TESTE DE HIPÓTESES NO CASO GENÉRICO DE K CRITÉRIOS CLASSIFICADORES SEM INTERAÇÃO *

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PREFÁCIO

É de todos bem conhecido que os problemas de testagem de hipóteses pertinentes àquele domínio estatístico que se costuma designar por análise de variância podem ser totalmente resolvidos a partir de um teorema estabelecido no estudo da chamada hipótese linear geral.

Acontece, porém, que a aplicação do referido teorema nos leva, em geral, a determinantes de ordem tão elevada, que a procura de uma expressão mais cômoda para os mesmos é, ao menos à primeira vista, desanimadora. Em conseqüência disto, ao estruturar os testes, os autores, via de regra, preferem fazê-lo utilizando apenas alguns dos princípios demonstrados no estudo da hipótese linear e não os resultados finais a que êstes mesmos princípios conduzem. É claro que com êste proceder são êles forçados a repetir, em cada caso, uma série de passagens intermediárias, adaptadas, naturalmente, à questão em foco.

Ao prelecionarmos, no ano findo, um curso de especialização relativo a matéria, chamamos atenção para êste estado de coisas, e propusemos aos alunos seguir orientação diversa, procurando atingir os objetivos visados por via mais imediata.

O presente trabalho é resultado do esfôrço de um dêles. Ver-se-á, aí, que mesmo no caso por nós considerado, o qual é bastante geral, é possível — utilizando artifícios que não oferecem tamanhas dificuldades como a que seria lícito entrever — obter, para os determinantes originários, expressões que vão depender de determinantes incomparàvelmente mais simples de serem manejados.

A propósito dêstes últimos cumpre assinalar que a sua ordem é menor do que a de determinantes que figuram em expressões dadas por outros autores, nos casos particulares em que êstes encararam o assunto da forma como aqui foi feita. Êste fato, que merece o devido realce, tem como corolário uma muito maior simplicidade na obtenção de certas expressões de marcado interêsse prático.

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1 — *O teorema* *

Seja $On : y_\alpha \mid x_{1\alpha}, \ldots, x_{k\alpha}, (\alpha = 1, \ldots, n)$ uma amostra casual simples de tamanho n de uma população

$$N \left(\sum_{p=1}^{k} a_p \, x_p, \quad \sigma^2 \right), \text{ onde os}$$

x_p não são variáveis aleatórias, são linearmente independentes e os pontos paramétricos possíveis pertencem à região:

$$\Omega : \begin{cases} -\infty < a_p < +\infty & (p = 1, \ldots, k) \\ \sigma^2 > 0 & (a = 1, 2, \ldots, r_1) \\ \text{onde: } \sum_{p=1}^{k} d_{pu} \, a_p = 0 & (u = 1, 2, \ldots, r_1) \\ \text{e a matriz } [\, d_{pu} \,] \text{ tem característica } r_1, \text{ isto é,} & \\ & C \, [\, d_{pu} \,] = r_1 \end{cases}$$

Suponhamos que se deseja pôr em prova a hipótese H de que o verdadeiro ponto paramétrico $(a_1, \ldots, a_k, \sigma^2)$ pertence a:

$$\omega : \begin{cases} -\infty < a_p < +\infty & (p = 1, \ldots, k) \\ \sigma^2 > 0 & \\ \text{onde: } \sum_{p=1}^{k} d_{pv} \quad a_p = 0 & (v = 1, \ldots, r_2) \\ \text{e} \qquad C \, [\, d_{pv} \,] = r_2 & r_1 < r_2 < k \end{cases}$$

Façamos:

$$(1.1) \qquad a_{pq} = \sum_{\alpha=1}^{n} x_{p\alpha} \; x_{q\alpha} \qquad (p, q = 1, 2, \ldots, k)$$

$$(1.2) \qquad a_{op} = \sum_{\alpha=1}^{n} y_\alpha \; x_{p\alpha}$$

$$(1.3) \qquad a_{oo} = \sum_{\alpha=1}^{n} y_\alpha^2$$

e designemos, respectivamente, por $\hat{\sigma}^2_\Omega$ *e* $\hat{\sigma}^2_\omega$ as estimativas de máxima verossimilhança de σ^2 para variações dos parâmetros sôbre Ω e ω, então:

* Com pequenas alterações o enunciado dêste teorema é o dado por Wilks, S. S. (Mathematical Statistics, 1946).

$$
1.^{\circ})\quad n\,\hat{\sigma}^2_{\Omega} = \frac{\Delta_{oo}}{1}
\begin{vmatrix}
a_{00} & a_{01} & \cdots & a_{0k} & 0 & 0 & \cdots & 0 \\
a_{01} & a_{11} & \cdots & a_{k1} & d_{11} & d_{21} & \cdots & d_{r_1 1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{0k} & a_{1k} & \cdots & a_{kk} & d_{1k} & d_{2k} & \cdots & d_{r_1 k} \\
0 & d_{11} & \cdots & d_{1k} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & d_{r_1 1} & \cdots & d_{r_1 k} & 0 & 0 & \cdots & 0
\end{vmatrix}
\tag{1.4}
$$

onde Δ_{oo} é o cofator de a_{oo} no determinante supra que designaremos por Δ e, portanto,

$$
n\,\hat{\sigma}^2_{\Omega} = \frac{\Delta}{\Delta_{oo}}
$$

$$
2.^{\circ})\quad n\,\hat{\sigma}^2_{\omega} = \frac{1}{\Delta'_{oo}}
\begin{vmatrix}
a_{00} & a_{01} & \cdots & a_{0k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{01} & a_{11} & \cdots & a_{k1} & d_{11} & d_{21} & \cdots & d_{r_2 1} & d_{r_1+1 1} & \cdots & d_{r_2 1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
a_{0k} & a_{1k} & \cdots & a_{kk} & d_{1k} & d_{2k} & \cdots & d_{r_1 k} & d_{r_1+1, k} & \cdots & d_{r_2 k} \\
0 & d_{11} & \cdots & d_{1k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & d_{21} & \cdots & d_{2k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & d_{r_1 1} & \cdots & d_{r_1 k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & d_{r_1+1, 1} & \cdots & d_{r_1+1, k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & d_{r_2 1} & \cdots & d_{r_2 k} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{vmatrix}
\tag{1.5}
$$

onde novamente Δ'_{oo} é o cofator de a_{oo} no determinante supra que designaremos por Δ' e, portanto,

$$
n\,\hat{\sigma}^2_{\omega} = \frac{\Delta'}{\Delta'_{oo}}
$$

3.º) As quantidades:

$$(1.6) \qquad \begin{cases} q_1 = \dfrac{n\,\hat{\sigma}^2_{\Omega}}{\sigma^2} \\[2ex] q_2 = \dfrac{n\,(\hat{\sigma}^2_{\omega} - \hat{\sigma}^2_{\Omega})}{\sigma^2} \end{cases}$$

são independentes e têm distribuição χ^2 com $(n - k + r_1)$ e $(r_2 - r_1)$ gráus de liberdade, respectivamente.

4.º) O critério λ para testar H é dado por:

$$(1.7) \qquad \lambda = \frac{1}{\left(1 + \dfrac{q_2}{q_1}\right)^{\frac{n}{2}}} = \left(\frac{1}{1 + \dfrac{r_2 - r_1}{n + r_1 - k}\,F} \right)^{\frac{n}{2}}$$

onde F tem distribuição de Snedecor com $(r_2 - r_1)$ e $(n - k + r_1)$ gráus de liberdade quando H é verdadeira".

2 — *k — critérios de classificação com números diferentes de observações por cela, sem interação.*

2a — *O problema*

$$\text{Seja} \quad O_n : y_{i_1 i_2 \ldots i_{k+1}} \begin{cases} i^*_\beta = 1,2, \ldots, r \\ \beta = 1,2, \ldots, k \\ i_{k+1} = 1, \ldots, n_{i_1 \ldots i_k} \end{cases}$$

uma amostra casual simples de tamanho $n = \sum_{i_1, \ldots, i_k} n_{i_1 \ldots i_k}$ suposta proveniente de uma população:

$$N\,(m + \sum_{\beta=1}^{k} E^{(\beta)}_{i_\beta}\,,\ \sigma^2)$$

com $\sigma^2 > 0$ e os parâmetros sujeitos às k restrições:

$$(2.1) \qquad \sum_{i_\beta = 1}^{r_\beta} E^{(\beta)}_{i_\beta} = 0.$$

* Em todo o desenvolvimento suporemos $r_1 \geq r_2 \geq \ldots \geq r_k$ o que não constitui restrição uma vez que a ordem dos critérios classificadores é arbitrária.

O nosso atual problema é, à base de tal amostra, estruturar testes sôbre hipóteses do tipo:

$$H_{\beta} \; : E^{(\beta)}_{i_\beta} = 0 \quad (i_{\beta} = 1, \; \dots, \; r_{\beta}) \tag{2.2}$$

para qualquer β.

2b — A fim de mostrar que estamos diante de um caso particular da hipótese linear geral, comecemos renomeando as variáveis $y_{i_1 \, \dots \, i_{k+1}}$ pondo:

$$y_{i_1 \, i_2 \, \dots \, i_k \, i_{k+1}} = y_{\alpha} \tag{2.3}$$

com

$$\begin{aligned}
\alpha = \; & n^{(1)}_{1} + n^{(1)}_{2} + \dots + n^{(1)}_{(i_1 - 1)} + \\
& + n^{(1,2)}_{i_1 1} + n^{(1,2)}_{i_2 2} + \dots + n^{(1,2)}_{i_1 (i_2 - 1)} + \\
& + n^{(1,2,3)}_{i_1 i_2 1} + n^{(1,2,3)}_{i_1 i_2 2} + \dots + n^{(1,2,3)}_{i_1 i_2 (i_3 - 1)} + \\
& + \dots\dots\dots\dots\dots\dots\dots\dots \\
& + n^{(1,2,\dots,k-1,k)}_{i_1 i_2 \dots i_{k-1} 1} + \dots + n^{(1,2,\dots,k-1,k)}_{i_1 i_2 \dots i_{k-1} (i_{1-1})} \\
& + i_{k+1}
\end{aligned}$$

onde:

$$n^{(\beta)}_{i_\beta} = \sum_{i_1 = 1}^{r_1} \sum_{i_2 = 1}^{r_2} \dots \sum_{i_{\beta-1} = 1}^{r_{\beta-1}} \sum_{i_{\beta+1} = 1}^{r_{\beta+1}} \dots \sum_{i_k = 1}^{r_k} n_{i_1 \, i_2 \, \dots \, i \, \dots \, i_k}$$

$$(\beta = 1, \; \dots, \; k)$$

$$n^{(\beta,\gamma)}_{i_\beta \, i_\gamma} = \sum_{i_1 = 1}^{r_1} \dots \sum_{i_{\beta-1} = 1}^{r_{\beta-1}} \sum_{i_{\beta+1} = 1}^{r_{\beta+1}} \dots \sum_{i_{\gamma-1} = 1}^{r_{\gamma-1}} \sum_{i_{\gamma+1} = 1}^{r_{\gamma+1}} \dots \sum_{i_k = 1}^{r_k} n_{i_1 \dots i_\beta \dots i_\gamma \dots i_k}$$

$$(\beta \neq \gamma = 1, 2, \; \dots, \; k)$$

. . . .

e onde n com ao menos um índice inferior igual a zero vale zero.

Em seguida, façamos:

$$\left.\begin{aligned}
E^{(\beta)}_{i_\beta} &= a_{r_1 + r_2 \dots + r_{\beta-1} + i_\beta} \\
m &= a_{r_1 + r_2 + \dots + r_k + 1}
\end{aligned}\right| \quad
\begin{aligned}
i_\beta &= 1, \; \dots, \; r_\beta \\
\beta &= 1, \; \dots, \; k
\end{aligned} \tag{2.4}$$

com

$$r_0 = 0$$

Com isto as restrições (2.1), tornam-se:

$$(2.5) \qquad \sum_{i_\beta'=1}^{r_\beta}{}' \; a_{r_1 + r_2 + \ldots + r_{\beta-1} + i_\beta} = 0 \qquad (\beta = 1, \ldots, k)$$

ou, ainda:

$$(2.6) \qquad \sum_{p=1}^{r_1 + r_2 + \ldots + r_k + 1}{}' \; d_{pu} \, a_p = 0 \qquad (u = 1, \ldots, k)$$

onde a matriz $[\, d_{pu} \,]$ é dada por:

$$\begin{array}{l|ccccccccccccccccccc|}
 & a_1 & a_2 & \ldots & a_{r_1} & a_{r_1+1} & a_{r_1+2} & \ldots & a_{r_1+r_2} & \ldots & a_{r_1+\ldots+r_{k-2}+1} & a_{r_1+\ldots+r_{k-2}+2} & \ldots & a_{r_1+\ldots+r_{k-1}} & a_{r_1+\ldots+r_{k-1}+1} & a_{r_1+\ldots+r_{k-1}+2} & \ldots & a_{r_1+\ldots+r_k} & a_{r_1+\ldots+r_k+1} \\
\hline
1.^a \text{ restrição} & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
2.^a \text{ restrição} & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
(k-1)^{ma} \;,, & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
k^{ma} \text{ restrição} & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 \\
\hline
\end{array}$$

(2.7)

a qual tem característica k; de fato, o menor formado pelas k linhas e as k colunas: 1.ª, $(r_1 + 1)^{ma}$, $(r_1 + r_2 + 1)^{ma} \ldots (r_1 + \ldots + r_{k-2} + 1)^{ma}$ $(r_1 + \ldots + r_{k-1} + 1)^{ma}$, é dado por:

$$\begin{vmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1
\end{vmatrix} = 1$$

Com isto, para que:

$$E\ (y_{i_1\, i_2\, \ldots\, i_k\, i_{k+1}}) = m + \sum_{\beta=1}^{k} E_{i_\beta}^{(\beta)}$$

possa ser posta sob a forma canônica:

$$E\ (y_\alpha) = \sum_{p=1}^{r_1 + \ldots + r_k + 1} a_p\ x_{p\alpha}$$

resta definir convenientemente os xx, o que é feito da seguinte forma:

Se $p = r + \ldots + r_{\beta-1} + i_\beta$:

$$x_{p\alpha} = \begin{cases} 1 & \text{para os } n_{i_\beta}^{(\beta)} \text{ valores de } \alpha \text{ tais que os } y_\alpha \\ & \text{tem correspondentes} \\ & y_{i_1\, i_2 \ldots i_\beta \ldots i_k} \text{ com } i_\beta = p - r_1 - \ldots - r_{\beta-1} \\ 0 & \text{em contrário} \end{cases} \tag{2.8}$$

e se $p = r_1 + \ldots + r_k + 1$:

(2.9) $\quad x_{p\alpha} = 1 \quad$ para todos os n valores de α.

