A Class of Topological Foliations on $S^2$
That Are Topologically Equivalent to Polynomial Vector fields

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Abstract. Let $\mathcal{F}$ be an oriented topological foliation on $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$ having only a finite number of singularities. If $\mathcal{F}$ has only a finite number of closed orbits and satisfies one additional condition, then it is shown that $\mathcal{F}$ is topologically equivalent to (the foliation induced by) a polynomial vector field.

1. Introduction

In this note we extend the main result of Schecter-Singer [4] from the $C^1$-class to the $C^0$-class. While the statement of our result is a little more general than that of [4], when we restrict to the $C^1$-class, the proofs given in [4] apply to the situation stated here (see Remark 2.1 below). Besides extending to the $C^0$-topology, we wanted to present, in a concise way, this very nice result of Schecter-Singer whose complete statement takes the first 16 pages of the referred article. We must say that this work depends on the results and arguments given in [4].

Two (one-dimensional) oriented topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, with or without singularities, defined on 2-manifolds $M_1$ and $M_2$, respectively, with corresponding set of singularities $S_1 \subset M_1$ and $S_2 \subset M_2$ are called topologically equivalent if there is a homeomorphism $h : M_1 \rightarrow M_2$ that takes $S_1$ onto $S_2$ and sends orbits (i.e. leaves) of $\mathcal{F}_1$ onto orbits of $\mathcal{F}_2$, preserving the direction of the orbits.

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In this paper, we consider a class of oriented topological foliation, with singularities, on $S^2$ that are topologically equivalent to (the foliations induced by) polynomial vector fields. Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. “Vector field on $S^2$” always means a tangent vector field to $S^2$; a polynomial vector field on $S^2$ is, in addition, one each of whose coordinates is a polynomial in $x, y, z$.

Isolated singularities $p$ and $q$ of oriented topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on 2–manifolds $M$ and $N$ are called topologically equivalent if there are neighborhoods $U$ and $V$ of $p$ and $q$ such that $\mathcal{F}_1|_U$ is topologically equivalent to $\mathcal{F}_2|_V$ via a homeomorphism that takes $p$ to $q$.

Orient $S^2$ by its unit outer normal vector field. In this paper, an isolated singularity $p$ of an oriented topological foliation $\mathcal{F}$ on $S^2$ is said to be of finite type if (i) it is not topologically equivalent to a node; (ii) the local phase portrait of $p$ is the union of finitely many hyperbolic, elliptic and parabolic sectors in the sense of [1, page 315]; in particular the elliptic sectors have no hyperbolic parts and the hyperbolic sectors have no elliptic parts ([3, Chapter VII – page 161]).

2. Singularities of finite type

Let $p$ be an isolated singularity of an oriented topological foliation of finite type $\mathcal{F}$ on $S^2$. Then $p$ has arbitrarily small canonical neighborhoods homeomorphic to compact discs whose boundaries are circles having the least possible number of tangencies with the foliation $\mathcal{F}$. In all figures, $C$ will denote one of these circles [1, pp. 313-314], [3, Chapter VII – page 161]; see Fig. 1.

