V^t : \mathcal{L}^\infty(\Omega, P) \to \mathcal{L}^\infty(\Omega, P) \) acting on measurable functions \( f \) (of the form \( f = I_{\{X_0=a_0,X_{t_1}=a_1, \ldots,X_{t_r}=a_r\}} \)) by

\[
\mathcal{L}_V^t(f)(w) := \mathcal{L}_t^t(G_t f) = \\
\mathcal{L}_t^t(e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} f) = \sum_{b=1}^n \mathcal{L}_t^t(e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} I_{\{X_{t_i}=b\}} f)(w).
\]

Note that \( e^{\int_0^t (V \circ \Theta_s)(\cdot)ds} I_{\{X_{t_i}=b\}} \) does not depend on information for time larger than \( t \). In the case \( f \) is such that \( t_r \leq t \) (in the above notation), then \( \mathcal{L}_V^t(f)(w) \) depends only on \( w(0) \).

The integration on \( s \) above is consider over the open interval \((0,t)\).

We will show next the existence of an eigenfunction and an eigen-measure for such operator \( \mathcal{L}_V^t \). First we need the following:

**Theorem (Perron-Frobenius for continuous time).** ([S] page 111)

Given \( L \), \( p_0 \) and \( V \) as above, there exists
a) a unique positive function \( u_V : \Omega \to \mathbb{R} \), constant equal to the value \( u_V(i) \) in each cylinder \( X_0 = i, i \in \{1, 2, \ldots, n\} \), (sometimes we will consider \( u_V \) as a vector in \( \mathbb{R}^n \)).

b) a unique probability vector \( \mu_V \) in \( \mathbb{R}^n \) (a probability over over the set \( \{1, 2, \ldots, n\} \) such that \( \mu_V(\{i\}) > 0, \forall i \)), that is,

\[
\sum_{i=1}^{n} (u_V)_i (\mu_V)_i = 1,
\]

c) a real positive value \( \lambda(V) \), such that for any positive \( s \)

\[
e^{-s\lambda(V)} u_V e^{s(L+V)} = u_V.
\]

d) Moreover, for any \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \)

\[
\lim_{t \to \infty} e^{-t\lambda(V)} v e^{t(L+V)} = (\sum_{i=1}^{n} v_i (\mu_V)_i) u_V,
\]

e) for any positive \( s \)

\[
e^{-s\lambda(V)} e^{s(L+V)} \mu_V = \mu_V.
\]

From property e) it follows that

\[(L+V) \mu_V = \lambda(V) \mu_V,
\]
or

\[(L+V - \lambda(V) I) \mu_V = 0.
\]

From c) it follows that

\[u_V (L+V) = \lambda(V) u_V,
\]
or

\[u_V (L+V - \lambda(V) I) = 0.
\]

We point out that e) means that for any positive \( t \) we have \((P_V^t)^* \mu_V = e^{\lambda(V)t} \mu_V\).

Note that when \( V = 0 \), then \( \lambda(V) = 0 \), \( \mu_V = p^0 \) and \( u_V \) is constant equal to 1.

Now we return to our setting: for each \( i_0 \) and \( t \) fixed one can consider the probability \( \mu_{i_0}^t \) defined over the sigma-algebra \( \mathcal{F}_t = \sigma(\{X_s | s \leq t\}) \) with support on \( \{X_0 = i_0\} \) such that for cylinder sets with \( 0 < t_1 < \ldots < t_r \leq t \)

\[
\mu_{i_0}^t (\{X_0 = i_0, X_{t_1} = a_1, \ldots, X_{t_r-1} = a_{r-1}, X_t = j_0\}) = P_{j_0a_r} \cdots P_{a_2a_1} P_{a_1i_0}.
\]

The probability \( \mu_{i_0}^t \) is not stationary.
We denote by $Q(j, i)_{t}$ the $i, j$ entry of the matrix $e^{t(L+V)}$, that is $(e^{t(L+V)})_{j,i}$. It is known ([K] page 52 or [S] Lemma 5.15) that

$$Q(j_0, i_0)_{t} = E_{(X_0 = i_0)}\{ e^{\int_0^t (V \circ \Theta_s)(w) ds} : X(t) = j_0 \} = \int_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w) ds} d\mu_{t}(w).$$

For example,

$$\int_{\{X_t = j_0\}} e^{\int_0^t (V \circ \Theta_s)(w) ds} dP = \sum_{i=1, 2, \ldots, n} Q(j_0, i)_{t} p^0_i.$$ 

In the particular case where $V$ is constant equal 0, then $p^0 = \mu_V$ and $\lambda(V) = 0$.