Notemos, finalmente, que uma hipótese do tipo genérico $E_{i_\beta}^{(\beta)} = 0$,

ou seja, $a_{r_1 + \ldots r_{\beta-1} + i_\beta} = 0 \quad (i_\beta = 1, \ldots, r_\beta)$

equivale a acrescentar às restrições da pressuposição, $(r_\beta - 1)$ restrições lineares da forma:

$$a_{r_1 + r_2 + \ldots + r_{\beta-1} + 1} - a_{r_1 + r_2 + \ldots + r_{\beta-1} + i_\beta} = 0 \qquad (i_\beta = 2, \ldots, r_\beta) \tag{2.10}$$

as quais são linearmente independentes entre si e das restrições da pressuposição. De fato, a (2.8) pode ser escrita sob a forma:

$$\sum_{p=1}^{r_1 + \ldots + r_k + 1} d_{p\nu_\beta}\ a_p = 0 \qquad (\nu_\beta = 1, 2, \ldots, r_\beta - 1) \tag{2.11}$$

onde a matriz $d_{p\nu_\beta}$ é dada por:

$$
\begin{array}{l}
 \\
\text{1.ª restrição} \\
\text{2.ª restrição} \\
\vdots \\
(r_\beta - 1)^{ma} \text{ ,,}
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccc}
a_1 & a_2 & \cdots & a_{r_1} & a_{r_1+1} & a_{r_2+2} & \cdots & a_{r_1+r_2} & \cdots & a_{r_1+r_2+\cdots+r_{\beta-1}+1} & a_{r_2+r_2+\cdots+r_{\beta-1}+2} & a_{r_1+r_2+\cdots+r_{\beta-1}+3} & \cdots & a_{r_1+r_2+\cdots+r_{k-1}+r} & \cdots & a_{r_1+\cdots+r_{k-1}+1} & a_{r_1+\cdots+r_{k-1}+2} & \cdots & a_{r_1+\cdots+r_{k-1}+r_k} \quad a_{r_1+\cdots+r_k+1}
\end{array} \\
\left[\begin{array}{ccccccccccccccccccc}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & -1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \quad 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & -1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \quad 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \quad \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & \cdots & 0 \quad 0
\end{array}\right]
\end{array}
= \left[d_{pv_\beta} \right]
$$

(2.12)

Desde que se verifica imediatamente que a (2.12) tem característica $(r_\beta - 1)$, as $(r_\beta - 1)$ novas restrições são linearmente independentes entre si. Além disto, uma vez, que todo o vetor linha da (2.12) é ortogonal a todo o vetor linha da (2.7), então, as $(r_\beta - 1)$ novas restrições lineares são indepententes das k restrições lineares da pressuposição. Dêste resultado, segue que a matriz formada pela reunião da (2.7) e (2.12) e que representaremos por $[d_{pl_\beta}]$ onde $l_\beta = 1,2, \ldots, k + r_\beta - 1$ terá característica $(k + r_\beta - 1)$ para qualquer $\beta = 1,2, \ldots, k$.

2c — Sintetizando, o atual problema pode ser posto como segue:

— À base de uma amostra casual simples de tamanho n:

$$y_\alpha \mid x_{1\alpha} , x_{2\alpha} , \ldots, x_{r_1 + \cdots + r_k + 1\alpha}$$

$$(\alpha = 1, \ldots, n)$$

de uma população:

$$N \left(\sum_{p=1}^{r_1 + \cdots + r_k + 1} a_p \, x_{p\alpha} , \sigma^2 \right)$$

onde os xp são definidos pela (2.8), (2.9) e (2.10) e os pontos paramétricos admissíveis pertencem à região:

$$
\Omega : \begin{cases} -\infty < a_p < +\infty & (p = 1, 2, \ldots, r_1 + r_2 + \ldots + r_k + 1) \\ \sigma^2 > 0 & \\ \text{com} \quad \sum_{p=1}^{r_1 + \ldots + r_k + 1} d_{pu} \; a_p = 0 & (u = 1, 2, \ldots, k) \\ \text{onde} \quad [\, d_{pu} \,] \text{ é dada pela } (2.7) \text{ e } c \; [d_{pu}\,] = \mathrm{k} & \end{cases}
$$

ôr em prova as hipóteses (2.2) de que o verdadeiro ponto paramétrico ertence à região:

$$
\begin{array}{r} \omega_\beta : \\ \beta = 1, 2, \ldots, k) \end{array} \left| \begin{array}{l} \text{Sub-espaço de } \Omega \text{ para o qual:} \\ \sum_{p=1}^{r_1+\ldots+r_k+1} d_{pl_\beta} \; a_p = 0 \quad (l_\beta = 1,2,\ldots,\; k + r_\beta - 1) \\ \text{onde} \quad c \, [\, d_{pl_\beta} \;] = k + r_\beta - 1 \end{array} \right.
$$

Como vemos, o nosso problema nada mais é do que um caso partiular da hipótese linear geral e, como tal, pode ser resolvido pela aplicação o teorema do parágrafo 1, o que equivale a dizer que para a sua solução asta calcular os valores de $n \; \hat{\sigma}^2_\Omega$ e $n \; \hat{\sigma}^2_{\omega\beta}$ $(\beta = 1, \ldots, k)$.

Para alcançar êste objetivo notemos que com as definições feitas, as xpressões (1.1), 1.2) e (1.3) tornam-se, respectivamente:

$$
{}_{pp} = \sum_{\alpha=1}^{n} x^2_{p\alpha} = \begin{cases} n^{(\beta)}_{i_\beta} & \text{se} \quad p = r_1 + \ldots + r_{\beta-1} + i_\beta \\ n & \text{se} \quad p = r_1 + \ldots + r_k + 1 \end{cases}
$$

$$
{}_{pq} = \sum_{\alpha=1}^{n} x_{p\alpha} \, x_{q\alpha} = \begin{cases} 0 & \text{se} \quad p \neq q = r_1 + \ldots + r_{\beta-1} + i_\beta \\ n^{(\beta,\,\gamma)}_{i_\beta \; i_\gamma} & \text{se} \quad \begin{cases} p = r_1 + \ldots + r_{\beta-1} + i_\beta \\ q = r_1 + \ldots + r_{\beta-1} + i_\beta \\ \gamma \neq \beta = 1,2, \ldots, k \end{cases} \\ n^{(\beta)}_{i_\beta} & \text{se} \quad \begin{cases} p = r_1 + \ldots + r_{\beta-1} + i_\beta \\ q = r_1 + \ldots + r \end{cases} \end{cases}
$$

$(p \neq q)$

$$a_{op} = \sum_{\alpha=1}^{n} x_{p\alpha}\, y_\alpha = \begin{cases} \displaystyle\sum_{i_1=1}^{r_1} \cdots \sum_{i_{\beta-1}=1}^{r_{\beta-1}} \sum_{i_{\beta+1}=1}^{r_{\beta-1}} \cdots \sum_{i_k=1}^{r_k} y_{i_1 \ldots i_\beta \ldots i_k} = y_{i_\beta}^{(\beta)} \text{ se} \\ \qquad\qquad p = r_1 + \ldots + r_{\beta-1} + i_\beta \\ \displaystyle\sum_{i_1=1}^{r_1} \cdots \sum_{i_k=1}^{r_k} y_{i_1 \ldots i_k} = Y \text{ se } p = r_1 + \ldots + r_k + 1 \end{cases}$$

$$a_{oo} = \sum_{\alpha=1}^{n} y_\alpha^2 = \sum_{i_1=1}^{r_1} \cdots \sum_{i_{k+1}=1}^{n_{i_1 \ldots i_k}} y^2_{i_1 \ldots i_k i_{k+1}}$$

2d — Com isto, temos

$$n\, \hat{\sigma}^2_\Omega = \frac{\Delta}{\Delta_{oo}} ,$$

onde Δ é dado por:

$$
\Delta = \left|\begin{array}{cccccccccccccc}
\sum y^2_{i_1\cdots i_{k+1}} & y_1^{(1)} & y_2^{(1)} \cdots y_{n_1}^{(1)} & y_1^{(2)} & y_2^{(2)} \cdots y_{n_2}^{(2)} & \cdots\cdots & y_1^{(k-1)} & y_2^{(k-1)} \cdots y_{n_{k-1}}^{(k-1)} & y_1^{(k)} & y_2^{(k)} \cdots y_{n_k}^{(k)} & y & 0 & 0 \cdots 0 & 0 \\
y_1^{(1)} & n_1^{(1)} & 0 \cdots 0 & n_{11}^{(1,2)} & n_{12}^{(1,2)} \cdots n_{1n_2}^{(1,2)} & \cdots\cdots & n_{11}^{(1,k-1)} & n_{12}^{(1,k-1)} \cdots n_{1n_{k-1}}^{(1,k-1)} & n_{11}^{(1,k)} & n_{12}^{(1,k)} \cdots n_{1n_k}^{(1,k)} & n_1^{(1)} & 1 & 0 \cdots 0 & 0 \\
y_2^{(1)} & 0 & n_2^{(1)} \cdots 0 & n_{21}^{(1,2)} & n_{22}^{(1,2)} \cdots n_{2n_2}^{(1,2)} & \cdots\cdots & n_{21}^{(1,k-1)} & n_{22}^{(1,k-1)} \cdots n_{2n_{k-1}}^{(1,k-1)} & n_{21}^{(1,k)} & n_{22}^{(1,k)} \cdots n_{2n_k}^{(1,k)} & n_2^{(1)} & 1 & 0 \cdots 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_1}^{(1)} & 0 & 0 \cdots n_{n_1}^{(1)} & n_{n_11}^{(1,2)} & n_{n_12}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & \cdots\cdots & n_{n_11}^{(1,k-1)} & n_{n_12}^{(1,k-1)} \cdots n_{n_1n_{k-1}}^{(1,k-1)} & n_{n_11}^{(1,k)} & n_{n_12}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & n_{n_1}^{(1)} & 1 & 0 \cdots 0 & 0 \\
y_1^{(2)} & n_{11}^{(1,2)} & n_{21}^{(1,2)} \cdots n_{n_11}^{(1,2)} & n_1^{(2)} & 0 \cdots 0 & \cdots\cdots & n_{11}^{(2,k-1)} & n_{12}^{(2,k-1)} \cdots n_{1n_{k-1}}^{(2,k-1)} & n_{11}^{(2,k)} & n_{12}^{(2,k)} \cdots n_{1n_k}^{(2,k)} & n_1^{(2)} & 0 & 1 \cdots 0 & 0 \\
y_2^{(2)} & n_{12}^{(1,2)} & n_{22}^{(1,2)} \cdots n_{n_12}^{(1,2)} & 0 & n_2^{(2)} \cdots 0 & \cdots\cdots & n_{21}^{(2,k-1)} & n_{22}^{(2,k-1)} \cdots n_{2n_{k-1}}^{(2,k-1)} & n_{21}^{(2,k)} & n_{22}^{(2,k)} \cdots n_{2n_k}^{(2,k)} & n_2^{(2)} & 0 & 1 \cdots 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_2}^{(2)} & n_{1n_2}^{(1,2)} & n_{2n_2}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & 0 & 0 \cdots n_{n_2}^{(2)} & \cdots\cdots & n_{n_21}^{(2,k-1)} & n_{n_22}^{(2,k-1)} \cdots n_{n_2n_{k-1}}^{(2,k-1)} & n_{n_21}^{(2,k)} & n_{n_22}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & n_{n_2}^{(2)} & 0 & 1 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(k-1)} & n_{11}^{(1,k-1)} & n_{21}^{(1,k-1)} \cdots n_{n_11}^{(1,k-1)} & n_{11}^{(2,k-1)} & n_{21}^{(2,k-1)} \cdots n_{n_21}^{(2,k-1)} & \cdots\cdots & n_1^{(k-1)} & 0 \cdots 0 & n_{11}^{(k-1,k)} & n_{12}^{(k-1,k)} \cdots n_{1n_k}^{(k-1,k)} & n_1^{(k-1)} & 0 & 0 \cdots 1 & 0 \\
y_2^{(k-1)} & n_{12}^{(1,k-1)} & n_{22}^{(1,k-1)} \cdots n_{n_12}^{(1,k-1)} & n_{12}^{(2,k-1)} & n_{22}^{(2,k-1)} \cdots n_{n_22}^{(2,k-1)} & \cdots\cdots & 0 & n_2^{(k-1)} \cdots 0 & n_{21}^{(k-1,k)} & n_{22}^{(k-1,k)} \cdots n_{2n_k}^{(k-1,k)} & n_2^{(k-1)} & 0 & 0 \cdots 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_{k-1}}^{(k-1)} & n_{1n_{k-1}}^{(1,k-1)} & n_{2n_{k-1}}^{(1,k-1)} \cdots n_{n_1n_{k-1}}^{(1,k-1)} & n_{1n_{k-1}}^{(2,k-1)} & n_{2n_{k-1}}^{(2,k-1)} \cdots n_{n_2n_{k-1}}^{(2,k-1)} & & 0 & 0 \cdots n_{n_{k-1}}^{(k-1)} & n_{n_{k-1}1}^{(k-1,k)} & n_{n_{k-1}2}^{(k-1,k)} \cdots n_{n_{k-1}n_k}^{(k-1,k)} & n_{n_{k-1}}^{(k-1)} & 0 & 0 \cdots 1 & 0 \\
y_1^{(k)} & n_{11}^{(1,k)} & n_{21}^{(1,k)} \cdots n_{n_11}^{(1,k)} & n_{11}^{(2,k)} & n_{21}^{(2,k)} \cdots n_{n_21}^{(2,k)} & \cdots\cdots & n_{11}^{(k-1,k)} & n_{21}^{(k-1,k)} \cdots n_{n_{k-1}1}^{(k-1,k)} & n_1^{(k)} & 0 \cdots 0 & n_1^{(k)} & 0 & 0 \cdots 0 & 1 \\
y_2^{(k)} & n_{12}^{(1,k)} & n_{22}^{(1,k)} \cdots n_{n_12}^{(1,k)} & n_{12}^{(2,k)} & n_{22}^{(2,k)} \cdots n_{n_22}^{(2,k)} & \cdots\cdots & n_{12}^{(k-1,k)} & n_{22}^{(k-1,k)} \cdots n_{n_{k-1}2}^{(k-1,k)} & 0 & n_2^{(k)} \cdots 0 & n_2^{(k)} & 0 & 0 \cdots 0 & 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_k}^{(k)} & n_{1n_k}^{(1,k)} & n_{2n_k}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & n_{1n_k}^{(2,k)} & n_{2n_k}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & \cdots\cdots & n_{1n_k}^{(k-1,k)} & n_{2n_k}^{(k-1,k)} \cdots n_{n_{k-1}n_k}^{(k-1,k)} & 0 & 0 \cdots n_{n_k}^{(k)} & n_{n_k}^{(k)} & 0 & 0 \cdots 0 & 1 \\
y & n_1^{(1)} & n_2^{(1)} \cdots n_{n_1}^{(1)} & n_1^{(2)} & n_2^{(2)} \cdots n_{n_2}^{(2)} & \cdots\cdots & n_1^{(k-1)} & n_2^{(k-1)} \cdots n_{n_{k-1}}^{(k-1)} & n_1^{(k)} & n_2^{(k)} \cdots n_{n_k}^{(k)} & n & 0 & 0 \cdots 0 & 0 \\
0 & 1 & 1 \cdots 1 & 0 & 0 \cdots 0 & \cdots\cdots & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 & 0 \\
0 & 0 & 0 \cdots 0 & 1 & 1 \cdots 1 & \cdots\cdots & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & \cdots\cdots & 1 & 1 \cdots 1 & 0 & 0 \cdots 0 & 0 & 0 & 0 \cdots 0 & 0
\end{array}\right| \tag{2.13}
$$

e $\triangle_{00}$, lembremos, é o cofator de $\Sigma\, y^2_{i_1 \ldots i_k + 1}$ em $\triangle$.