The restrictions of $\mathcal{F}$ to any two canonical neighborhoods of $p$ are topologically equivalent. There is a familiar division of any canonical neighborhood of $p$ into a finite number of elliptic, hyperbolic, and parabolic sectors [1, Chap. 8]; see Fig. 1. If $\gamma$ is an orbit of $\mathcal{F}$, we shall denote by $\gamma(t)$ an arbitrary parametrization of $\gamma$, with $t$ varying in $\mathbb{R}$ and such that, for increasing $t$, $\gamma(t)$ moves in conformity with the orientation of $\mathcal{F}$. The definitions below do no depend on the particular parametrization $\gamma(t)$ of $\gamma$. An $\alpha$-(resp. $\omega$-)separatrix at $p$ is a semiorbit $\gamma(t)$ of $\mathcal{F}$ that approaches $p$ as $t \to -\infty$ (resp. as $t \to \infty$) and that bounds a hyperbolic sector at $p$. We shall use the shorter expression separatrix to refer to an orbit of $\mathcal{F}$ that includes an $\alpha$- or $\omega$- separatrix at any singularity. If $\gamma = \gamma(t)$ is the orbit of $\mathcal{F}$ that passes through $p$ at $t = 0$, then $q$ belongs to the $\alpha$-limit set (resp. $\omega$-limit set) of $p$ if and only if there is a sequence $t_n \to -\infty$ (resp. $t_n \to \infty$) such that $||\gamma(t_n) - q|| \to 0$. A limit set $K$ is the $\alpha$- or $\omega$-limit set of some
point; a limit set is always a compact connected union of orbits. Moreover, if \( \mathcal{F} \) has only a finite number of singularities and closed orbits, then by the Poincaré Bendixson Theorem, each limit set of \( \mathcal{F} \) is either a singularity or a single closed orbit or else a compact connected union of singularities and orbits that are \( \alpha \)-separatrices at one end and \( \omega \)-separatrices at the other. A limit set of the latter type is called a separatrix cycle. If \( \Gamma \) is an attracting separatrix cycle (resp. a repelling separatrix cycle), there exists an open cylinder \( A \) such that \( A \cap \Gamma = \emptyset, \Gamma \subset \overline{A} \), and for all \( p \in A \), the \( \omega \)-limit set of \( p \) is \( \Gamma \) (resp. the \( \alpha \)-limit set of \( p \) is \( \Gamma \)).

Let \( S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \). Given an subinterval \( I \) of \([0,\infty)\) we shall denote by

\[ I \cdot S^1 = \{(ax,ay) : a \in I, (x,y) \in S^1\} \]

The set \( I \cdot S^1 \) will be simply denoted by \( S^1 \). An oriented topological foliation \( \mathcal{F} \) over \((0,2) \cdot S^1 \) is said to be of type 1 (and degree \( s \in \mathbb{N} \setminus \{0\} \)) if (i) the terms of the sequence \( \{p_k = (\cos(2\pi(k-1)/s), \sin(2\pi(k-1)/s)) : k = 1,2,\ldots,s\} \) of \( S^1 \) make up the set \( S \) of singularities of \( \mathcal{F} \); (ii) every such singularity \( p_k \) is topologically equivalent to either a hyperbolic saddle or to a node, and (iii) \( S^1 \setminus S \) is made up of (full) orbits of \( \mathcal{F} \). We shall say that \((p_1,p_2,\ldots,p_s)\) is the sequence of singularities of \( \mathcal{F} \).

Let \( \mathcal{F} \) be an oriented topological foliation on \( S^2 \). If \( p \) is an isolated singularity of finite type, there exists an open neighborhood \( V \) of \( p \) and a type one foliation \( \mathcal{F}_1 \) on \((0,2) \cdot S^1 \) such that, for some \( \varepsilon > 0 \), \( \mathcal{F}_1|_{(1,1+\varepsilon)\cdot S^1} \) is

topologically equivalent to $\mathcal{F}|_{V\setminus\{p\}}$. Since $\mathcal{F}_1|_{S^1}$ is a one–dimensional oriented foliation having only attracting and repelling singularities, it (and so $\mathcal{F}_1$) has an even number $s$ of singularities. The foliation $\mathcal{F}_1$ will be said to be a topological blown up of $p$. Let $(p_1, p_2, \ldots, p_s)$ be the sequence of singularities of $\mathcal{F}_1$. The saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \ldots, p_s))$, is the sequence of $s$ symbols from the set $\{S_\alpha, S_\omega, N_\alpha, N_\omega\}$. The $j$th symbol is determined by the behavior of $\mathcal{F}_1$ in $[1, 2) \cdot S^1$ near $p_j$. The $j$th symbol of the saddle-node sequence is

- $S_\alpha$ (resp. $S_\omega$) if there are two hyperbolic sectors of $\mathcal{F}_1$ at $p_j$ in $[1, 2) \cdot S^1$, bounded by $S^1$ and an $\alpha$- (resp. $\omega$-) separatix at $p_j$;
- $N_\alpha$ (resp. $N_\omega$) if a neighborhood of $p_j$ in $[1, 2) \cdot S^1$ is the union of negative (resp. positive) semi-orbits of $\mathcal{F}_1$ that converge to $p_j$.

The saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \ldots, p_s))$, will be said to be a saddle-node sequence of $p$. The saddle-node cycle of a singularity is just the saddle-node sequence thought of as a cycle: the first term in the sequence follows the last. In the following lemma, which is immediate, if $\delta$ denotes $\alpha$ (resp. denotes $\omega$), then $\delta^*$ will denote $\omega$ (resp. will denote $\alpha$).

**Lemma 2.1.** Let $\mathcal{F}$ be a topological foliation on $S^2$ having an isolated singularity $p$ of finite type. Let $\mathcal{F}_1$ be a topological blown up of $p$ and let $(p_1, p_2, \ldots, p_s)$ be the sequence of singularities of $\mathcal{F}_1$. Let $\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$, be the saddle-node sequence of $(\mathcal{F}_1, (p_1, p_2, \ldots, p_s))$. Then, the first symbol in a saddle-node cycle $\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$, of a finite type singularity $p$, can be taken to be $S_\alpha$ or $N_\omega$. Moreover, for $\delta \in \{\alpha, \omega\}$,

1. $S_\delta$ (resp. $N_\delta$) is always followed by $S_\delta^*$ or $N_\delta^*$ (resp. by $S_\delta$ or $N_\delta^*$).
2. Each pair of consecutive terms $\sigma_i, \sigma_{i+1}$ of the form $S_\delta S_\delta^*$ corresponds to exactly one hyperbolic sector $\text{Sec}(\sigma_i, \sigma_{i+1})$ of $p$. See Fig. 2.
3. Each pair of consecutive terms $\sigma_i, \sigma_{i+1}$ of the form $N_\delta N_\delta^*$ corresponds to exactly one elliptic sector $\text{Sec}(\sigma_i, \sigma_{i+1})$ of $p$;
4. Let $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k}$ be a subsequence of $\Sigma$ such that (i) $\sigma_i, \sigma_{i+k} \in \{S_\delta, N_\delta\}$ and, (ii) every term $\sigma_i, \sigma_{i+2}, \ldots, \sigma_{i+k-1}$ belongs to $\{S_\delta, N_\delta\}$ (and so $S_\delta S_\delta^*$ and $N_\delta N_\delta^*$ alternate). Then

    - (4.1) if $k \geq 3$ is odd and $\text{Sec}(\sigma_i, \sigma_{i+1}), \text{Sec}(\sigma_{i+k-1}, \sigma_{i+k})$ are elliptic then $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k}$ corresponds to exactly one parabolic sector $\text{Sec}(\sigma_i, \sigma_{i+k})$ separating the referred two elliptic sectors. See Fig. 3.
    - (4.2) if $k \geq 4$ is even and one between $\text{Sec}(\sigma_i, \sigma_{i+2}), \text{Sec}(\sigma_{i+k-2}, \sigma_{i+k})$ is elliptic and the other hyperbolic, then $\sigma_i, \sigma_{i+1}, \ldots, \sigma_{i+k}$ corresponds to exactly one parabolic sector $\text{Sec}(\sigma_i, \sigma_{i+k})$ separating the referred two sectors. See Fig. 4.
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(4.3) if $k \geq 5$ is odd and $\text{Sec}(\sigma_{i+1}, \sigma_{i+2})$, $\text{Sec}(\sigma_{i+k-1}, \sigma_{i+k})$ are hyperbolic then $\sigma_{i+1}, \ldots, \sigma_{i+k}$ corresponds to exactly one parabolic sector $\text{Sec}(\sigma_{i+1}, \ldots, \sigma_{i+k})$ separating the referred two hyperbolic sectors. See Fig. 5.

(5) The topological blown up $\mathcal{F}_1$ of $p$ can be taken so that, for any parabolic sector $P$ of $p$, and modulo the restrictions imposed by (4) above, we may select the length $k$ of the subsequence $\sigma_{i+1}, \ldots, \sigma_{i+k}$ of $\Sigma$ which satisfies $P = \text{Sec}(\sigma_{i+1}, \ldots, \sigma_{i+k})$.