We denote by $f = f_V$, where $f(w) = f(w(0))$, the density of probability $\mu_V$ in $S$ with respect to the probability $p^0$ in $S$.

Therefore, $\int f dp^0 = 1$.

**Proposition 3.3.** $f_V(w) = \frac{\mu_V(w)}{p^0(w)} = \frac{(\mu_V)_0}{(p^0)_0}$, $f_V : \Omega \to \mathbb{R}$, is an eigen-function for $\mathcal{L}^t_V$ with eigenvalue $e^t \lambda(V)$.

**Proof:** Note that $\frac{\mu_V(w)}{p^0(w)} = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} I\{X_0 = c\}$.

For a given $w$, denote $w(0)$ by $j_0$, then conditioning

$$\mathcal{L}_V^t (\frac{\mu_V}{p^0})(w) = \sum_{c=1}^n \sum_{b=1}^n \mathcal{L}_V^t \left( \frac{\mu_V(c)}{p^0(c)} I\{X_0 = c\} I\{X_t = b\} \right) (w).$$

Consider $c$ fixed, then for $b = j_0$ we have

$$\mathcal{L}_V^t ( I\{X_0 = c\} I\{X_t = j_0\} ) (w) = \frac{Q(j_0, c)_{t} p_0^j}{p_0^j},$$

and for $b \neq j_0$, we have $\mathcal{L}_V^t ( I\{X_0 = c\} I\{X_t = b\} ) (w) = 0$.

Finally, for any $t > 0$

$$\mathcal{L}_V^t (\frac{\mu_V}{p^0})(w) = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} Q(j_0, c_{t} p_0^j) = \sum_{c=1}^n \frac{\mu_V(c)}{p^0(c)} (e^{t(L+V)})_{j_0} = e^{t\lambda(V)} (\mu_V)_{j_0} = e^{t\lambda(V)} (\frac{\mu_V}{p^0}) (w),$$

because $e^{t(L+V)}(\mu_V) = e^{t\lambda(V)}(\mu_V)$. 

Therefore, for any $t > 0$ the function $f_V = \frac{\mu_V}{p}$ (that depends only on $w(0)$) is an eigenfunction for the operator $\hat{L}^t_V$ associated to the eigenvector $e^{t(\lambda(V))}$.

The above result shows that the eigenfunction for the Ruelle operator associated to the eigenvector $V$ is the Radon-Nykodin derivative for $\mu_V$ with respect to $p^0$.

**Definition 3.4.** Consider for each $t$ the operator acting on $g$ given by

$$\hat{L}^t_V(g)(w) = \left[ \frac{1}{f_V} \hat{L}^t(e^{t(\lambda(V))}g\Theta_t)(1)ds \right] g f_V \right](w)$$

From the above $\hat{L}^t_V(1) = 1$, for all positive $t$.

We present some examples: note that by conditioning, if $g = I_{\{X_0=a_0, X_{t_1}=a_1, X_{t_2}=a_2, X_t=a_3\}}$, with $0 < t_1 < t_2 < t$, then

$$\hat{L}^t_V(g)(w) = \frac{1}{p_{a_3}^0} \frac{\mu_V(a_3)}{\mu_V(a_3)} e^{(t-t_2)(L+V-\lambda I)}e^{(t_2-t_1)(L+V-\lambda I)}e^{t_1(L+V-\lambda I)}e^{t_1(L+V-\lambda I)} \frac{\mu_V(a_0)}{p_{a_0}^0} =$$

$$\frac{1}{p_{a_3}^0} \frac{\mu_V(a_3)}{\mu_V(a_3)} e^{(t-t_2)(L+V-\lambda I)}e^{(t_2-t_1)(L+V-\lambda I)}e^{t_1(L+V-\lambda I)}e^{t_1(L+V-\lambda I)} \frac{\mu_V(a_0)}{p_{a_0}^0},$$

for $w$ such that $w_0 = a_3$, and $\hat{L}^t_V(g)(w) = 0$ otherwise.

Moreover, for $g = I_{\{X_0=a_0, X_{t_1}=a_1, X_{t_2}=a_2\}}$, with $0 < t_1 < t < t_2$, then

$$\hat{L}^t_V(g)(w) = \frac{1}{\mu_V(a)} e^{(t-t_1)(L+V-\lambda I)}e^{t_1(L+V-\lambda I)} \frac{\mu_V(a)}{p_{a_0}^0},$$

for $w$ such that $w_0 = a$, $w_{t_2-t} = a_2$, and $\hat{L}^t_V(g)(w) = 0$ otherwise.