Em $\triangle$ subtraindo da $(r_1 + \ldots + r_k + 2)^{ma}$ linha (coluna) a soma das 2.ª, 3.ª, ..., $(r_1 + 1)^{ma}$ linhas (colunas) e, desenvolvendo o determinante em função da $(r_1 + \ldots + r_k + 2)^{ma}$ linha e, seguida, o menor assim obtido em função da $(r_1 + \ldots r_k + 2)^{ma}$ coluna, vem:

$$
\Delta = r_1^2
\begin{vmatrix}
\sum y^2_{i_1\cdots i_{k+1}} & y_1^{(1)} & y_2^{(1)} \cdots y_{n_1}^{(1)} & y_1^{(2)} & y_2^{(2)} \cdots y_{n_2}^{(2)} & \cdots & y_1^{(k-1)} & y_2^{(k-1)} \cdots y_{n_{k-1}}^{(k-1)} & y_1^{(k)} & y_2^{(k)} \cdots y_{n_k}^{(k)} & 0 \cdots 0 & 0 \\
y_1^{(1)} & n_1^{(1)} & 0 \cdots 0 & n_{11}^{(1,2)} & n_{12}^{(1,2)} \cdots n_{1n_2}^{(1,2)} & \cdots & n_{11}^{(1,k-1)} & n_{12}^{(1,k-1)} \cdots n_{1n_{k-1}}^{(1,k-1)} & n_{11}^{(1,k)} & n_{12}^{(1,k)} \cdots n_{1n_k}^{(1,k)} & 0 \cdots 0 & 0 \\
y_2^{(1)} & 0 & n_2^{(1)} \cdots 0 & n_{21}^{(1,2)} & n_{22}^{(1,2)} \cdots n_{2n_2}^{(1,2)} & \cdots & n_{21}^{(1,k-1)} & n_{22}^{(1,k-1)} \cdots n_{2n_{k-1}}^{(1,k-1)} & n_{21}^{(1,k)} & n_{22}^{(1,k)} \cdots n_{2n_k}^{(1,k)} & 0 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_1}^{(1)} & 0 & 0 \cdots n_{n_1}^{(1)} & n_{n_11}^{(1,2)} & n_{n_12}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & \cdots & n_{n_11}^{(1,k-1)} & n_{n_12}^{(1,k-1)} \cdots n_{n_1n_{k-1}}^{(1,k-1)} & n_{n_11}^{(1,k)} & n_{n_12}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & 0 \cdots 0 & 0 \\
y_1^{(2)} & n_{11}^{(1,2)} & n_{21}^{(1,2)} \cdots n_{n_11}^{(1,2)} & n_1^{(2)} & 0 \cdots 0 & \cdots & n_{11}^{(2,k-1)} & n_{12}^{(2,k-1)} \cdots n_{1n_{k-1}}^{(2,k-1)} & n_{11}^{(2,k)} & n_{12}^{(2,k)} \cdots n_{1n_k}^{(2,k)} & 1 \cdots 0 & 0 \\
y_2^{(2)} & n_{12}^{(1,2)} & n_{22}^{(1,2)} \cdots n_{n_12}^{(1,2)} & 0 & n_2^{(2)} \cdots 0 & \cdots & n_{21}^{(2,k-1)} & n_{22}^{(2,k-1)} \cdots n_{2n_{k-1}}^{(2,k-1)} & n_{21}^{(2,k)} & n_{22}^{(2,k)} \cdots n_{2n_k}^{(2,k)} & 1 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_2}^{(2)} & n_{1n_2}^{(1,2)} & n_{2n_2}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & 0 & 0 \cdots n_{n_2}^{(2)} & \cdots & n_{n_21}^{(2,k-1)} & n_{n_22}^{(2,k-1)} \cdots n_{n_2n_{k-1}}^{(2,k-1)} & n_{n_21}^{(2,k)} & n_{n_22}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & 1 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(k-1)} & n_{11}^{(1,k-1)} & n_{21}^{(1,k-1)} \cdots n_{n_11}^{(1,k-1)} & n_{11}^{(2,k-1)} & n_{21}^{(2,k-1)} \cdots n_{n_21}^{(2,k-1)} & \cdots & n_1^{(k-1)} & 0 \cdots 0 & n_{11}^{(k-1,k)} & n_{12}^{(k-1,k)} \cdots n_{1n_k}^{(k-1,k)} & 0 \cdots 1 & 0 \\
y_2^{(k-1)} & n_{12}^{(1,k-1)} & n_{22}^{(1,k-1)} \cdots n_{n_12}^{(1,k-1)} & n_{12}^{(2,k-1)} & n_{22}^{(2,k-1)} \cdots n_{n_22}^{(2,k-1)} & \cdots & 0 & n_2^{(k-1)} \cdots 0 & n_{21}^{(k-1,k)} & n_{22}^{(k-1,k)} \cdots n_{2n_k}^{(k-1,k)} & 0 \cdots 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_{k-1}}^{(k-1)} & n_{1n_{k-1}}^{(1,k-1)} & n_{2n_{k-1}}^{(1,k-1)} \cdots n_{n_1n_{k-1}}^{(1,k-1)} & n_{1n_{k-1}}^{(2,k-1)} & n_{2n_{k-1}}^{(2,k-1)} \cdots n_{n_2n_{k-1}}^{(2,k-1)} & \cdots & 0 & 0 \cdots n_{n_{k-1}}^{(k-1)} & n_{n_{k-1}1}^{(k-1,k)} & n_{n_{k-1}2}^{(k-1,k)} \cdots n_{n_{k-1}n_k}^{(k-1,k)} & 0 \cdots 1 & 0 \\
y_1^{(k)} & n_{11}^{(1,k)} & n_{21}^{(1,k)} \cdots n_{n_11}^{(1,k)} & n_{11}^{(2,k)} & n_{21}^{(2,k)} \cdots n_{n_21}^{(2,k)} & \cdots & n_{11}^{(k-1,k)} & n_{21}^{(k-1,k)} \cdots n_{n_{k-1}1}^{(k-1,k)} & n_1^{(k)} & 0 \cdots 0 & 0 \cdots 0 & 1 \\
y_2^{(k)} & n_{12}^{(1,k)} & n_{22}^{(1,k)} \cdots n_{n_12}^{(1,k)} & n_{12}^{(2,k)} & n_{22}^{(2,k)} \cdots n_{n_22}^{(2,k)} & \cdots & n_{12}^{(k-1,k)} & n_{22}^{(k-1,k)} \cdots n_{n_{k-1}2}^{(k-1,k)} & 0 & n_2^{(k)} \cdots 0 & 0 \cdots 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_k}^{(k)} & n_{1n_k}^{(1,k)} & n_{2n_k}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & n_{1n_k}^{(2,k)} & n_{2n_k}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & \cdots & n_{1n_k}^{(k-1,k)} & n_{2n_k}^{(k-1,k)} \cdots n_{n_{k-1}n_k}^{(k-1,k)} & 0 & 0 \cdots n_{n_k}^{(k)} & 0 \cdots 0 & 1 \\
0 & 0 & 0 \cdots 0 & 1 & 1 \cdots 1 & \cdots & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & \cdots & 1 & 1 \cdots 1 & 0 & 0 \cdots 0 & 0 \cdots 0 & 0 \\
0 & 0 & 0 \cdots 0 & 0 & 0 \cdots 0 & \cdots & 0 & 0 \cdots 0 & 1 & 1 \cdots 1 & 0 \cdots 0 & 0
\end{vmatrix}
$$

Vamos, a seguir, realizar sôbre $\triangle$ a seguinte transformação genérica constante de r_β operações: de cada uma das linhas (colunas) encabeçadas por $Y^{(\beta)}_{i_\beta}$ $\quad (i_\beta = 1,2, \ldots, r_\beta)$

subtraiamos a expressão:

$$\text{linha (coluna) encabeçada por } Y^{(1)}_1 \times \frac{n^{(1,\beta)}_{1\; i_\beta}}{n^{(1)}_1} +$$

$$+ \text{ linha (coluna) encabeçada por } Y^{(1)}_2 \times \frac{n^{(1,\beta)}_{2\; i_\beta}}{n^{(1)}_2} +$$

....

$$+ \text{ linha (coluna) encabeçada por } Y^{(1)}_{r_1} \times \frac{n^{(1,\beta)}_{r_1\; i_\beta}}{n^{(1)}_{r_1}}$$

Façamos, para facilidade:

$$Q^{(\beta)}_{i_\beta} = Y^{(\beta)}_{i_\beta} - \sum_{i_1=1}^{r_1} \frac{n^{(1,\beta)}_{i_1\; i_\beta}}{n^{(1)}_{i_1}} \, Y^{(1)}_{i_1} \tag{2.13}$$

$$m^{[\beta,\gamma]}_{i_\beta\; i_\gamma} = \begin{cases} \text{se } \gamma \neq \beta : n^{(\beta,\gamma)}_{i_\beta\; i_\gamma} - \displaystyle\sum_{i_1=1}^{r_1} \frac{r^{(1,\beta)}_{i_1\; i_\beta} \, n^{(1,\gamma)}_{i_1\; i_\gamma}}{n^{(1)}_{i_1}} \\ \text{se } \gamma = \beta \text{ e } \begin{cases} \text{se } i_\gamma = i_\beta : n^{(\beta)}_{i_\beta} - \displaystyle\sum_{i_1=1}^{r_1} \frac{n^{(1,\beta)}_{i_1\; i_\beta} \, n^{(1,\gamma)}_{i_1\; i_\gamma}}{n^{(1)}_{i_1}} \\ \text{se } i_\gamma \neq i_\beta : - \displaystyle\sum_{i_1=1}^{r_1} \frac{n^{(1,\beta)}_{i_1\; i_\beta} \, n^{(1,\gamma)}_{i_1\; i_\gamma}}{n^{(1)}_{i_1}} \end{cases} \\ \qquad\qquad \gamma = \beta,\ \beta + 1, \ldots, k \\ \qquad\qquad i_\gamma = 1, 2, \ldots, r_\gamma \end{cases} \tag{2.14}$$

Repetindo $(k - 1)$ vêzes esta operações ao se fazer $\beta = 2, 3, \ldots, k$, vem:

$$
\Delta = r_1^2 \left|
\begin{array}{cccccccccccccccccccccc}
\sum y^2_{i_1\cdots i_{k+1}} & Y_1^{(1)} & Y_2^{(1)} & \cdots & Y_{n_1}^{(1)} & Q_1^{(2)} & Q_2^{(2)} & \cdots & Q_{n_2}^{(2)} & Q_1^{(3)} & Q_2^{(3)} & \cdots & Q_{n_3}^{(3)} & \cdots & Q_1^{(k)} & Q_2^{(k)} & \cdots & Q_{n_k}^{(k)} & 0 & 0 & \cdots & 0 \\
Y_1^{(1)} & n_1^{(1)} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
Y_2^{(1)} & 0 & n_2^{(1)} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
Y_{n_1}^{(1)} & 0 & 0 & \cdots & n_{n_1}^{(1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
Q_1^{(2)} & 0 & 0 & \cdots & 0 & m^{[2,2]}_{11} & m^{[2,2]}_{12} & \cdots & m^{[2,2]}_{1n_2} & m^{[2,3]}_{11} & m^{[2,3]}_{12} & \cdots & m^{[2,3]}_{1n_3} & \cdots & m^{[2,k]}_{11} & m^{[2,k]}_{12} & \cdots & m^{[2,k]}_{1n_k} & 1 & 0 & \cdots & 0 \\
Q_2^{(2)} & 0 & 0 & \cdots & 0 & m^{[2,2]}_{21} & m^{[2,2]}_{22} & \cdots & m^{[2,2]}_{2n_2} & m^{[2,3]}_{21} & m^{[2,3]}_{22} & \cdots & m^{[2,3]}_{2n_3} & \cdots & m^{[2,k]}_{21} & m^{[2,k]}_{22} & \cdots & m^{[2,k]}_{2n_k} & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
Q_{n_2}^{(2)} & 0 & 0 & \cdots & 0 & m^{[2,2]}_{n_21} & m^{[2,2]}_{n_22} & \cdots & m^{[2,2]}_{n_2n_2} & m^{[2,3]}_{n_21} & m^{[2,3]}_{n_22} & \cdots & m^{[2,3]}_{n_2n_3} & \cdots & m^{[2,k]}_{n_21} & m^{[2,k]}_{n_22} & \cdots & m^{[2,k]}_{n_2n_k} & 1 & 0 & \cdots & 0 \\
Q_1^{(3)} & 0 & 0 & \cdots & 0 & m^{[2,3]}_{11} & m^{[2,3]}_{21} & \cdots & m^{[2,3]}_{n_21} & m^{[3,3]}_{11} & m^{[3,3]}_{12} & \cdots & m^{[3,3]}_{1n_3} & \cdots & m^{[3,k]}_{11} & m^{[3,k]}_{12} & \cdots & m^{[3,k]}_{1n_k} & 0 & 1 & \cdots & 0 \\
Q_2^{(3)} & 0 & 0 & \cdots & 0 & m^{[2,3]}_{12} & m^{[2,3]}_{22} & \cdots & m^{[2,3]}_{n_22} & m^{[3,3]}_{21} & m^{[3,3]}_{22} & \cdots & m^{[3,3]}_{2n_3} & \cdots & m^{[3,k]}_{21} & m^{[3,k]}_{22} & \cdots & m^{[3,k]}_{2n_k} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
Q_{n_3}^{(3)} & 0 & 0 & \cdots & 0 & m^{[2,3]}_{1n_3} & m^{[2,3]}_{2n_3} & \cdots & m^{[2,3]}_{n_2n_3} & m^{[3,3]}_{n_31} & m^{[3,3]}_{n_32} & \cdots & m^{[3,3]}_{n_3n_3} & \cdots & m^{[3,k]}_{n_31} & m^{[3,k]}_{n_32} & \cdots & m^{[3,k]}_{n_3n_k} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
Q_1^{(k)} & 0 & 0 & \cdots & 0 & m^{[2,k]}_{11} & m^{[2,k]}_{21} & \cdots & m^{[2,k]}_{n_21} & m^{[3,k]}_{11} & m^{[3,k]}_{21} & \cdots & m^{[3,k]}_{n_31} & \cdots & m^{[k,k]}_{11} & m^{[k,k]}_{12} & \cdots & m^{[k,k]}_{1n_k} & 0 & 0 & \cdots & 1 \\
Q_2^{(k)} & 0 & 0 & \cdots & 0 & m^{[2,k]}_{12} & m^{[2,k]}_{22} & \cdots & m^{[2,k]}_{n_22} & m^{[3,k]}_{12} & m^{[3,k]}_{22} & \cdots & m^{[3,k]}_{n_32} & \cdots & m^{[k,k]}_{21} & m^{[k,k]}_{22} & \cdots & m^{[k,k]}_{2n_k} & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
Q_{n_k}^{(k)} & 0 & 0 & \cdots & 0 & m^{[2,k]}_{1n_k} & m^{[2,k]}_{2n_k} & \cdots & m^{[2,k]}_{n_2n_k} & m^{[3,k]}_{1n_k} & m^{[3,k]}_{2n_k} & \cdots & m^{[3,k]}_{n_3n_k} & \cdots & m^{[k,k]}_{n_k1} & m^{[k,k]}_{n_k2} & \cdots & m^{[k,k]}_{n_kn_k} & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0
\end{array}
\right|
$$

