Existence of solutions for singular fully nonlinear equations

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Abstract. In this note we describe how to approximate some classes of singular equations by nonsingular equations. We obtain a solution to each nonsingular problem and estimates guaranteeing that the limiting function is a solution of the original problem.

1. Introduction

The following problem was studied in [5]

\[
\begin{aligned}
-\Delta u &= \chi_{\{u>0\}}\left(-u^{-\beta} + \lambda u^p\right) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
\]  

(1)

0 < β < 1 and 0 < p < 1.

Theorem 1.1. There exists a maximal solution for every λ > 0. There is constant λ* > 0 such that for λ > λ* the maximal solution is positive. And for λ < λ*, the maximal solution vanishes on a set of positive measure.

We solve problem (1) by perturbing the equation as \( -\Delta u + \frac{u}{(u+\epsilon)^{1+\beta}} = \lambda u^p \). The solutions \( u_\epsilon \searrow u \) pointwise and

\[
\int_\Omega u(-\Delta \varphi) + \int_{\{u>0\}} \frac{1}{u^{1+\beta}} \varphi \leq \lambda \int_\Omega u^p \varphi, \quad \forall \varphi \in C^2(\Omega), \ varphi \geq 0, \ varphi = 0 \ on \ \partial\Omega.
\]  

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There are two approaches to show that \( u \) is indeed a solution of (1). Relation (2) tells us that \( u \) is a maximal subsolution. We then regularize it and show that \( u \in C^{1, \frac{1-\beta}{2}} \) and indeed solves the problem (1). In doing this, we need to obtain a local estimate \( |\nabla u| \leq C u^{1-\beta} \) in \( \Omega' \subset \subset \Omega \). One of the main ingredients is the following Harnack type lemma.

**Lemma 1.2.** For every ball \( B_r(p) \subset \Omega \) there are constants \( c_0, \tau > 0 \) depending only on \( n \) and \( \beta \) such that if

\[
\int_{\partial B_r(p)} u \geq c_0 r^{1+\beta}, \text{ then } u(x) \geq \tau \int_{\partial B_r(p)} u \text{ a.e. in } B_{r/2}(p)
\]

The second approach relies on an estimate for \( u_\varepsilon \) by the maximum principle, namely \( |\nabla u_\varepsilon| \leq C u_\varepsilon^{1-\beta} \) in \( \Omega' \subset \subset \Omega \). The idea to obtain such an estimate is to define \( v = \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^{2-\beta} \varphi_1^2} \), where \( \varphi_1 \) is the first eigenfunction of the Laplacian with zero boundary condition. The function \( v \) has a maximum at \( x_0 \in \Omega \), and then \( \Delta v(x_0) \leq 0 \). If the estimate is not true, it is possible to take a constant \( C > 0 \) independently of \( \varepsilon \) such that \( \sup v > C \) and by computation \( \Delta v(x_0) > 0 \), a contradiction. Using the estimate and multiplying the equation by an adequate test function, we let \( \varepsilon \to 0 \) in the equation to get a weak solution.

The next problem was studied in [7]

\[
\begin{aligned}
-\Delta u &= \chi_{\{u > 0\}} \left( \log u + \lambda u^p \right) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

Both approaches described above work in this case and an analogous result to Theorem 1.1 holds true. The estimate obtained for the maximal subsolution (which is shown to be a solution) is \( |\nabla u| \leq C u^{1-\beta} \) and \( u \in C^{1,1} \), a better regularity than the one for (1). This is roughly explained by the fact that \( \log u \) is less singular than \(-1/u^\beta\). The estimate by maximum principle is \( |\nabla u_\varepsilon| \leq C u_\varepsilon \) in \( \Omega' \subset \subset \Omega \).

2. **Fully nonlinear elliptic equations**

We proceed to discuss in more detail the following fully nonlinear problem which was addressed in a work in progress with E. Teixeira [8]. We consider

\[
\begin{aligned}
F(D^2 u) &= G \left( x, u, |\nabla u|^2 \right) \quad \text{in } \Omega \\
u &= f \quad \text{on } \partial \Omega
\end{aligned}
\]

with \( f \in C^{1,\alpha}(\partial \Omega) \) and \( G: \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) a \( C^1 \) function. Following [3], we define \( F: \text{Sym}(d \times d) \to \mathbb{R} \) and assume \( F(0) = 0 \). The uniform ellipticity
reads as follows: \( \exists \lambda, \Lambda, 0 < \lambda \leq \Lambda \) such that
\[
F(M + \mathcal{N}) \leq F(M) + \Lambda \|\mathcal{N}^+\| - \lambda \|\mathcal{N}^-\|, \quad \forall M, \mathcal{N} \in \text{Sym}(d \times d).
\]