Consider now the dual operator $(\hat{L}^t_V)^*$. For $t$ fixed consider the transformation in the set of measures $\mu$ on $\Omega$ given by $(\hat{L}^t_V)^*(\mu) = \nu$.

**Theorem 1.** For each $t$ there exists a probability measure $\nu_V$ on $(\Omega, B)$ which is fixed for such transformation $(\hat{L}^t_V)^*$. The probability $\nu_V$ does not depend on $t$.

**Proof:**

We have to show that there exists $\nu_V$ such that for all $t > 0$ and for all $g$ we have

$$\int \hat{L}^t_V(g) d\nu_V = \int g d\nu_V.$$
Remember that, \( \hat{L}_V^t(1) = 1 \), therefore, if \( \mu \) is a probability, then \((\hat{L}_V^t)^*(\mu) = \nu \) is also a probability.

Denote by \( \nu = \nu_V \) the probability obtained in the following way, for
\[
g = I\{X_0 = a_0, X_{t_1} = a_1, X_{t_2} = a_2, \ldots, X_{t_{r-1}} = a_{r-1}, X_r = a_r\},
\]
with \( 0 < t_1 < t_2 < \ldots < t_{s-1} < t_s < \ldots < t_r \), we define
\[
\int g(w) \, d\nu(w) = e^{(t_r-t_{r-1})(L+V-\lambda I)} \cdots e^{(t_2-t_1)(L+V-\lambda I)} e^{t_1(L+V-\lambda I)} \mu_V(a_0).
\]

It is easy to see that these probability transitions satisfy the Kolmogorov compatibility conditions. In section 4.1 and in Theorem 1.1 (same section) in [EK] it is described these conditions. We point out that the space we consider is separable and complete according to chapter 3 in the same book.

In order to show that \( \nu \) is a probability we have to use the fact that \( \sum_{c \in S} \mu_V(c) = 1 \).

For example, \( \int I_{\{X_0 = c\}} \, d\nu = \mu_V(c) \). Moreover,
\[
\int 1 \, d\nu = \sum_c \sum_a \int I_{\{X_t = c, X_0 = a\}} \, d\nu = \sum_c \sum_a e^{t(L-V-\lambda I)} \mu_V(a) = \sum_c \mu_V(c) = 1.
\]

Suppose \( t \) is such that \( 0 < t_1 < t_2 < \ldots < t_{s-1} < t \leq t_s < \ldots < t_r \), then
\[
z(w) = \hat{L}_V^t(g)(w) = \frac{1}{\mu_V(a)} e^{(t-t_{s-1})(L+V-\lambda I)} \cdots e^{(t_2-t_1)(L+V-\lambda I)} e^{t_1(L+V-\lambda I)} \mu_V(a_0),
\]
for \( w \) such that \( w(0) = a, w_{t_{s-1}} = a_s, w_{t_{s+1}} = a_{s+1}, \ldots, w_{t_{r-t}} = a_r \), and \( \hat{L}_V^t(g)(w) = 0 \) otherwise. Note that \( z(w) = \hat{L}_V^t(g)(w) \) depends only on \( w_0, w_{t_1}, w_{t_2}, \ldots, w_{t_r-t} \).

We have to show that for any \( g \) we have \( \int g \, d\nu = \int \hat{L}_V^t(g) \, d\nu \).

Now,
\[
\int z(w) \, d\nu(w) = \int \sum_{c \in S} I_{\{X_0 = c, X_{t_2} = a_s, X_{t_{s+1}} = a_{s+1}, \ldots, X_{t_r} = a_r\}} \, z(w) \, d\nu(w) = \sum_{c \in S} \mu(\{X_0 = c, X_{t_2} = a_s, X_{t_{s+1}} = a_{s+1}, \ldots, X_{t_r} = a_r\}) \frac{1}{\mu_V(c)} e^{(t-t_{s-1})(L+V-\lambda I)} \cdots e^{(t_2-t_1)(L+V-\lambda I)} e^{t_1(L+V-\lambda I)} \mu_V(a_0) = \int \hat{L}_V^t(g) \, d\nu.
\]
Theorem A. For any integrable $w \in \mathcal{L}^\infty(P)$ and any positive $t$
\[ \mathcal{L}^t(f_V(w)) = \sum_j \mathcal{L}^t(I_{\{X_0=j\}}) f_V(j) = \sum_j \mathcal{L}^t(I_{\{X_0=j\}}) \frac{\mu_V(j)}{p_j^0} = \sum_j \frac{1}{p_c^0} P_{cj} \frac{\mu_V(j)}{p_j^0} = \sum_j \frac{1}{p_c^0} P_{cj} \mu_V(j) = \frac{\mu_V(c)}{p_c^0} = f_V(w). \]