Adicionando, agora, à linha (coluna) encabeçada por $Q_{r_\beta}^{(\beta)}$ a soma das linhas (colunas) encabeçadas por $Q_1^{(\beta)}$, $Q_2^{(\beta)}$, ..., $Q_{r_\beta}^{(\beta)} - 1$ $(\beta = 2, \ldots, k)$ e, notando que:

$$\sum_{i_\beta=1}^{r_\beta} Q_{i_\beta}^{(\beta)} = \sum_{i_\beta=1}^{r_\beta} \left(Y_{i_\beta}^{(\beta)} - \sum_{i_1=1}^{r_1} \frac{n_{i_1 i_\beta}^{(1,\beta)}}{n_{i_1}^{(1)}} Y_{i_1}^{(1)} \right)$$

$$= Y - \sum_{i_1=1}^{r_1} \frac{Y_{i_1}^{(1)}}{n_{i_1}^{(1)}} \cdot n_{i_1}^{(1)} = 0 \tag{2.15}$$

$$\sum_{i_\beta=1}^{r} m_{i_\gamma i_\beta}^{[\gamma,\beta]} = \begin{cases} \displaystyle\sum_{i_\beta=1}^{r_\beta} n_{i_\gamma i_\beta}^{(\gamma,\beta)} - \sum_{i_\beta=1}^{r_\beta} \sum_{i_1=1}^{r_1} \frac{n_{i_1 i_\gamma}^{(1,\gamma)} \, n_{i_1 i_\beta}^{(1,\beta)}}{n_{i_1}^{(1)}} = \\ \displaystyle = n_{i_\gamma}^{(\gamma)} - \sum_{i_1=1}^{r_1} n_{i_1}^{(1,\gamma)} = 0 \quad (\gamma \neq \beta = 2, \ldots, k) \\ \\ \displaystyle m_{i_\gamma i_\gamma}^{[\gamma,\gamma]} + \sum_{i_\beta \neq i_\beta = 1}^{r_\beta} m_{i_\gamma i_\beta}^{[\gamma,\gamma]} = \\ \displaystyle = n_{i_\gamma}^{(\gamma)} - \sum_{i_1=1}^{r_1} \frac{n_{i_1 i_\gamma}^{(1,\gamma)} \, n_{i_1 i_\gamma}^{(1,\gamma)}}{n_{i_1}^{(1)}} - \sum_{i_\beta \neq i_\gamma = 1}^{r_\beta} \sum_{i_1=1}^{r_1} \frac{n_{i_1 i_\gamma}^{(1,\gamma)} \, n_{i_1 i_\beta}^{(1,\gamma)}}{n_{i_1}^{(1)}} \\ \displaystyle = n_{i_\gamma}^{(\gamma)} - \sum_{i_\beta=1}^{r_\beta} \sum_{i_1=1}^{r_1} \frac{n_{i_1 i_\gamma}^{(1,\gamma)} \, n_{i_1 i_\beta}^{(1,\gamma)}}{n_{i_1}^{(1)}} \\ \displaystyle = n_{i_\gamma}^{(\gamma)} - \sum_{i_1=1}^{r_1} n_{i_1 i_\gamma}^{(1,\gamma)} = 0 \quad (\gamma \parallel \beta = 2, \ldots, k) \end{cases}$$

$(i_\gamma = 1, \ldots, r_\gamma)$

vem:

$$
\Delta = r_1^2
\begin{vmatrix}
\sum y^2_{i_1\ldots i_{k+1}} & Y_1^{(1)} & Y_2^{(1)}\cdots Y_{n_1}^{(1)} & Q_1^{(2)} & Q_2^{(2)}\cdots Q^{(2)}_{n_2-1} & 0 & Q_1^{(3)} & Q_2^{(3)}\cdots Q^{(3)}_{n_3-1} & 0\cdots & Q_1^{(k)} & Q_2^{(k)}\cdots Q^{(k)}_{n_k-1} & 0 & 0 & 0\cdots 0 \\
Y_1^{(1)} & n_1^{(1)} & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
Y_2^{(1)} & 0 & n_2^{(1)}\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Y_{n_1}^{(1)} & 0 & 0\cdots n_{n_1}^{(1)} & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
Q_1^{(2)} & 0 & 0\cdots 0 & m^{[2,2]}_{11} & m^{[2,2]}_{12}\cdots m^{[2,2]}_{1n_2-1} & 0 & m^{[2,3]}_{11} & m^{[2,3]}_{12}\cdots m^{[2,3]}_{1n_3-1} & 0\cdots & m^{[2,k]}_{11} & m^{[2,k]}_{12}\cdots m^{[2,k]}_{1n_k-1} & 0 & 1 & 0\cdots 0 \\
Q_2^{(2)} & 0 & 0\cdots 0 & m^{[2,2]}_{21} & m^{[2,2]}_{22}\cdots m^{[2,2]}_{2n_2-1} & 0 & m^{[2,3]}_{21} & m^{[2,3]}_{22}\cdots m^{[2,3]}_{2n_3-1} & 0\cdots & m^{[2,k]}_{21} & m^{[2,k]}_{22}\cdots m^{[2,k]}_{2n_k-1} & 0 & 1 & 0\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q^{(2)}_{n_2-1} & 0 & 0\cdots 0 & m^{[2,2]}_{n_2-1,1} & m^{[2,2]}_{n_2-1,2}\cdots m^{[2,2]}_{n_2-1,n_2-1} & 0 & m^{[2,3]}_{n_2-1,1} & m^{[2,3]}_{n_2-1,2}\cdots m^{[2,3]}_{n_2-1,n_3-1} & 0\cdots & m^{[2,k]}_{n_2-1,1} & m^{[2,k]}_{n_2-1,2}\cdots m^{[2,k]}_{n_2-1,n_k-1} & 0 & 1 & 0\cdots 0 \\
0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & r_2 & 0\cdots 0 \\
Q_1^{(3)} & 0 & 0\cdots 0 & m^{[2,3]}_{11} & m^{[2,3]}_{21}\cdots m^{[2,3]}_{n_2-1,1} & 0 & m^{[3,3]}_{11} & m^{[3,3]}_{12}\cdots m^{[3,3]}_{1n_3-1} & 0\cdots & m^{[3,k]}_{11} & m^{[3,k]}_{12}\cdots m^{[3,k]}_{1n_k-1} & 0 & 0 & 1\cdots 0 \\
Q_2^{(3)} & 0 & 0\cdots 0 & m^{[2,3]}_{12} & m^{[2,3]}_{22}\cdots m^{[2,3]}_{n_2-1,2} & 0 & m^{[3,3]}_{21} & m^{[3,3]}_{22}\cdots m^{[3,3]}_{2n_3-1} & 0\cdots & m^{[3,k]}_{21} & m^{[3,k]}_{22}\cdots m^{[3,k]}_{2n_k-1} & 0 & 0 & 1\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q^{(3)}_{n_3-1} & 0 & 0\cdots 0 & m^{[2,3]}_{1n_3-1} & m^{[2,3]}_{2n_3-1}\cdots m^{[2,3]}_{n_2-1,n_3-1} & 0 & m^{[3,3]}_{n_3-1,1} & m^{[3,3]}_{n_3-1,2}\cdots m^{[3,3]}_{n_3-1,n_3-1} & 0\cdots & m^{[3,k]}_{n_3-1,1} & m^{[3,k]}_{n_3-1,2}\cdots m^{[3,k]}_{n_3-1,n_k-1} & 0 & 0 & 1\cdots 0 \\
0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & r_3\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Q_1^{(k)} & 0 & 0\cdots 0 & m^{[2,k]}_{11} & m^{[2,k]}_{21}\cdots m^{[2,k]}_{n_2-1,1} & 0 & m^{[3,k]}_{11} & m^{[3,k]}_{21}\cdots m^{[3,k]}_{n_3-1,1} & 0\cdots & m^{[k,k]}_{11} & m^{[k,k]}_{12}\cdots m^{[k,k]}_{1n_k-1} & 0 & 0 & 0\cdots 1 \\
Q_2^{(k)} & 0 & 0\cdots 0 & m^{[2,k]}_{12} & m^{[2,k]}_{22}\cdots m^{[2,k]}_{n_2-1,2} & 0 & m^{[3,k]}_{12} & m^{[3,k]}_{22}\cdots m^{[3,k]}_{n_3-1,2} & 0\cdots & m^{[k,k]}_{21} & m^{[k,k]}_{22}\cdots m^{[k,k]}_{2n_k-1} & 0 & 0 & 0\cdots 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q^{(k)}_{n_k-1} & 0 & 0\cdots 0 & m^{[2,k]}_{1n_k-1} & m^{[2,k]}_{2n_k-1}\cdots m^{[2,k]}_{n_2-1,n_2-1} & 0 & m^{[3,k]}_{1n_k-1} & m^{[3,k]}_{2n_k-1}\cdots m^{[3,k]}_{n_3-1,n_k-1} & 0\cdots & m^{[k,k]}_{n_k-1,1} & m^{[k,k]}_{n_k-1,2}\cdots m^{[k,k]}_{n_k-1,n_k-1} & 0 & 0 & 0\cdots 1 \\
0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots r_k \\
0 & 0 & 0\cdots 0 & 1 & 1\cdots 1 & r_2 & 0 & 0\cdots 0 & 0\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 1 & 1\cdots 1 & r_3\cdots & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0 & 0\cdots 0 & 0\cdots & 1 & 1\cdots 1 & r_k & 0 & 0\cdots 0
\end{vmatrix}
$$

Desenvolvendo êste determinante pelo método de Cauchy, vem:

$$\triangle = r_1^2 \, r_2^2 \ldots r_k^2 \Big\{ n_1^{(1)} \, n_2^{(2)} \ldots n_{r_1}^{(1)} \; D \sum_{i_1, \ldots, i_{k+1}} y^2_{i_1 \, i_2 \ldots i_{k+1}} -$$

$$- \left| y_1^{(1)} \right|^2 n_2^{(1)} \ldots n_{r_1}^{(1)} \; D - \left| y_2^{(1)} \right|^2 n_1^{(1)} \, n_3^{(1)} \ldots n_{r_1}^{(1)} \; D - \ldots$$

$$\ldots - \left| y_{r_1}^{(1)} \right|^2 n_1^{(1)} \ldots n_{r_1 - 1}^{(1)} \; D -$$

$$- n_1^{(1)} \, n_2^{(1)} \ldots n_{r_1}^{(1)} \sum_{\delta = 2}^{k} \sum_{\gamma = 2}^{k} \sum_{i_\delta = 1}^{r_\beta - 1} \sum_{i_\gamma = 1}^{r_\gamma - 1} Q_{i_\gamma}^{(\gamma)} \, Q_{i_\delta}^{(\delta)} \times$$

$$D^*_{(r_2 + \ldots + r_{\gamma - 1} - (\gamma - 2) + (i_\gamma) \, , \, (r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)} \Big\}$$

onde

$$D^*_{(r_2 + \ldots r_{\gamma - 1} - (\gamma - 2) + i_\gamma) \, . \, (r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)}$$

é o cofator do elemento que ocupa a linha $(r_2 + \ldots + r_{\gamma - 1} - (\gamma - 2) + i_\gamma)$ e a coluna $(r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)$ no determinante D dado por:

$$
D = \left|\begin{array}{ccccccccccccc}
m_{11}^{[2,2]} & m_{12}^{[2,2]} \cdots & m_{1n_2-1}^{[2,2]} & m_{11}^{[2,3]} & m_{12}^{[2,3]} \cdots & m_{1n_3-1}^{[2,3]} & \cdots & m_{11}^{[2,k-1]} & m_{12}^{[2,k-1]} \cdots & m_{1n_{k-1}-1}^{[2,k-1]} & m_{11}^{[2,k]} & m_{12}^{[2,k]} \cdots & m_{1n_k-1}^{[2,k]} \\
m_{21}^{[2,2]} & m_{22}^{[2,2]} \cdots & m_{2n_2-1}^{[2,2]} & m_{21}^{[2,3]} & m_{22}^{[2,3]} \cdots & m_{2n_3-1}^{[2,3]} & \cdots & m_{21}^{[2,k-1]} & m_{22}^{[2,k-1]} \cdots & m_{2n_{k-1}-1}^{[2,k-1]} & m_{21}^{[2,k]} & m_{22}^{[2,k]} \cdots & m_{2n_k-1}^{[2,k]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
m_{n_2-1,1}^{[2,2]} & m_{n_2-1,2}^{[2,2]} \cdots & m_{n_2-1,n_2-1}^{[2,2]} & m_{n_2-1,1}^{[2,3]} & m_{n_2-1,2}^{[2,3]} \cdots & m_{n_2-1,n_3-1}^{[2,3]} & \cdots & m_{n_2-1,1}^{[2,k-1]} & m_{n_2-1,2}^{[2,k-1]} \cdots & m_{n_2-1,n_{k-1}-1}^{[2,k-1]} & m_{n_2-1,1}^{[2,k]} & m_{n_2-1,2}^{[2,k]} \cdots & m_{n_2-1,n_k-1}^{[2,k]} \\
m_{11}^{[2,3]} & m_{21}^{[2,3]} \cdots & m_{n_2-1,1}^{[2,3]} & m_{11}^{[3,3]} & m_{12}^{[3,3]} \cdots & m_{1n_3-1}^{[3,3]} & \cdots & m_{11}^{[3,k-1]} & m_{12}^{[3,k-1]} \cdots & m_{1n_{k-1}-1}^{[3,k-1]} & m_{11}^{[3,k]} & m_{12}^{[3,k]} \cdots & m_{1n_k-1}^{[3,k]} \\
m_{12}^{[2,3]} & m_{22}^{[2,3]} \cdots & m_{n_2-1,2}^{[2,3]} & m_{21}^{[3,3]} & m_{22}^{[3,3]} \cdots & m_{2n_3-1}^{[3,3]} & \cdots & m_{21}^{[3,k-1]} & m_{22}^{[3,k-1]} \cdots & m_{2n_{k-1}-1}^{[3,k-1]} & m_{21}^{[3,k]} & m_{22}^{[3,k]} \cdots & m_{2n_k-1}^{[3,k]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
m_{1n_3-1}^{[2,3]} & m_{2n_3-1}^{[2,3]} \cdots & m_{n_2-1,n_3-1}^{[2,3]} & m_{n_3-1,1}^{[3,3]} & m_{n_3-1,2}^{[3,3]} \cdots & m_{n_3-1,n_3-1}^{[3,3]} & \cdots & m_{n_3-1,1}^{[3,k-1]} & m_{n_3-1,2}^{[3,k-1]} \cdots & m_{n_3-1,n_{k-1}-1}^{[3,k-1]} & m_{n_3-1,1}^{[3,k]} & m_{n_3-1,2}^{[3,k]} \cdots & m_{n_3-1,n_k-1}^{[3,k]} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
m_{11}^{[2,k-1]} & m_{21}^{[2,k-1]} \cdots & m_{n_2-1,1}^{[2,k-1]} & m_{11}^{[3,k-1]} & m_{21}^{[3,k-1]} \cdots & m_{n_3-1,1}^{[3,k-1]} & \cdots & m_{11}^{[k-1,k-1]} & m_{12}^{[k-1,k-1]} \cdots & m_{1n_{k-1}-1}^{[k-1,k-1]} & m_{11}^{[k-1,k]} & m_{12}^{[k-1,k]} \cdots & m_{1n_k-1}^{[k-1,k]} \\
m_{12}^{[2,k-1]} & m_{22}^{[2,k-1]} \cdots & m_{n_2-1,2}^{[2,k-1]} & m_{12}^{[3,k-1]} & m_{22}^{[3,k-1]} \cdots & m_{n_3-1,2}^{[3,k-1]} & \cdots & m_{21}^{[k-1,k-1]} & m_{22}^{[k-1,k-1]} \cdots & m_{2n_{k-1}-1}^{[k-1,k-1]} & m_{21}^{[k-1,k]} & m_{22}^{[k-1,k]} \cdots & m_{2n_k-1}^{[k-1,k]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
m_{1n_{k-1}-1}^{[2,k-1]} & m_{2n_{k-1}-1}^{[2,k-1]} \cdots & m_{n_2-1,n_{k-1}-1}^{[2,k-1]} & m_{1n_{k-1}-1}^{[3,k-1]} & m_{2n_{k-1}-1}^{[3,k-1]} \cdots & m_{n_3-1,n_{k-1}-1}^{[3,k-1]} & \cdots & m_{n_{k-1}-1,1}^{[k-1,k-1]} & m_{n_{k-1}-1,2}^{[k-1,k-1]} \cdots & m_{n_{k-1}-1,n_{k-1}-1}^{[k-1,k-1]} & m_{n_{k-1}-1,1}^{[k-1,k]} & m_{n_{k-1}-1,2}^{[k-1,k]} \cdots & m_{n_{k-1}-1,n_k-1}^{[k-1,k]} \\
m_{11}^{[2,k]} & m_{21}^{[2,k]} \cdots & m_{n_2-1,1}^{[2,k]} & m_{11}^{[3,k]} & m_{21}^{[3,k]} \cdots & m_{n_3-1,1}^{[3,k]} & \cdots & m_{11}^{[k-1,k]} & m_{21}^{[k-1,k]} \cdots & m_{n_{k-1}-1,1}^{[k-1,k]} & m_{11}^{[k,k]} & m_{12}^{[k,k]} \cdots & m_{1n_k-1}^{[k,k]} \\
m_{12}^{[2,k]} & m_{22}^{[2,k]} \cdots & m_{n_2-1,2}^{[2,k]} & m_{12}^{[3,k]} & m_{22}^{[3,k]} \cdots & m_{n_3-1,2}^{[3,k]} & \cdots & m_{12}^{[k-1,k]} & m_{22}^{[k-1,k]} \cdots & m_{n_{k-1}-1,2}^{[k-1,k]} & m_{21}^{[k,k]} & m_{22}^{[k,k]} \cdots & m_{2n_k-1}^{[k,k]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
m_{1n_k-1}^{[2,k]} & m_{2n_k-1}^{[2,k]} \cdots & m_{n_2-1,n_k-1}^{[2,k]} & m_{1n_k-1}^{[3,k]} & m_{2n_k-1}^{[3,k]} \cdots & m_{n_3-1,n_k-1}^{[3,k]} & \cdots & m_{1n_k-1}^{[k-1,k]} & m_{2n_k-1}^{[k-1,k]} \cdots & m_{n_{k-1}-1,n_k-1}^{[k-1,k]} & m_{n_k-1,1}^{[k,k]} & m_{n_k-1,2}^{[k,k]} \cdots & m_{n_k n_k-1}^{[k,k]}
\end{array}\right|,
$$

Finalmente, notando que

$$\triangle_{00} = r_1^2 \; r_2^2 \; \ldots \; r_k^2 \; n_1^{(1)} \; n_2^{(1)} \; \ldots \; n_{r_1}^{(1)} \; . \; D \; ,$$

então, temos:

$$n \; \hat{\sigma}^2_{\Omega} = \sum_{i_1, \ldots, i_{k+1}} y^2_{i_1 \, i_2 \ldots i_{k+1}} - \sum_{i_1 = 1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]}{n^{(1)}_{i_1}} -$$

$$- \sum_{\delta = 2}^{k} \; \sum_{\gamma = 2}^{k} \; \sum_{i_\delta = 1}^{r_\delta - 1} \; \sum_{i_\gamma = 1}^{r_\gamma - 1} Q^{(\gamma)}_{i_\gamma} \; Q^{(\delta)}_{i_\delta} \; .$$

(2.17)

$$. \; D^{(r_2 + \ldots + r_{\gamma - 1} - (\gamma - 2) + i_\gamma), (r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)}$$

onde:

$$D^{(r_2 + \ldots + r_{\gamma - 1} - (\gamma - 2) + i_\gamma), (r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)} =$$

$$= \frac{1}{D} \; \overset{*}{D}{}^{(r_2 + \ldots + r_{\gamma - 1} - (\gamma - 2) + i_\gamma), (r_2 + \ldots + r_{\delta - 1} - (\delta - 2) + i_\delta)}$$

Vejamos, agora, $n \; \hat{\sigma}^2_{\omega_\beta} = \dfrac{\triangle'}{\triangle'_{00}}$, onde $\triangle'_{00}$ é o cofator de $\Sigma \; y^2_{i_1 \ldots i_{k+1}}$ que por sua vez é dado por:

$$
\Delta_{\omega_\beta} = \begin{vmatrix}
\sum y^2_{i_1\cdots i_{k+1}} & y_1^{(1)}\; y_2^{(1)} \cdots y_{n_1}^{(1)} & y_1^{(2)}\; y_2^{(2)} \cdots y_{n_2}^{(2)} & \cdots & y_1^{(\beta)}\; y_2^{(\beta)} \cdots y_{n_\beta}^{(\beta)} & \cdots & y_1^{(k)}\; y_2^{(k)} \cdots y_{n_k}^{(k)} & y & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
y_1^{(1)} & n_1^{(1)}\; 0 \cdots 0 & n_{11}^{(1,2)}\; n_{12}^{(1,2)} \cdots n_{1n_2}^{(1,2)} & \cdots & n_{11}^{(1,\beta)}\; n_{12}^{(1,\beta)} \cdots n_{1n_\beta}^{(1,\beta)} & \cdots & n_{11}^{(1,k)}\; n_{12}^{(1,k)} \cdots n_{1n_k}^{(1,k)} & n_1^{(1)} & 1 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
y_2^{(1)} & 0\; n_2^{(1)} \cdots 0 & n_{21}^{(1,2)}\; n_{22}^{(1,2)} \cdots n_{2n_2}^{(1,2)} & \cdots & n_{21}^{(1,\beta)}\; n_{22}^{(1,\beta)} \cdots n_{2n_\beta}^{(1,\beta)} & \cdots & n_{21}^{(1,k)}\; n_{22}^{(1,k)} \cdots n_{2n_k}^{(1,k)} & n_2^{(1)} & 1 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_1}^{(1)} & 0\; 0 \cdots n_{n_1}^{(1)} & n_{n_11}^{(1,2)}\; n_{n_12}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & \cdots & n_{n_11}^{(1,\beta)}\; n_{n_12}^{(1,\beta)} \cdots n_{n_1n_\beta}^{(1,\beta)} & \cdots & n_{n_11}^{(1,k)}\; n_{n_12}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & n_{n_1}^{(1)} & 1 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
y_1^{(2)} & n_{11}^{(1,2)}\; n_{21}^{(1,2)} \cdots n_{n_11}^{(1,2)} & n_1^{(2)}\; 0 \cdots 0 & \cdots & n_{11}^{(2,\beta)}\; n_{12}^{(2,\beta)} \cdots n_{1n_\beta}^{(2,\beta)} & \cdots & n_{11}^{(2,k)}\; n_{12}^{(2,k)} \cdots n_{1n_k}^{(2,k)} & n_1^{(2)} & 0 & 1 \cdots 0 \cdots 0 & 0 \cdots 0 \\
y_2^{(2)} & n_{12}^{(1,2)}\; n_{22}^{(1,2)} \cdots n_{n_12}^{(1,2)} & 0\; n_2^{(2)} \cdots 0 & \cdots & n_{21}^{(2,\beta)}\; n_{22}^{(2,\beta)} \cdots n_{2n_\beta}^{(2,\beta)} & \cdots & n_{21}^{(2,k)}\; n_{22}^{(2,k)} \cdots n_{2n_k}^{(2,k)} & n_2^{(2)} & 0 & 1 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_2}^{(2)} & n_{1n_2}^{(1,2)}\; n_{2n_2}^{(1,2)} \cdots n_{n_1n_2}^{(1,2)} & 0\; 0 \cdots n_{n_2}^{(2)} & \cdots & n_{n_21}^{(2,\beta)}\; n_{n_22}^{(2,\beta)} \cdots n_{n_2n_\beta}^{(2,\beta)} & \cdots & n_{n_21}^{(2,k)}\; n_{n_22}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & n_{n_2}^{(2)} & 0 & 1 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(\beta)} & n_{11}^{(1,\beta)}\; n_{21}^{(1,\beta)} \cdots n_{n_11}^{(1,\beta)} & n_{11}^{(2,\beta)}\; n_{21}^{(2,\beta)} \cdots n_{n_21}^{(2,\beta)} & \cdots & n_1^{(\beta)}\; 0 \cdots 0 & \cdots & n_{11}^{(\beta,k)}\; n_{12}^{(\beta,k)} \cdots n_{1n_k}^{(\beta,k)} & n_1^{(\beta)} & 0 & 0 \cdots 1 \cdots 0 & 1 \cdots 1 \\
y_2^{(\beta)} & n_{12}^{(1,\beta)}\; n_{22}^{(1,\beta)} \cdots n_{n_12}^{(1,\beta)} & n_{12}^{(2,\beta)}\; n_{22}^{(2,\beta)} \cdots n_{n_22}^{(2,\beta)} & \cdots & 0\; n_2^{(\beta)} \cdots 0 & \cdots & n_{21}^{(\beta,k)}\; n_{22}^{(\beta,k)} \cdots n_{2n_k}^{(\beta,k)} & n_2^{(\beta)} & 0 & 0 \cdots 1 \cdots 0 & -1 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_\beta}^{(\beta)} & n_{1n_\beta}^{(1,\beta)}\; n_{2n_\beta}^{(1,\beta)} \cdots n_{n_1n_\beta}^{(1,\beta)} & n_{1n_\beta}^{(2,\beta)}\; n_{2n_\beta}^{(2,\beta)} \cdots n_{n_2n_\beta}^{(2,\beta)} & \cdots & 0\; 0 \cdots n_{n_\beta}^{(\beta)} & \cdots & n_{n_\beta1}^{(\beta,k)}\; n_{n_\beta2}^{(\beta,k)} \cdots n_{n_\beta n_k}^{(\beta,k)} & n_{n_\beta}^{(\beta)} & 0 & 0 \cdots 1 \cdots 0 & 0 \cdots 1 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(k)} & n_{11}^{(1,k)}\; n_{21}^{(1,k)} \cdots n_{n_11}^{(1,k)} & n_{11}^{(2,k)}\; n_{21}^{(2,k)} \cdots n_{n_21}^{(2,k)} & \cdots & n_{11}^{(\beta,k)}\; n_{21}^{(\beta,k)} \cdots n_{n_\beta1}^{(\beta,k)} & \cdots & n_1^{(k)}\; 0 \cdots 0 & n_1^{(k)} & 0 & 0 \cdots 0 \cdots 1 & 0 \cdots 0 \\
y_2^{(k)} & n_{12}^{(1,k)}\; n_{22}^{(1,k)} \cdots n_{n_12}^{(1,k)} & n_{12}^{(2,k)}\; n_{22}^{(2,k)} \cdots n_{n_22}^{(2,k)} & \cdots & n_{12}^{(\beta,k)}\; n_{22}^{(\beta,k)} \cdots n_{n_\beta2}^{(\beta,k)} & \cdots & 0\; n_2^{(k)} \cdots 0 & n_2^{(k)} & 0 & 0 \cdots 0 \cdots 1 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_k}^{(k)} & n_{1n_k}^{(1,k)}\; n_{2n_k}^{(1,k)} \cdots n_{n_1n_k}^{(1,k)} & n_{1n_k}^{(2,k)}\; n_{2n_k}^{(2,k)} \cdots n_{n_2n_k}^{(2,k)} & \cdots & n_{1n_k}^{(\beta,k)}\; n_{2n_k}^{(\beta,k)} \cdots n_{n_\beta n_k}^{(\beta,k)} & \cdots & 0\; 0 \cdots n_{n_k}^{(k)} & n_{n_k}^{(k)} & 0 & 0 \cdots 0 \cdots 1 & 0 \cdots 0 \\
y & n_1^{(1)}\; n_2^{(1)} \cdots n_{n_1}^{(1)} & n_1^{(2)}\; n_2^{(2)} \cdots n_{n_2}^{(2)} & \cdots & n_1^{(\beta)}\; n_2^{(\beta)} \cdots n_{n_\beta}^{(\beta)} & \cdots & n_1^{(k)}\; n_2^{(k)} \cdots n_{n_k}^{(k)} & n & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
0 & 1\; 1 \cdots 1 & 0\; 0 \cdots 0 & \cdots & 0\; 0 \cdots 0 & \cdots & 0\; 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
0 & 0\; 0 \cdots 0 & 1\; 1 \cdots 1 & \cdots & 0\; 0 \cdots 0 & \cdots & 0\; 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0\; 0 \cdots 0 & 0\; 0 \cdots 0 & \cdots & 1\; 1 \cdots 1 & \cdots & 0\; 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0\; 0 \cdots 0 & 0\; 0 \cdots 0 & \cdots & 0\; 0 \cdots 0 & \cdots & 1\; 1 \cdots 1 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
0 & 0\; 0 \cdots 0 & 0\; 0 \cdots 0 & \cdots & 1\; -1 \cdots 0 & \cdots & 0\; 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0\; 0 \cdots 0 & 0\; 0 \cdots 0 & \cdots & 1\; 0 \cdots -1 & \cdots & 0\; 0 \cdots 0 & 0 & 0 & 0 \cdots 0 \cdots 0 & 0 \cdots 0
\end{vmatrix}
$$