![Figure 2](image1.png)

![Figure 3](image2.png)

![Figure 4](image3.png)

We shall say that the topological blown up $\mathcal{F}_1$ of the singularity $p$ as above is *tight* if the subsequences of $\Sigma$ associated to parabolic sectors have lengths 3, 4 and 5 according as they correspond to the situations considered in (4.1), (4.2) and (4.3), respectively.

Let $\mathcal{F}$ be a oriented topological foliation on $S^2$ having finitely many singularities, each of which is either of finite type or topologically equivalent to a node. Let $\{p_1, p_2, \ldots, p_s\}$ be the singularities of $\mathcal{F}$ of finite type. For each such singularity $p_i$, we consider a topological blown up $\mathcal{F}_i$ of $p_i$ and construct a corresponding saddle–node sequence $\Sigma_i = \sigma_{i1}\sigma_{i2}, \ldots, \sigma_{im_i}$ as above. Set $d_i = (m_i + 2)/2$. Each separatrix cycle $K$ of $\mathcal{F}$ corresponds to a cycle $C_K$ of some of the $\sigma_{ij}$. Any $\sigma_{ij}$ in such a cycle is an $S_\alpha$ or an $S_\omega$.

Let $\mathcal{L}$ denote the set of all $\sigma_{ij}$ such that $\sigma_{ij} \in \{S_\alpha, S_\omega\}$ and $\sigma_{ij+d_i-1} \in \{S_\alpha, S_\omega\}$. Here the second subscript is mod $m_i$. We say $\mathcal{F}$, $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_s)$) satisfies the *separatrix cycle condition* provided there is a function $f(\sigma_{ij})$ from $\mathcal{L}$ to the positive reals such that

(F1) $f(\sigma_{ij}) = f(\sigma_{ij+d_i-1})$ if $d_i - 1$ is even; $f(\sigma_{ij}) = [f(\sigma_{ij+d_i-1})]^{-1}$ if $d_i - 1$ is odd.

(F2) For every one-sided limit set $K$ of $\mathcal{F}$ that is a separatrix cycle, either

1. all $\sigma_{ij}$ in $C_K$ are in $\mathcal{L}$ and $\prod_{\sigma_{ij} \in C_K} f(\sigma_{ij}) > 1$ (resp. $< 1$) if $K$ is attracting (resp. repelling);
2. some $\sigma_{ij}$ in $C_K$ are not in $\mathcal{L}$; if $K$ is attracting (resp. repelling), all such $\sigma_{ij}$ are $S_\alpha$’s (resp. $S_\omega$’s).
Our main result is

**Theorem 2.1.** Let $\mathcal{F}$ be a one-dimensional oriented topological foliation on $S^2$ such that

(H1) it has only a finite number of closed orbits and it has finitely many singularities; every singularity is either of finite type or topologically equivalent to a node;

(H2) if $p_1, p_2, \cdots, p_s$, are the finite type singularities of $\mathcal{F}$, then, for every such $p_i$ there exists a topological blown up $\mathcal{F}_i$ of $p_i$ such that $(\mathcal{F}, (\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_s))$ satisfies the separatrix cycle condition.

Then $\mathcal{F}$ is topologically equivalent to a polynomial vector field.

**Remark 2.1.** S. Schecter and M. F. Singer state and prove the above theorem in the case that $\mathcal{F}$ is induced by a $C^1$-vector field and every $\mathcal{F}_i$ is a tight blown up of $p_i$. Nevertheless, within the $C^1$-class, their proof applies to the situation stated here. This fact was observed in [4, Example 3 – page 423].

The proof of the following proposition follows immediately from the Smoothing Theorem and the Smoothing Corollary of [2].

**Proposition 2.1.** Let $\mathcal{F}$ be a continuous one dimensional orientable foliation with singularities on the 2-sphere $S^2$. If the set of singularities of $\mathcal{F}$ is compact, then there exists a $C^\infty$ vector field $X$ on $S^2$ which is topologically equivalent to $\mathcal{F}$.

**Proof of Theorem 2.1.** It follows from Proposition 2.1 that the exists a smooth vector field $Y$ topologically equivalent to $\mathcal{F}$.

By Schecter-Singer main result [4] (see Remark 2.1) $Y$ is topologically equivalent to a polynomial vector field.

**References**