\[ \sum_{c \in S} e^{(t_1-t_0-r_0)} (L+V-\lambda I) \ldots e^{(t_{s+1}-t_s)(L+V-\lambda I)} e_{a_{s+1}a_s}^{t_s-t} (L+V-\lambda I) \mu_V(c) \]
\[ \frac{1}{\mu_V(c)} e^{(t-t_0-r_0)} (L+V-\lambda I) \ldots e^{(t_{s+1}-t_s)(L+V-\lambda I)} e_{a_{s+1}a_s}^{t_s-t} (L+V-\lambda I) \mu_V(c) = \]
\[ \sum_{c \in S} e^{(t_1-t_0-r_0)} (L+V-\lambda I) \ldots e^{(t_{s+1}-t_s)(L+V-\lambda I)} e_{a_{s+1}a_s}^{t_s-t} (L+V-\lambda I) \mu_V(c) = \]
\[ \int g \, d\nu. \]

The claim for the general $g$ follows from the above result.
Therefore, $(\hat{\mathcal{L}}_V^t)^*(\nu_V) = \nu_V$.

□

**Definition 3.5.** Consider the stationary probability $\rho_V = f_V \nu_V$ on $\Omega$. We call it the equilibrium state for $V$.

**Definition 3.6.** We call the probability $\nu_V$ on $\Omega$ the Gibbs state for $V$.

**Proposition 3.7.** For any integrable $f, g \in \mathcal{L}^\infty(P)$ and any positive $t$
\[ \int \hat{\mathcal{L}}_V^t(f) \, g \, d\nu_V = \int \hat{\mathcal{L}}_V^t(f(g \circ \theta_t)) \, d\nu_V = \int f(g \circ \theta_t) \, d\nu_V. \]

Now we can prove our main result.

**Theorem A.** For any integrable $g \in \mathcal{L}^\infty(P)$ and any positive $t$
\[ \int e^{-\int_0^t (V \circ \theta_s) \, ds} \left[ \left( \frac{1}{f_V} \mathcal{L}^t \left( e^{\int_0^t (V \circ \theta_s) \, ds} g \, f_V \right) \right) \circ \theta_t \right] \, d\nu_V = \int g \, d\nu_V. \]

**Proof:**
Note first that if $w$ is such that $w(0) = c$, then
\[ \mathcal{L}^t(f_V(w)) = \sum_j \mathcal{L}^t(I_{\{X_0=j\}}) f_V(j) = \sum_j \mathcal{L}^t(I_{\{X_0=j\}}) \frac{\mu_V(j)}{p_j^0} = \]
\[ \sum_j \frac{1}{p_c^0} P_{cj} \frac{\mu_V(j)}{p_j^0} = \sum_j \frac{1}{p_c^0} P_{cj} \mu_V(j) = \frac{\mu_V(c)}{p_c^0} = f_V(w). \]
Therefore, for all $w$
\[
\frac{1}{f_V} \mathcal{L}^t(f_V(w)) = 1.
\]

Finally,
\[
\int e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} \left( \frac{1}{f_V} \mathcal{L}^t \left( e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f_V \right) \right) d\nu_V =
\int e^{-\int_0^t (V \circ \Theta_s - \lambda)(\cdot) ds} \left( \frac{1}{f_V} \mathcal{L}^t \left( e^{\int_0^t (V \circ \Theta_s - \lambda)(\cdot) ds} f_V \right) \right) d\nu_V =
\int [\hat{L}_V(e^{-\int_0^t (V \circ \Theta_s - \lambda)(\cdot) ds} f_V)] \left( \frac{1}{f_V} \mathcal{L}^t \left( e^{\int_0^t (V \circ \Theta_s - \lambda)(\cdot) ds} f_V \right) \right) d\nu_V =
\int \left[ \frac{1}{f_V} \mathcal{L}^t \left( e^{-\int_0^t (V \circ \Theta_s)(\cdot) ds} e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f_V \right) \right] \left( \frac{1}{f_V} \mathcal{L}^t \left( e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f_V \right) \right) d\nu_V =
\int \left[ \frac{1}{f_V} \mathcal{L}^t(f_V) \right] \left[ \frac{1}{f_V} \mathcal{L}^t(e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f_V) \right] d\nu_V =
\int \frac{1}{f_V} \mathcal{L}^t(e^{\int_0^t (V \circ \Theta_s)(\cdot) ds} f_V) d\nu_V = \int \hat{L}_V(g) d\nu_V = \int g d\nu_V.
\]

\[\square\]

Referências