Adicionando-se à $(r_1 + r_2 + \ldots + r_{\beta-1} + 2)^{ma}$ linha (coluna), isto é, à encabeçada por $Y_1^{(\beta)}$ a soma das linhas (colunas) encabeçadas por $Y_2^{(\beta)}$, $Y_3^{(\beta)}$, ..., $Y_{r_\beta}^{(\beta)}$ e da linha (coluna) assim obtida subtraindo-se a linha (coluna) encabeçada por Y vem, depois de desenvolvido o determinante sucessivamente em função da $(r_1 + r_2 + \ldots + r_{\beta-1} + 2)^{ma}$ linha (coluna) e das colunas (linhas) $(r_1 + r_2 + \ldots + r_k + k + 3)$, $(r_1 + r_2 + \ldots + r_k + k + 4)$, ..., $(r_1 + r_2 + \ldots + r_k + k + r_\beta + 1)$:

$$
\Delta\omega_\beta = r_\beta^2
\begin{array}{|cccccccccccccccccccc|}
\sum y^2_{i_1\cdots i_{k+1}} & y_1^{(1)} & y_2^{(1)}\cdots & y_{n_1}^{(1)} & y_1^{(2)} & y_2^{(2)}\cdots & y_{n_2}^{(2)}\cdots & y_1^{(\beta-1)} & y_2^{(\beta-1)}\cdots & y_{n_{\beta-1}}^{(\beta-1)} & y_1^{(\beta+1)} & y_2^{(\beta+1)}\cdots & y_{n_{\beta+1}}^{(\beta+1)}\cdots & y_1^{(k)} & y_2^{(k)}\cdots & y_{n_k}^{(k)} & y & 0 & 0\cdots 0 & 0\cdots 0 \\
y_1^{(1)} & n_1^{(1)} & 0\cdots & 0 & n_{11}^{(1,2)} & n_{12}^{(1,2)}\cdots & n_{1n_2}^{(1,2)}\cdots & n_{11}^{(1,\beta-1)} & n_{12}^{(1,\beta-1)}\cdots & n_{1n_{\beta-1}}^{(1,\beta-1)} & n_{11}^{(1,\beta+1)} & n_{12}^{(1,\beta+1)}\cdots & n_{1n_{\beta+1}}^{(1,\beta+1)}\cdots & n_{11}^{(1,k)} & n_{12}^{(1,k)}\cdots & n_{1n_k}^{(1,k)} & n_1^{(1)} & 1 & 0\cdots 0 & 0\cdots 0 \\
y_2^{(1)} & 0 & n_2^{(1)}\cdots & 0 & n_{21}^{(1,2)} & n_{22}^{(1,2)}\cdots & n_{2n_2}^{(1,2)}\cdots & n_{21}^{(1,\beta-1)} & n_{22}^{(1,\beta-1)}\cdots & n_{2n_{\beta-1}}^{(1,\beta-1)} & n_{21}^{(1,\beta+1)} & n_{22}^{(1,\beta+1)}\cdots & n_{2n_{\beta+1}}^{(1,\beta+1)}\cdots & n_{21}^{(1,k)} & n_{22}^{(1,k)}\cdots & n_{2n_k}^{(1,k)} & n_2^{(1)} & 1 & 0\cdots 0 & 0\cdots 0 \\
\vdots & \vdots & & \ddots & \vdots & & \ddots & \vdots & & \ddots & \vdots & & \ddots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_1}^{(1)} & 0 & 0\cdots & n_{n_1}^{(1)} & n_{n_11}^{(1,2)} & n_{n_12}^{(1,2)}\cdots & n_{n_1n_2}^{(1,2)}\cdots & n_{n_11}^{(1,\beta-1)} & n_{n_12}^{(1,\beta-1)}\cdots & n_{n_1n_{\beta-1}}^{(1,\beta-1)} & n_{n_11}^{(1,\beta+1)} & n_{n_12}^{(1,\beta+1)}\cdots & n_{n_1n_{\beta+1}}^{(1,\beta+1)}\cdots & n_{n_11}^{(1,k)} & n_{n_12}^{(1,k)}\cdots & n_{n_1n_k}^{(1,k)} & n_{n_1}^{(1)} & 1 & 0\cdots 0 & 0\cdots 0 \\
y_1^{(2)} & n_{11}^{(1,2)} & n_{21}^{(1,2)}\cdots & n_{n_11}^{(1,2)} & n_1^{(2)} & 0\cdots & 0\cdots & n_{11}^{(2,\beta-1)} & n_{12}^{(2,\beta-1)}\cdots & n_{1n_{\beta-1}}^{(2,\beta-1)} & n_{11}^{(2,\beta+1)} & n_{12}^{(2,\beta+1)}\cdots & n_{1n_{\beta+1}}^{(2,\beta+1)}\cdots & n_{11}^{(2,k)} & n_{12}^{(2,k)}\cdots & n_{1n_k}^{(2,k)} & n_1^{(2)} & 0 & 1\cdots 0 & 0\cdots 0 \\
y_2^{(2)} & n_{12}^{(1,2)} & n_{22}^{(1,2)}\cdots & n_{n_12}^{(1,2)} & 0 & n_2^{(2)}\cdots & 0\cdots & n_{21}^{(2,\beta-1)} & n_{22}^{(2,\beta-1)}\cdots & n_{2n_{\beta-1}}^{(2,\beta-1)} & n_{21}^{(2,\beta+1)} & n_{22}^{(2,\beta+1)}\cdots & n_{2n_{\beta+1}}^{(2,\beta+1)}\cdots & n_{21}^{(2,k)} & n_{22}^{(2,k)}\cdots & n_{2n_k}^{(2,k)} & n_2^{(2)} & 0 & 1\cdots 0 & 0\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots & & \ddots & \vdots & & \ddots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\
y_{n_2}^{(2)} & n_{1n_2}^{(1,2)} & n_{2n_2}^{(1,2)}\cdots & n_{n_1n_2}^{(1,2)} & 0 & 0\cdots & n_{n_2}^{(2)}\cdots & n_{n_21}^{(2,\beta-1)} & n_{n_22}^{(2,\beta-1)}\cdots & n_{n_2n_{\beta-1}}^{(2,\beta-1)} & n_{n_21}^{(2,\beta+1)} & n_{n_22}^{(2,\beta+1)}\cdots & n_{n_2n_{\beta+1}}^{(2,\beta+1)}\cdots & n_{n_21}^{(2,k)} & n_{n_22}^{(2,k)}\cdots & n_{n_2n_k}^{(2,k)} & n_{n_2}^{(2)} & 0 & 1\cdots 0 & 0\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(\beta-1)} & n_{11}^{(1,\beta-1)} & n_{21}^{(1,\beta-1)}\cdots & n_{n_11}^{(1,\beta-1)} & n_{11}^{(2,\beta-1)} & n_{21}^{(2,\beta-1)}\cdots & n_{n_21}^{(2,\beta-1)}\cdots & n_1^{(\beta-1)} & 0\cdots & 0 & n_{11}^{(\beta-1,\beta+1)} & n_{12}^{(\beta-1,\beta+1)}\cdots & n_{1n_{\beta+1}}^{(\beta-1,\beta+1)}\cdots & n_{11}^{(\beta-1,k)} & n_{12}^{(\beta-1,k)}\cdots & n_{1n_k}^{(\beta-1,k)} & n_1^{(\beta-1)} & 0 & 0\cdots 1 & 0\cdots 0 \\
y_2^{(\beta-1)} & n_{12}^{(1,\beta-1)} & n_{22}^{(1,\beta-1)}\cdots & n_{n_12}^{(1,\beta-1)} & n_{12}^{(2,\beta-1)} & n_{22}^{(2,\beta-1)}\cdots & n_{n_22}^{(2,\beta-1)}\cdots & 0 & n_2^{(\beta-1)}\cdots & 0 & n_{21}^{(\beta-1,\beta+1)} & n_{22}^{(\beta-1,\beta+1)}\cdots & n_{2n_{\beta+1}}^{(\beta-1,\beta+1)}\cdots & n_{21}^{(\beta-1,k)} & n_{22}^{(\beta-1,k)}\cdots & n_{2n_k}^{(\beta-1,k)} & n_2^{(\beta-1)} & 0 & 0\cdots 1 & 0\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_{\beta-1}}^{(\beta-1)} & n_{1n_{\beta-1}}^{(1,\beta-1)} & n_{2n_{\beta-1}}^{(1,\beta-1)}\cdots & n_{n_1n_{\beta-1}}^{(1,\beta-1)} & n_{1n_{\beta-1}}^{(2,\beta-1)} & n_{2n_{\beta-1}}^{(2,\beta-1)}\cdots & n_{n_2n_{\beta-1}}^{(2,\beta-1)}\cdots & 0 & 0\cdots & n_{n_{\beta-1}}^{(\beta-1)} & n_{n_{\beta-1}1}^{(\beta-1,\beta+1)} & n_{n_{\beta-1}2}^{(\beta-1,\beta+1)}\cdots & n_{n_{\beta-1}n_{\beta+1}}^{(\beta-1,\beta+1)}\cdots & n_{n_{\beta-1}1}^{(\beta-1,k)} & n_{n_{\beta-1}2}^{(\beta-1,k)}\cdots & n_{n_{\beta-1}n_k}^{(\beta-1,k)} & n_{n_{\beta-1}}^{(\beta-1)} & 0 & 0\cdots 1 & 0\cdots 0 \\
y_1^{(\beta+1)} & n_{11}^{(1,\beta+1)} & n_{21}^{(1,\beta+1)}\cdots & n_{n_11}^{(1,\beta+1)} & n_{11}^{(2,\beta+1)} & n_{21}^{(2,\beta+1)}\cdots & n_{n_21}^{(2,\beta+1)}\cdots & n_{11}^{(\beta-1,\beta+1)} & n_{21}^{(\beta-1,\beta+1)}\cdots & n_{n_{\beta-1}1}^{(\beta-1,\beta+1)} & n_1^{(\beta+1)} & 0\cdots & 0\cdots & n_{11}^{(\beta+1,k)} & n_{12}^{(\beta+1,k)}\cdots & n_{1n_k}^{(\beta+1,k)} & n_1^{(\beta+1)} & 0 & 0\cdots 0 & 1\cdots 0 \\
y_2^{(\beta+1)} & n_{12}^{(1,\beta+1)} & n_{22}^{(1,\beta+1)}\cdots & n_{n_12}^{(1,\beta+1)} & n_{12}^{(2,\beta+1)} & n_{22}^{(2,\beta+1)}\cdots & n_{n_22}^{(2,\beta+1)}\cdots & n_{12}^{(\beta-1,\beta+1)} & n_{22}^{(\beta-1,\beta+1)}\cdots & n_{n_{\beta-1}2}^{(\beta-1,\beta+1)} & 0 & n_2^{(\beta+1)}\cdots & 0\cdots & n_{21}^{(\beta+1,k)} & n_{22}^{(\beta+1,k)}\cdots & n_{2n_k}^{(\beta+1,k)} & n_2^{(\beta+1)} & 0 & 0\cdots 0 & 1\cdots 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_{\beta+1}}^{(\beta+1)} & n_{1n_{\beta+1}}^{(1,\beta+1)} & n_{2n_{\beta+1}}^{(1,\beta+1)}\cdots & n_{n_1n_{\beta+1}}^{(1,\beta+1)} & n_{1n_{\beta+1}}^{(2,\beta+1)} & n_{2n_{\beta+1}}^{(2,\beta+1)}\cdots & n_{n_2n_{\beta+1}}^{(2,\beta+1)}\cdots & n_{1n_{\beta+1}}^{(\beta-1,\beta+1)} & n_{2n_{\beta+1}}^{(\beta-1,\beta+1)}\cdots & n_{n_{\beta-1}n_{\beta+1}}^{(\beta-1,\beta+1)} & 0 & 0\cdots & n_{n_{\beta+1}}^{(\beta+1)}\cdots & n_{n_{\beta+1}1}^{(\beta+1,k)} & n_{n_{\beta+1}2}^{(\beta+1,k)}\cdots & n_{n_{\beta+1}n_k}^{(\beta+1,k)} & n_{n_{\beta+1}}^{(\beta+1)} & 0 & 0\cdots 0 & 1\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_1^{(k)} & n_{11}^{(1,k)} & n_{21}^{(1,k)}\cdots & n_{n_11}^{(1,k)} & n_{11}^{(2,k)} & n_{21}^{(2,k)}\cdots & n_{n_21}^{(2,k)}\cdots & n_{11}^{(\beta-1,k)} & n_{21}^{(\beta-1,k)}\cdots & n_{n_{\beta-1}1}^{(\beta-1,k)} & n_{11}^{(\beta+1,k)} & n_{21}^{(\beta+1,k)}\cdots & n_{n_{\beta+1}1}^{(\beta+1,k)}\cdots & n_1^{(k)} & 0\cdots & 0 & n_1^{(k)} & 0 & 0\cdots 0 & 0\cdots 1 \\
y_2^{(k)} & n_{12}^{(1,k)} & n_{22}^{(1,k)}\cdots & n_{n_12}^{(1,k)} & n_{12}^{(2,k)} & n_{22}^{(2,k)}\cdots & n_{n_22}^{(2,k)}\cdots & n_{12}^{(\beta-1,k)} & n_{22}^{(\beta-1,k)}\cdots & n_{n_{\beta-1}2}^{(\beta-1,k)} & n_{12}^{(\beta+1,k)} & n_{22}^{(\beta+1,k)}\cdots & n_{n_{\beta+1}2}^{(\beta+1,k)}\cdots & 0 & n_2^{(k)}\cdots & 0 & n_2^{(k)} & 0 & 0\cdots 0 & 0\cdots 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
y_{n_k}^{(k)} & n_{1n_k}^{(1,k)} & n_{2n_k}^{(1,k)}\cdots & n_{n_1n_k}^{(1,k)} & n_{1n_k}^{(2,k)} & n_{2n_k}^{(2,k)}\cdots & n_{n_2n_k}^{(2,k)}\cdots & n_{1n_k}^{(\beta-1,k)} & n_{2n_k}^{(\beta-1,k)}\cdots & n_{n_{\beta-1}n_k}^{(\beta-1,k)} & n_{1n_k}^{(\beta+1,k)} & n_{2n_k}^{(\beta+1,k)}\cdots & n_{n_{\beta+1}n_k}^{(\beta+1,k)}\cdots & 0 & 0\cdots & n_{n_k}^{(k)} & n_{n_k}^{(k)} & 0 & 0\cdots 0 & 0\cdots 1 \\
y & n_1^{(1)} & n_2^{(1)}\cdots & n_{n_1}^{(1)} & n_1^{(2)} & n_2^{(2)}\cdots & n_{n_2}^{(2)}\cdots & n_1^{(\beta-1)} & n_2^{(\beta-1)}\cdots & n_{n_{\beta-1}}^{(\beta-1)} & n_1^{(\beta+1)} & n_2^{(\beta+1)}\cdots & n_{n_{\beta+1}}^{(\beta+1)}\cdots & n_1^{(k)} & n_2^{(k)}\cdots & n_{n_k}^{(k)} & n & 0 & 0\cdots 0 & 0\cdots 0 \\
0 & 1 & 1\cdots & 1 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots 0 & 0\cdots 0 \\
0 & 0 & 0\cdots & 0 & 1 & 1\cdots & 1\cdots & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots 0 & 0\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 1 & 1\cdots & 1 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots 0 & 0\cdots 0 \\
0 & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 1 & 1\cdots & 1\cdots & 0 & 0\cdots & 0 & 0 & 0 & 0\cdots 0 & 0\cdots 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 0 & 0\cdots & 0 & 0 & 0\cdots & 0\cdots & 1 & 1\cdots & 1 & 0 & 0 & 0\cdots 0 & 0\cdots 0
\end{array}
$$

Comparando êste determinante com o $\triangle$ dado pela (2.11), vemos que êles são formalmente iguais, diferindo apenas na ordem.

