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Partial orders related to the Hom-order and degenerations.

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Abstract. Given a finite length module $M$ over a $K$-algebra $\Lambda$, an $n \times n$-matrix $A$ over $\Lambda$ induces a $K$-homomorphism $A_M : M^n \to M^n$. We then define the relation $\leq_n$ by

$$M \leq_n N \iff \ell(\text{coker } A_M) \leq \ell(\text{coker } A_N).$$

We will show that $\leq_n$ is a partial order on the set of modules of length $d$ (modulo isomorphisms) if $n \geq d^2$.

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Throughout the paper let $\Lambda$ be an artin algebra with center $K$, and let $\text{mod } \Lambda$ denote the category of finitely generated left $\Lambda$-modules. For a $\Lambda$-module $X$, $\ell(X)$ denotes the length of $X$ as a $K$-module. For a homomorphism $\phi$, $\text{im } \phi$ denotes its image and $\text{coker } \phi$ denotes its cokernel.

For a natural number $d$, let $\text{rep}_d \Lambda = \{ X \in \text{mod } \Lambda \mid \ell(X) = d \}$. One can define several partial orders on $\text{rep}_d \Lambda$ modulo isomorphisms (see [3]). Here we will look at the Hom-order and the quasiorders $\leq_n$, and investigate for which $n \leq d^2$ is a partial order on $\text{rep}_d \Lambda$.

Definition. The relation $\leq_{\text{Hom}}$ on $\text{rep}_d \Lambda$ is defined by $M \leq_{\text{Hom}} N$ if $\ell(\text{Hom}_\Lambda(X,M)) \leq \ell(\text{Hom}_\Lambda(X,N))$ for all $X \in \text{mod } \Lambda$. This relation is called the Hom-order.

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We immediately see that \( \leq_{\text{Hom}} \) is reflexive and transitive. That it is also antisymmetric was first shown by Auslander in [1], a result that will be stated here as Corollary 2.

Let \( \mathcal{M}_n(\Lambda) \) denote the ring of \( n \times n \)-matrices over \( \Lambda \). For a matrix \( A \in \mathcal{M}_n(\Lambda) \) and a \( \Lambda \)-module \( M \), let \( A_M : M^n \to M^n \) denote the \( K \)-homomorphism given by multiplying a column vector of elements from \( M \) with the matrix \( A \).

**Definition.** For a natural number \( n \), the relation \( \leq_n \) on \( \text{rep}_d \Lambda \) is defined by \( M \leq_n N \) if \( \ell(\text{coker} \ A_M) \leq \ell(\text{coker} \ A_N) \) for all \( A \in \mathcal{M}_n(\Lambda) \).

Again it is obvious that this relation is reflexive and transitive, but it is not in general antisymmetric. We have that \( \ell(\text{coker} \ A_M) = \ell((\text{Hom}_\Lambda(\Lambda^n/\Lambda^n A, M))) \), where \( \Lambda^n A \) denotes the image of the \( \Lambda \) - endomorphism induced by matrix multiplication from the right with \( A \). Hence, \( M \leq_{\text{Hom}} N \) implies \( M \leq_n N \). We also have that \( M \leq_{n+1} N \) implies \( M \leq_n N \) for any \( n \). It follows that if \( \leq_i \) is a partial order, then so is \( \leq_n \) for all \( n \geq i \). If \( K \) is an algebraically closed field, then we have by Hilbert’s Basis Theorem that there must exist a minimal \( n \) such that \( \leq_n \) is equivalent to \( \leq_{\text{Hom}} \) on \( \text{rep}_d(\Lambda) \). However, the Basis Theorem does not give any clue about how large this \( n \) is, and there is no known procedure for finding it. It is also not known if this is the same as the minimal \( n \) that makes \( \leq_n \) a partial order.

This latter \( n \) we can at least give a bound on. The following result was stated without proof in [3].