In order to state our Lipschitz estimate, let \( \phi: (0, \infty) \to \mathbb{R} \) be such that \( \liminf_{s \to \infty} \phi(s) \geq 0 \). We define the asymptotic behavior of \( \phi \) passing 0, \( \kappa: (0, 1) \to (0, \infty) \) by \( \kappa(\varepsilon) := \inf \{ s : \phi(s) > -\varepsilon \} \).

**Theorem 2.1.** Let \( u \in C^3(\Omega) \) be a solution. Define
\[
\sigma(|p|) := \inf_{(x, u)} \frac{D_p G(x, u, |p|^2)|p|^2 - |D_p G(x, u, |p|^2)| |p|}{G^2(x, u, |p|^2)}
\]
assume \( S := \liminf_{|p| \to \infty} \sigma(|p|) \geq 0 \). Then \( \max_{\Omega} |\nabla u| \leq C \), where \( C \) depends only on \( d, \lambda, \Lambda, \|f\|_{C^{1, \alpha}} \) and the asymptotic behavior of \( \sigma \) passing 0.

The proof runs by defining \( v = |\nabla u|^2 \). We compute \( D_{i,j} v \) and use the equation. Since \( v \) has a maximum at \( x_0 \in \Omega \), we use the asymptotic behavior to conclude the estimate. It is not a proof by contradiction.

Specializing the function \( G \) we study the problem
\[
\begin{align*}
F(D^2 u) &= \beta(u) \Gamma(|\nabla u|^2) & \text{in } \Omega \\
u &= f & \text{on } \partial\Omega,
\end{align*}
\]
where \( \beta: \mathbb{R} \to \mathbb{R} \) and \( \Gamma: [0, \infty) \to \mathbb{R} \) are \( C^{1, \alpha} \) functions. We have two consequences of Theorem 2.1.

**Corollary 2.2.** If \( \inf_{\Omega} \frac{\beta'(u)}{\beta(u)} > -\infty \) and \( \frac{\Gamma(\tau)}{\tau} \to +\infty \) as \( \tau \to +\infty \), then \( \max_{\Omega} |\nabla u| \leq C \).

**Corollary 2.3.** If \( \beta \) is nondecreasing, \( |\beta| + |\beta'| > 0 \) and \( \liminf_{\tau \to \infty} \frac{\Gamma(\tau)}{\tau} > 0 \), then \( \max_{\Omega} |\nabla u| \leq C \).

Definition: \( u \) is a viscosity subsolution in \( \Omega \) if \( F(D^2 u) \geq g \) in the viscosity sense in \( \Omega \) if, that is, for every \( x_0 \in \Omega, V_{x_0} \) neighborhood, \( \varphi \in C^2(V_{x_0}) \), \( u \leq \varphi \) in \( V_{x_0} \), \( u(x_0) = \varphi(x_0) \), then \( F(D^2 \varphi(x_0)) \geq g(x_0) \).

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A viscosity solution is a continuous function \( u \) which is a subsolution and a supersolution.
Definition: a continuous function $u$ satisfies $F(D^2u) = +\infty$ at a point $X_0$ if $u$ cannot be touched from above by a smooth function at $X_0$.

The idea behind the above definition relies in the fact that if $u(x) = |x|$, then $\Delta u = (n - 1)/|x|$ in the distributional sense. We could say that $\Delta u(0) = +\infty$. In the light of the viscosity theory, given an arbitrary positive number $K$, $P_K(x) = \frac{K}{2^n}|x|^2$ touches $u$ at 0 from below. Indeed, $P(0) = u(0)$ and in $0 < |x| < \frac{1}{K}$, we have $u(x) > P_K(x)$. Thus, “$\Delta u(0) \geq K$” for every $K$.

Definition: a continuous function $u$ is a viscosity solution in the topological sense if it satisfies $F(D^2u) = +\infty$ at a point $X_0$.