Admitindo-se que $\beta \neq 1$, podemos, então, baseados no valor de $\triangle$, escrever para valor de $\triangle_{\omega_\beta}$:

$$n\,\hat{\sigma}^2_{\omega_\beta} = \sum_{i_1 i_2 \dots i_{k+1}} y^2_{i_1 i_2 \dots i_{k+1}} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} -$$

$$- \sum_{\gamma=2}^{\beta-1} \sum_{\delta=2}^{\beta-1} \sum_{i_\gamma=1}^{r_\gamma-1} \sum_{i_\delta=1}^{r_\delta-1} Q^{(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta}$$

$$D_\beta^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),\,(r_2+\dots+r_{\delta-1}-(\delta-2)+i_\delta)}$$

$$-2 \sum_{\gamma=2}^{\beta-1} \sum_{\delta=\beta+1}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \sum_{i_\delta=1}^{r_\delta-1} Q^{(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta}$$

$$D_\beta^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),\,(r_2+\dots+r_{\delta-1}-(\delta-3)+i_\delta)}$$

$$- \sum_{\delta=\beta+1}^{k} \sum_{\gamma=\beta+1}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \sum_{i_\delta=1}^{r_\delta-1} Q^{(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta}$$

$$D_\beta^{(r_2+\dots+r_{\gamma-1}-(\gamma-3)+i_\gamma),\,(r_2+\dots+r_{\delta-1}-(\delta-3)+i_\delta)}$$

$$(\beta = 2, \dots, k)$$

ou, mais genèricamente:

$$n\,\hat{\sigma}^2_{\omega_\beta} = \sum_{i_1 \dots i_{k+1}} y^2_{i_1 \dots i_{k+1}} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} -$$

$$- \sum_{\delta=2}^{k} \sum_{\gamma=2}^{k} \sum_{i_\gamma=2}^{r_\gamma-1} \sum_{i_\delta=1}^{r_\delta-1} Q^{(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta} \; .$$

$$. \; D_\beta^{(r_2+\dots+r_{\beta-1}+r_{\beta+1}+\dots+r_{\gamma-1}-(\gamma-2-\theta_{\gamma\beta})+i_\gamma),\;(r_2+\dots+r_{\beta-1}+r_{\beta+1}+\dots+r_{\delta-1}-(\delta-2-\theta_{\delta\beta})+i_\delta)} \qquad (2.18)$$

$$(\beta = 2, \dots, k)$$

onde

$$\theta_{\gamma\beta} = \begin{cases} 1 & \text{se} \quad \gamma > \beta \\ 0 & \text{se} \quad \gamma \leq \beta \end{cases}$$

e D_β é dado por:

$$
D_\beta = \left|\begin{array}{cccccccccccccccccccccc}
m^{[2,2]}_{11} & m^{[2,2]}_{12} & \cdots & m^{[2,2]}_{1\,n_2-1} & m^{[2,3]}_{11} & m^{[2,3]}_{12} & \cdots & m^{[2,3]}_{1\,n_3-1} & \cdots & m^{[2,\beta-1]}_{11} & m^{[2,\beta-1]}_{12} & \cdots & m^{[2,\beta-1]}_{1\,n_{\beta-1}-1} & m^{[2,\beta+1]}_{11} & m^{[2,\beta+1]}_{12} & \cdots & m^{[2,\beta+1]}_{1\,n_{\beta+1}-1} & \cdots & m^{[2,k]}_{11} & m^{[2,k]}_{12} & \cdots & m^{[2,k]}_{1\,n_k-1} \\
m^{[2,2]}_{21} & m^{[2,2]}_{22} & \cdots & m^{[2,2]}_{2\,n_2-1} & m^{[2,3]}_{21} & m^{[2,3]}_{22} & \cdots & m^{[2,3]}_{2\,n_3-1} & \cdots & m^{[2,\beta-1]}_{21} & m^{[2,\beta-1]}_{22} & \cdots & m^{[2,\beta-1]}_{2\,n_{\beta-1}-1} & m^{[2,\beta+1]}_{21} & m^{[2,\beta+1]}_{22} & \cdots & m^{[2,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[2,k]}_{21} & m^{[2,k]}_{22} & \cdots & m^{[2,k]}_{2\,n_k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
m^{[2,2]}_{n_2-1,1} & m^{[2,2]}_{n_2-1,2} & \cdots & m^{[2,2]}_{n_2-1,n_2-1} & m^{[2,3]}_{n_2-1,1} & m^{[2,3]}_{n_2-1,2} & \cdots & m^{[2,3]}_{n_2-1,n_3-1} & \cdots & m^{[2,\beta-1]}_{n_2-1,1} & m^{[2,\beta-1]}_{n_2-1,2} & \cdots & m^{[2,\beta-1]}_{n_2-1,n_{\beta-1}-1} & m^{[2,\beta+1]}_{n_2-1,1} & m^{[2,\beta+1]}_{n_2-1,2} & \cdots & m^{[2,\beta+1]}_{n_2-1,n_{\beta+1}-1} & \cdots & m^{[2,k]}_{n_2-1,1} & m^{[2,k]}_{n_2-1,2} & \cdots & m^{[2,k]}_{n_2-1,n_k-1} \\
m^{[2,3]}_{11} & m^{[2,3]}_{21} & \cdots & m^{[2,3]}_{n_2-1,1} & m^{[3,3]}_{11} & m^{[3,3]}_{12} & \cdots & m^{[3,3]}_{1\,n_3-1} & \cdots & m^{[3,\beta-1]}_{11} & m^{[3,\beta-1]}_{12} & \cdots & m^{[3,\beta-1]}_{1\,n_{\beta-1}-1} & m^{[3,\beta+1]}_{11} & m^{[3,\beta+1]}_{12} & \cdots & m^{[3,\beta+1]}_{1\,n_{\beta+1}-1} & \cdots & m^{[3,k]}_{11} & m^{[3,k]}_{12} & \cdots & m^{[3,k]}_{1\,n_k-1} \\
m^{[2,3]}_{12} & m^{[2,3]}_{22} & \cdots & m^{[2,3]}_{n_2-1,2} & m^{[3,3]}_{21} & m^{[3,3]}_{22} & \cdots & m^{[3,3]}_{2\,n_3-1} & \cdots & m^{[3,\beta-1]}_{21} & m^{[3,\beta-1]}_{22} & \cdots & m^{[3,\beta-1]}_{2\,n_{\beta-1}-1} & m^{[3,\beta+1]}_{21} & m^{[3,\beta+1]}_{22} & \cdots & m^{[3,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[3,k]}_{21} & m^{[3,k]}_{22} & \cdots & m^{[3,k]}_{2\,n_k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
m^{[2,3]}_{1\,n_3-1} & m^{[2,3]}_{2\,n_3-1} & \cdots & m^{[2,3]}_{n_2-1,n_3-1} & m^{[3,3]}_{n_3-1,1} & m^{[3,3]}_{n_3-1,2} & \cdots & m^{[3,3]}_{n_3-1,n_3-1} & \cdots & m^{[3,\beta-1]}_{n_3-1,1} & m^{[3,\beta-1]}_{n_3-1,2} & \cdots & m^{[3,\beta-1]}_{n_3-1,n_{\beta-1}-1} & m^{[3,\beta+1]}_{n_3-1,1} & m^{[3,\beta+1]}_{n_3-1,2} & \cdots & m^{[3,\beta+1]}_{n_3-1,n_{\beta+1}-1} & \cdots & m^{[3,k]}_{n_3-1,1} & m^{[3,k]}_{n_3-1,2} & \cdots & m^{[3,k]}_{n_3-1,n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m^{[2,\beta-1]}_{11} & m^{[2,\beta-1]}_{21} & \cdots & m^{[2,\beta-1]}_{n_2-1,1} & m^{[3,\beta-1]}_{11} & m^{[3,\beta-1]}_{21} & \cdots & m^{[3,\beta-1]}_{n_3-1,1} & \cdots & m^{[\beta-1,\beta-1]}_{11} & m^{[\beta-1,\beta-1]}_{12} & \cdots & m^{[\beta-1,\beta-1]}_{1\,n_{\beta-1}-1} & m^{[\beta-1,\beta+1]}_{11} & m^{[\beta-1,\beta+1]}_{12} & \cdots & m^{[\beta-1,\beta+1]}_{1\,n_{\beta+1}-1} & \cdots & m^{[\beta-1,k]}_{11} & m^{[\beta-1,k]}_{12} & \cdots & m^{[\beta-1,k]}_{1\,n_k-1} \\
m^{[2,\beta-1]}_{12} & m^{[2,\beta-1]}_{22} & \cdots & m^{[2,\beta-1]}_{n_2-1,2} & m^{[3,\beta-1]}_{12} & m^{[3,\beta-1]}_{22} & \cdots & m^{[3,\beta-1]}_{n_3-1,2} & \cdots & m^{[\beta-1,\beta-1]}_{21} & m^{[\beta-1,\beta-1]}_{22} & \cdots & m^{[\beta-1,\beta-1]}_{2\,n_{\beta-1}-1} & m^{[\beta-1,\beta+1]}_{21} & m^{[\beta-1,\beta+1]}_{22} & \cdots & m^{[\beta-1,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[\beta-1,k]}_{21} & m^{[\beta-1,k]}_{22} & \cdots & m^{[\beta-1,k]}_{2\,n_k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
m^{[2,\beta-1]}_{1\,n_{\beta-1}-1} & m^{[2,\beta-1]}_{2\,n_{\beta-1}-1} & \cdots & m^{[2,\beta-1]}_{n_2-1,n_{\beta-1}-1} & m^{[3,\beta-1]}_{1\,n_{\beta-1}-1} & m^{[3,\beta-1]}_{2\,n_{\beta-1}-1} & \cdots & m^{[3,\beta-1]}_{n_3-1,n_{\beta-1}-1} & \cdots & m^{[\beta-1,\beta-1]}_{n_{\beta-1}-1,1} & m^{[\beta-1,\beta-1]}_{n_{\beta-1}-1,2} & \cdots & m^{[\beta-1,\beta-1]}_{n_{\beta-1}-1,n_{\beta-1}-1} & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,1} & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,2} & \cdots & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,n_{\beta+1}-1} & \cdots & m^{[\beta-1,k]}_{n_{\beta-1}-1,1} & m^{[\beta-1,k]}_{n_{\beta-1}-1,2} & \cdots & m^{[\beta-1,k]}_{n_{\beta-1}-1,n_k-1} \\
m^{[2,\beta+1]}_{11} & m^{[2,\beta+1]}_{21} & \cdots & m^{[2,\beta+1]}_{n_2-1,1} & m^{[3,\beta+1]}_{11} & m^{[3,\beta+1]}_{21} & \cdots & m^{[3,\beta+1]}_{n_3-1,1} & \cdots & m^{[\beta-1,\beta+1]}_{11} & m^{[\beta-1,\beta+1]}_{21} & \cdots & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,1} & m^{[\beta+1,\beta+1]}_{11} & m^{[\beta+1,\beta+1]}_{12} & \cdots & m^{[\beta+1,\beta+1]}_{1\,n_{\beta+1}-1} & \cdots & m^{[\beta+1,k]}_{11} & m^{[\beta+1,k]}_{12} & \cdots & m^{[\beta+1,k]}_{1\,n_k-1} \\
m^{[2,\beta+1]}_{12} & m^{[2,\beta+1]}_{22} & \cdots & m^{[2,\beta+1]}_{n_2-1,2} & m^{[3,\beta+1]}_{12} & m^{[3,\beta+1]}_{22} & \cdots & m^{[3,\beta+1]}_{n_3-1,2} & \cdots & m^{[\beta-1,\beta+1]}_{12} & m^{[\beta-1,\beta+1]}_{22} & \cdots & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,2} & m^{[\beta+1,\beta+1]}_{21} & m^{[\beta+1,\beta+1]}_{22} & \cdots & m^{[\beta+1,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[\beta+1,k]}_{21} & m^{[\beta+1,k]}_{22} & \cdots & m^{[\beta+1,k]}_{2\,n_k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
m^{[2,\beta+1]}_{1\,n_{\beta+1}-1} & m^{[2,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[2,\beta+1]}_{n_2-1,n_{\beta+1}-1} & m^{[3,\beta+1]}_{1\,n_{\beta+1}-1} & m^{[3,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[3,\beta+1]}_{n_3-1,n_{\beta+1}-1} & \cdots & m^{[\beta-1,\beta+1]}_{1\,n_{\beta+1}-1} & m^{[\beta-1,\beta+1]}_{2\,n_{\beta+1}-1} & \cdots & m^{[\beta-1,\beta+1]}_{n_{\beta-1}-1,n_{\beta+1}-1} & m^{[\beta+1,\beta+1]}_{n_{\beta+1}-1,1} & m^{[\beta+1,\beta+1]}_{n_{\beta+1}-1,2} & \cdots & m^{[\beta+1,\beta+1]}_{n_{\beta+1}-1,n_{\beta+1}-1} & \cdots & m^{[\beta+1,k]}_{n_{\beta+1}-1,1} & m^{[\beta+1,k]}_{n_{\beta+1}-1,2} & \cdots & m^{[\beta+1,k]}_{n_{\beta+1}-1,n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m^{[2,k]}_{11} & m^{[2,k]}_{21} & \cdots & m^{[2,k]}_{n_2-1,1} & m^{[3,k]}_{11} & m^{[3,k]}_{21} & \cdots & m^{[3,k]}_{n_3-1,1} & \cdots & m^{[\beta-1,k]}_{11} & m^{[\beta-1,k]}_{21} & \cdots & m^{[\beta-1,k]}_{n_{\beta-1}-1,1} & m^{[\beta+1,k]}_{11} & m^{[\beta+1,k]}_{21} & \cdots & m^{[\beta+1,k]}_{n_{\beta+1}-1,1} & \cdots & m^{[k,k]}_{11} & m^{[k,k]}_{12} & \cdots & m^{[k,k]}_{1\,n_k-1} \\
m^{[2,k]}_{12} & m^{[2,k]}_{22} & \cdots & m^{[2,k]}_{n_2-1,2} & m^{[3,k]}_{12} & m^{[3,k]}_{22} & \cdots & m^{[3,k]}_{n_3-1,2} & \cdots & m^{[\beta-1,k]}_{12} & m^{[\beta-1,k]}_{22} & \cdots & m^{[\beta-1,k]}_{n_{\beta-1}-1,2} & m^{[\beta+1,k]}_{12} & m^{[\beta+1,k]}_{22} & \cdots & m^{[\beta+1,k]}_{n_{\beta+1}-1,2} & \cdots & m^{[k,k]}_{21} & m^{[k,k]}_{22} & \cdots & m^{[k,k]}_{2\,n_k-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
m^{[2,k]}_{1\,n_k-1} & m^{[2,k]}_{2\,n_k-1} & \cdots & m^{[2,k]}_{n_2-1,n_k-1} & m^{[3,k]}_{1\,n_k-1} & m^{[3,k]}_{2\,n_k-1} & \cdots & m^{[3,k]}_{n_3-1,n_k-1} & \cdots & m^{[\beta-1,k]}_{1\,n_k-1} & m^{[\beta-1,k]}_{2\,n_k-1} & \cdots & m^{[\beta-1,k]}_{n_{\beta-1}-1,n_k-1} & m^{[\beta+1,k]}_{1\,n_k-1} & m^{[\beta+1,k]}_{2\,n_k-1} & \cdots & m^{[\beta+1,k]}_{n_{\beta+1}-1,n_k-1} & \cdots & m^{[k,k]}_{n_k-1,1} & m^{[k,k]}_{n_k-1,2} & \cdots & m^{[k,k]}_{n_k-1,n_k-1}
\end{array}\right|
$$