**Theorem 1.** For \( n \geq d^5 \) the relation \( \leq_n \) is a partial order on \( \text{rep}_d \Lambda \).

**Proof.** We need to show that \( \leq_n \) is antisymmetric when \( n \geq d^5 \). Let \( M \) and \( N \) be non-isomorphic \( \Lambda \)-modules of length \( d \). First we want to show that there exists a \( \Lambda \)-module \( X \) with \( \ell((\text{Hom}_\Lambda(X, M))) \neq \ell((\text{Hom}_\Lambda(X, N))) \) and \( \ell(X) \leq d^5 \). If \( M \) and \( N \) have any nonzero common direct summands, we can cancel them, so we assume they have none.

Let \( \{f_1, f_2, \ldots, f_m\} \) be a generating set for \( \text{Hom}_\Lambda(M, N) \) as a \( K \)-module. Letting \( C = \text{coker}(f_1, f_2, \ldots, f_m)^{tr} \) we have an exact sequence

\[
M \xrightarrow{(f_1, f_2, \ldots, f_m)^{tr}} N^m \rightarrow C \rightarrow 0.
\]

We have $\text{Hom}_\Lambda(M, N) \subseteq \text{Hom}_K(M, N)$ as a $K$-module, so $m \leq d^2$. Hence $\ell(C) \leq \ell(N^m) \leq d^3$. Applying $\text{Hom}_\Lambda(-, M)$ and $\text{Hom}_\Lambda(-, N)$ to the above sequence we get

$$0 \to \text{Hom}_\Lambda(C, M) \to \text{Hom}_\Lambda(N^m, M) \xrightarrow{\text{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, M)} \text{Hom}_\Lambda(M, M) \quad (1)$$

and

$$0 \to \text{Hom}_\Lambda(C, N) \to \text{Hom}_\Lambda(N^m, N) \xrightarrow{\text{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, N)} \text{Hom}_\Lambda(M, N). \quad (2)$$

$\text{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, N)$ is an epimorphism by construction. Now we assume

$$\ell(\text{Hom}_\Lambda(C, M)) = \ell(\text{Hom}_\Lambda(C, N)), \quad (3)$$

$$\ell(\text{Hom}_\Lambda(M, M)) = \ell(\text{Hom}_\Lambda(M, N)), \quad (4)$$

$$\ell(\text{Hom}_\Lambda(N, M)) = \ell(\text{Hom}_\Lambda(N, N)). \quad (5)$$

From the sequences (1) and (2) we then get

$$\ell(\text{im \ Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, M))$$

$$= \ell(\text{Hom}_\Lambda(N^m, M)) - \ell(\text{Hom}_\Lambda(C, M))$$

$$= \ell(\text{Hom}_\Lambda(N^m, N)) - \ell(\text{Hom}_\Lambda(C, N))$$

$$= \ell(\text{Hom}_\Lambda(M, N)) = \ell(\text{Hom}_\Lambda(M, M))$$

and hence $\text{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, M)$ is an epimorphism.

In particular the identity on $M$ factors through $(f_1, f_2, \ldots, f_m)^{tr}$, so $(f_1, f_2, \ldots, f_m)^{tr}$ must be a split monomorphism. Then $M$ and $N$ must have a common nonzero direct summand, which is a contradiction. Consequently one of the assumptions (3), (4) and (5) must fail and we have found the desired $X$.

We now show that there exists a $d^5 \times d^5$-matrix $A$ with entries from $\Lambda$ such that $\ell(\text{coker} A_M) \neq \ell(\text{coker} A_N)$. If $\text{ann} M \neq \text{ann} N$ there is a $\lambda \in \Lambda$ with $\ell(\text{coker} \lambda_M) \neq \ell(\text{coker} \lambda_N)$, so we assume $\text{ann} M = \text{ann} N$. Let $\Gamma = \Lambda/\text{ann} M$. We have $\text{Hom}_\Lambda(X, M) \cong$
$\text{Hom}_\Lambda(X/(\text{ann} \, M)X, M)$, so we may assume that $X$ is annihilated by $\text{ann} \, M$. Then we have $\text{Hom}_\Lambda(X, M) \cong \text{Hom}_\Gamma(X, M)$.