The approach to solve (3) is by considering again a perturbed problem

$$
\begin{cases}
F(D^2 u_\varepsilon) = \beta_\varepsilon(u_\varepsilon) \Gamma(|\nabla u_\varepsilon|^2) & \text{in } \Omega \\
u_\varepsilon = f & \text{on } \partial\Omega.
\end{cases}
$$

(4)

Using Corollary 2.2 we derive existence of a Lipschitz viscosity solution in the topological sense for

$$
\begin{cases}
F(D^2 u) = \frac{1}{|u|^q} \Gamma(|\nabla u|^2) & \text{in } \Omega \\
u = f & \text{on } \partial\Omega,
\end{cases}
$$

(5)

with $q \geq 1$, $\Gamma \geq 0$, $\Gamma$ superlinear and $F$ concave. In this case $\beta_\varepsilon(u) = 1/|u|^q$ for $u > \varepsilon$ and $\beta_\varepsilon(u) = \varepsilon$ for $u < -\varepsilon$. Between $-\varepsilon$ and $\varepsilon$, $\beta_\varepsilon(u)$ is a fourth order polynomial. Since $\beta_\varepsilon(u)$ is not monotone, Perron’s method should be adapted by adding a term $ku$ in both sides of the equation. This gives a solution $u_\varepsilon$ to (4). The estimate of Theorem 2.1 allows us to let $u_\varepsilon \to u$, thus obtaining a viscosity solution of (5).

Another existence of viscosity solution result can be obtained using Corollary 2.3 for the problem

$$
\begin{cases}
F(D^2 u) = \chi_{\{u>0\}} \Gamma(|\nabla u|^2) & \text{in } \Omega \\
u = f & \text{on } \partial\Omega.
\end{cases}
$$

(6)

In this case $\beta_\varepsilon$ is defined as follows. Let $\rho$ be a smooth function supported in $[0,1]$, $\rho > 0$ in $(0,1)$ and normalized as to $\int_R \rho = 1$. We define

$$
\beta_\varepsilon(s) := \frac{1}{2} \int_0^{s/\varepsilon} \rho(\tau)d\tau - \frac{1}{2} \int_0^{-s/\varepsilon} \rho(\tau)d\tau + \frac{1}{2} + \varepsilon,
$$

which satisfy the assumptions of Corollary 2.3.

Equations similar to (5) and (6) have been treated in [4, 6]. The solutions of the equations may exhibit a free boundary, whose regularity can be studied with techniques from [1].

Another problem that could be treated with our techniques is
\[
\begin{cases}
F(D^2 u) = \frac{1}{|u|^q} |\nabla u|^2 & \text{in } \Omega \\
u = f & \text{on } \partial \Omega
\end{cases}
\] (7)
where \(0 < q < 1\) and \(F\) is concave. There is a viscosity solution in the pointwise topological sense. Moreover, \(u \in C^{1,1-\frac{q}{2}}\), the regularity of the first problem (1).

In the proof we use a version of Theorem 2.1 and ideas from the proof of existence of solution to problem (5). Here \(\beta\) is the same used to solve (5) and is not monotone.

3. Examples and comparison to our results

In problem (5) we have shown existence of solution to
\[
\begin{cases}
F(D^2 u) = \frac{1}{|u|^q} |\nabla u|^2 & \text{in } \Omega \\
u = f & \text{on } \partial \Omega
\end{cases}
\]
if \(q \geq 1\). There is a result in [2] saying that
\[
\begin{cases}
\Delta u = \frac{1}{|u|^q} |\nabla u|^2 - h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has a \(\geq 0, \neq 0\) solution if and only if \(q \leq 2\), provided \(q > 0\) and \(h\) is smooth and positive at every compact subset of \(\Omega\).

By problem (7) we know that
\[
\begin{cases}
F(D^2 u) = \frac{1}{|u|^q} & \text{in } \Omega \\
u = f & \text{on } \partial \Omega
\end{cases}
\]
has a solution if \(0 < q < 1\), remember \(f \geq 0\). The first problem (1)
\[
\begin{cases}
\Delta u = \frac{1}{|u|^q} & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has no positive solution if \(0 < q < 1\). Notice that
\[
\begin{cases}
-\Delta u = \chi_{\{u>0\}}(-u^{-q} + \lambda) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
has no solution if \(q \geq 1\).
But by problem (6)
\[
\begin{aligned}
F(D^2 u) &= \chi_{\{u>0\}} & \text{in } \Omega \\
u &= f & \text{on } \partial \Omega
\end{aligned}
\]
has a solution.

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References