We have $\Gamma \subseteq \text{End}_K M$ and thus $\ell(\Gamma) \leq d^2$. We can now make a free resolution of $X$ as a $\Gamma$-module:

$$\Gamma^{d^5} \rightarrow \Gamma^{d^3} \rightarrow X \rightarrow 0$$

and from this we get the exact sequence

$$\Gamma^{d^5} \xrightarrow{A} \Gamma^{d^5} \oplus X \rightarrow \text{coker} \, A_M \rightarrow 0$$

Applying $\text{Hom}_\Gamma(-, M)$ and $\text{Hom}_\Gamma(-, N)$ to this sequence we get

$$0 \rightarrow \text{Hom}_\Gamma(X, M) \oplus M^{d^5-d^3} \rightarrow M^{d^5} \xrightarrow{A_M} M^{d^5} \rightarrow \text{coker} \, A_M \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}_\Gamma(X, N) \oplus N^{d^5-d^3} \rightarrow N^{d^5} \xrightarrow{A_N} N^{d^5} \rightarrow \text{coker} \, A_N \rightarrow 0.$$
Proof. Let \( M \) and \( N \) be non-isomorphic modules. There exists an indecomposable module \( X \) such that \( \ell(\text{Hom}_\Lambda(X, M)) \neq \ell(\text{Hom}_\Lambda(X, N)) \). From the minimal projective presentation of \( X \) we can make an exact sequence

\[
\Lambda^n \to \Lambda^n \to X \oplus P \to 0
\]

where \( P \) is projective, and from there we proceed as in the proof of Theorem 1. \( \Box \)

If \( \Lambda \) is of infinite representation type, then for any \( n \) there exists an indecomposable module \( X \) such that \( P_X \) is not a direct summand of \( \Lambda^n \). Let

\[
0 \to D\text{Tr}X \to Y \to X \to 0
\]

be the almost split sequence ending in \( X \) and let \( Z = X \oplus D\text{Tr}X \). If there is a matrix \( A \in \mathcal{M}_n(\Lambda) \) such that \( \ell(\text{coker} A_Z) \neq \ell(\text{coker} A_Y) \), we get that \( X \) is a direct summand in \( \Lambda^n/\Lambda^n A \), which leads to a contradiction. We therefore have \( \ell(\text{coker} A_Z) = \ell(\text{coker} A_Y) \) for all \( n \times n \)-matrices \( A \). Hence it is impossible to find an \( n \) that makes \( \leq_n \) a partial order for all \( \text{rep}_d\Lambda \) in this case.

The hereditary algebras of finite type can be described as path algebras over a particular class of quivers, the Dynkin quivers (see III.1 and VIII.5 in [2]). Applying Proposition 3 to these, we get the following result:

**Proposition 4.** Let \( k \) be field and \( Q \) be a quiver. Then we have the following.

1. If the underlying graph of \( Q \) is \( A_n \), then \( \leq_1 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \).
2. If the underlying graph of \( Q \) is \( D_n \), then \( \leq_2 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \). Depending on the orientation of \( Q \), we may also have that \( \leq_1 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \).
3. If the underlying graph of \( Q \) is \( E_6 \), then \( \leq_3 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \). Depending on the orientation of \( Q \), we may also have that \( \leq_2 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \).
4. If the underlying graph of \( Q \) is \( E_7 \), then \( \leq_4 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \). Depending on the orientation of \( Q \), we may also have that \( \leq_3 \) and \( \leq_2 \) are partial orders on \( \text{rep}_d kQ \) for any \( d \).
5. If the underlying graph of \( Q \) is \( E_8 \), then \( \leq_6 \) is a partial order on \( \text{rep}_d kQ \) for any \( d \). Depending on the orientation of \( Q \),

we may also have that $\leq_5$, $\leq_4$ and $\leq_3$ are partial orders on rep$_d kQ$ for any $d$.

For the cases in Proposition 4, $\leq_n$ is a partial order if and only if it is equivalent to $\leq_{\text{Hom}}$. Whether this holds in rep$_d \Lambda$ for any algebra $\Lambda$ and any natural number $d$ is an open problem.

References