$(\beta = 2, \ldots, k)$

Um raciocínio totalmente análogo nos levaria a escrever para valor de $n\ \hat{\sigma}^2_{\omega_1}$, por exemplo *, a expressão:

$$n\ \hat{\sigma}_{\omega_1} = \sum_{i_1, \ldots, i_{k+1}} y^2_{i_1 \ldots i_{k+1}} - \sum_{i_2=1}^{r_2} \frac{\left[Y^{(2)}_{i_2} \right]^2}{n^{(2)}_{i_2}} -$$

(2.20)

$$- \sum_{\delta=3}^{k} \sum_{\gamma=3}^{k} \sum_{i_\delta=1}^{r_\delta-1} \sum_{i_\gamma=1}^{r_\gamma-1} Q'^{(\gamma)}_{i_\gamma}\ Q'^{(\delta)}_{i_\delta}$$

$$D_1{}^{(r_3+\cdots+r_{\gamma-1}-(\gamma-3)+i_\gamma),\ (r_3+\cdots+r_{\delta-1}-(\delta-3)+i_\delta)}$$

onde:

$$Q'^{(\gamma)}_{i_\gamma} = Y^{(\gamma)}_{i_\gamma} - \sum_{i_2=1}^{r_2} \frac{n^{(2,\gamma)}_{i_2\, i_\gamma}}{n^{(2)}_{i_2}}\ Y^{(2)}_{i_2} \quad \begin{cases} \gamma = 3, \ldots, k \\ i_\gamma = 1, \ldots, r_\gamma \end{cases}$$

(2.21)

e D_1 é dado por:

* Dissemos, por exemplo, porque o valor de $n\ \hat{\sigma}^2_{\omega_1}$ pode ser expresso usando como pivô não o 2.º critério classificador, mas qualquer outro dentre os $(k-1)$ critérios. Se preferimos o 2.º critério é apenas por facilidade notatória.

$$
D_1 = \begin{vmatrix}
m'^{[3,3]}_{11} & m'^{[3,3]}_{12} & \cdots & m'^{[3,3]}_{1,n_3-1} & m'^{[3,4]}_{11} & m'^{[3,4]}_{12} & \cdots & m'^{[3,4]}_{1,n_4-1} & \cdots & m'^{[3,k]}_{11} & m'^{[3,k]}_{12} & \cdots & m'^{[3,k]}_{1,n_k-1} \\
m'^{[3,3]}_{21} & m'^{[3,3]}_{22} & \cdots & m'^{[3,3]}_{2,n_3-1} & m'^{[3,4]}_{21} & m'^{[3,4]}_{22} & \cdots & m'^{[3,4]}_{2,n_4-1} & \cdots & m'^{[3,k]}_{21} & m'^{[3,k]}_{22} & \cdots & m'^{[3,k]}_{2,n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m'^{[3,3]}_{n_3-1,1} & m'^{[3,3]}_{n_3-1,2} & \cdots & m'^{[3,3]}_{n_3-1,n_3-1} & m'^{[3,4]}_{n_3-1,1} & m'^{[3,4]}_{n_3-1,2} & \cdots & m'^{[3,4]}_{n_3-1,n_4-1} & \cdots & m'^{[3,k]}_{n_3-1,1} & m'^{[3,k]}_{n_3-1,2} & & m'^{[3,k]}_{n_3-1,n_k-1} \\
m'^{[3,4]}_{11} & m'^{[3,4]}_{21} & \cdots & m'^{[3,4]}_{n_3-1,1} & m'^{[4,4]}_{11} & m'^{[4,4]}_{12} & \cdots & m'^{[4,4]}_{1n_4-1} & \cdots & m'^{[4,k]}_{11} & m'^{[4,k]}_{12} & \cdots & m'^{[4,k]}_{1n_k-1} \\
m'^{[3,4]}_{12} & m'^{[3,4]}_{22} & \cdots & m'^{[3,4]}_{n_3-1,2} & m'^{[4,4]}_{21} & m'^{[4,4]}_{22} & \cdots & m'^{[4,4]}_{2n_4-1} & \cdots & m'^{[4,k]}_{21} & m'^{[4,k]}_{22} & \cdots & m'^{[4,k]}_{2n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m'^{[3,4]}_{1,n_4-1} & m'^{[3,4]}_{2,n_4-1} & \cdots & m'^{[3,4]}_{n_3-1,n_4-1} & m'^{[4,4]}_{n_4-1,1} & m'^{[4,4]}_{n_4-1,2} & \cdots & m'^{[4,4]}_{n_4-1,n_4-1} & \cdots & m'^{[4,k]}_{n_4-1,1} & m'^{[4,k]}_{n_4-1,2} & \cdots & m'^{[4,k]}_{n_4-1,n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m'^{[3,k]}_{11} & m'^{[3,k]}_{21} & \cdots & m'^{[3,k]}_{n_3-1,1} & m'^{[4,k]}_{11} & m'^{[4,k]}_{21} & \cdots & m'^{[4,k]}_{n_4-1,1} & \cdots & m'^{[k,k]}_{11} & m'^{[k,k]}_{12} & \cdots & m'^{[k,k]}_{1n_k-1} \\
m'^{[3,k]}_{12} & m'^{[3,k]}_{22} & \cdots & m'^{[3,k]}_{n_3-1,2} & m'^{[4,k]}_{12} & m'^{[4,k]}_{22} & \cdots & m'^{[4,k]}_{n_4-1,2} & \cdots & m'^{[k,k]}_{21} & m'^{[k,k]}_{22} & \cdots & m'^{[k,k]}_{2n_k-1} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
m'^{[3,k]}_{1,n_k-1} & m'^{[3,k]}_{2,n_k-1} & \cdots & m'^{[3,k]}_{n_3-1,n_k-1} & m'^{[4,k]}_{1n_k-1} & m'^{[4,k]}_{2,n_k-1} & \cdots & m'^{[4,k]}_{n_4-1,n_k-1} & \cdots & m'^{[k,k]}_{n_k-1,1} & m'^{[k,k]}_{n_k-1,2} & \cdots & m'^{[k,k]}_{n_k-1,n_k-1}
\end{vmatrix}
$$

e

$$
m'^{[\beta,\gamma]}_{i_\beta\, i_\gamma} = \begin{cases} \text{se} \quad \gamma \neq \beta : n^{(\beta,\gamma)}_{i_\beta\, i_\gamma} - \sum\limits_{i_2=1}^{r_2} \dfrac{n^{(2,\beta)}_{i_2\, i_\beta}\; n^{(2,\gamma)}_{i_2\, i_\gamma}}{n^{(2)}_{i_2}} \\[2ex] \text{se} \quad \gamma = \beta \ \text{e} \begin{cases} i_\gamma = i_\beta : \ n^{(\beta)}_{i_\beta} - \sum\limits_{i_2=1}^{r_2} \dfrac{n^{(2,\beta)}_{i_2\, i_\beta}\; n^{(2,\gamma)}_{i_2\, i_\gamma}}{n^{(2)}_{i_2}} \\[2ex] i_\gamma \neq i_\beta : \ - \sum\limits_{i_2=1}^{r_2} \dfrac{n^{(2,\beta)}_{i_2\, i_\beta}\; n^{(2,\gamma)}_{i_2\, i_\gamma}}{n^{(2)}_{i_2}} \end{cases} \\[2ex] \qquad\qquad \gamma = \beta,\ \beta + 1,\ \ldots,\ k \\ \qquad\qquad \beta = 3,\ 4,\ \ldots\ k \end{cases}
$$

Os resultados (2.18) e (2.20) juntamente com a (2.17), nos permitem escrever:

$$
n\, \hat{\sigma}^2_{\omega_1} - n\, \hat{\sigma}^2_{\Omega} = \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} - \sum_{i_2=1}^{r_2} \frac{\left[Y^{(2)}_{i_2} \right]^2}{n^{(2)}_{i_2}} +
$$

$$
+ \sum_{\delta=2}^{k} \sum_{\gamma=2}^{k} \sum_{i_\delta=1}^{r_\delta - 1} \sum_{i_\gamma=1}^{r_\gamma - 1} Q'^{(\gamma)}_{i_\gamma}\; Q'^{(\delta)}_{i_\delta}
$$

$$
D^{(r_2 + \ldots + r_{\gamma-1} - (\gamma-2) + i_\gamma),\ (r_2 + \ldots + r_{\delta-1} - (\delta-2) + i_\delta)}
$$

$$
- \sum_{\delta=3}^{k} \sum_{\gamma=3}^{k} \sum_{i_\delta=1}^{r_\delta - 1} \sum_{i_\gamma=1}^{r_\gamma - 1} Q'^{(\gamma)}_{i_\gamma}\; Q'^{(\delta)}_{i_\delta}
$$

$$
D_1^{(r_3 + \ldots + r_{\gamma-1} - (\gamma-3) + i_\gamma),\ (r_3 + \ldots + r_{\delta-1} - (\delta-3) + i_\delta)}
$$

(2.23)

$$
n\, \hat{\sigma}^2_{\omega_\beta} - n\, \hat{\sigma}^2_{\Omega} = \sum_{\delta=2}^{k} \sum_{\gamma=2}^{k} \sum_{i_\delta=1}^{r_\delta - 1} \sum_{i_\gamma=1}^{r_\gamma - 1} Q^{(\gamma)}_{i_\gamma}\; Q^{(\delta)}_{i_\delta}
$$

$$
D^{(r_2 + \ldots + r_{\gamma-1} - (\gamma-2) + i_\gamma),\ (r_2 + \ldots + r_{\delta-1} - (\delta-2) + i_\delta)}
$$

(2.24)

$$- \sum_{\delta=2}^{k} \sum_{\gamma=2}^{k} \sum_{i_\delta=1}^{r_\delta - 1} \sum_{i_\gamma=1}^{r_\gamma - 1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)}$$

$$D_\beta (r_2 + \ldots + r_{\beta-1} + r_{\beta+1} + \ldots + r_{\gamma-1} - (\gamma - 2 - \theta_{\gamma\beta}) + i_\gamma),$$
$$(r_2 + \ldots + r_{\beta-1} + r_{\beta+1} + \ldots + r_{\delta-1} - (\delta - 2 - \theta_{\delta\beta}) + i_\delta)$$

$$(\beta = 2, \ \ldots, \ k)$$

e portanto, o teste da hipótese $H_1 : E_{i_1}^{(1)} = 0$, $(i_1 = 1, \dots, r_1)$ será realizado através de:

$$F = \frac{\dfrac{1}{r_1 - 1}\left\{ \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)}\right]^2}{n_{i_1}^{(1)}} - \sum_{i_2=1}^{r_2} \frac{\left[Y_{i_2}^{(2)}\right]^2}{n_{i_2}^{(2)}} + \sum_{\delta=2}^{k}\sum_{\gamma=2}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\dots+r_{\delta-1}-(\delta-2)+i_\delta)} - \sum_{\delta=3}^{k}\sum_{\gamma=3}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q'^{(\gamma)}_{i_\gamma} Q'^{(\delta)}_{i_\delta} D_1^{(r_3+\dots+r_{\gamma-1}-(\gamma-3)+i_\gamma),(r_3+\dots+r_{\delta-1}-(\delta-3)+i_\delta)} \right\}}{\dfrac{1}{n - r_1 - r_2 - \dots - r_k + k - 1}\left\{ \sum_{i_1,\dots,i_{k+1}} y^2_{i_1\dots i_{k+1}} - \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)}\right]^2}{n_{i_1}^{(1)}} - \sum_{\delta=2}^{k}\sum_{\gamma=2}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\dots+r_{\delta-1}-(\delta-2)+i_\delta)} \right\}}$$

(2.25)

a qual tem distribuição F com $(r_1 - 1)$ e $(n - r_1 - r_2 - \dots - r_k + k - 1)$ graus de liberdade.

O teste da hipótese $H_\beta \quad E_{i_\beta}^{(\beta)} = 0$, $(i_\beta = 1, \dots, r_\beta$ para $\beta = 2, \dots, k$ será feito através de:

$$F = \frac{\dfrac{1}{r_\beta - 1}\left[\sum_{\delta=2}^{k}\sum_{\gamma=2}^{k}\sum_{i_\gamma=1}^{r_\gamma-1}\sum_{i_\delta=1}^{r_\delta-1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\dots+r_{\delta-1}-(\delta-2)+i_\delta)} - \sum_{\delta=2}^{k}\sum_{\gamma=2}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D_\beta^{(r_2+\dots+r_{\beta-1}+r_{\beta+1}+\dots+r_{\gamma-1}-(\gamma-2-\theta_{\gamma\beta})+i_\gamma),(r_2+\dots+r_{\beta-1}+r_{\beta+1}+\dots+r_{\delta-1}-(\delta-2-\theta_{\delta\beta})+i_\delta)} \right]}{\dfrac{1}{n - r_1 - r_2 - \dots - r_k + k - 1}\left[\sum_{i_1\dots i_{k+1}} y^2_{i_1\dots i_{k+1}} - \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)}\right]^2}{n_{i_1}^{(1)}} - \sum_{\delta=2}^{k}\sum_{\gamma=2}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D^{(r_2+\dots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\dots+r_{\delta-1}-(\delta-2)+i_\delta)} \right]}$$

(2.26)

a qual tem distribuição F com $(r - 1)$ e $(n - r_1 - r_2 - \dots - r_k + k - 1)$ graus de liberdade

Estes resultados, para maior comodidade podem ser postos sob as formas tabelares que seguem:

Teste de $E^{(\beta)}_{i_\beta} = 0$ $\quad (i_\beta = 1, \ldots, r_\beta ; \ \beta = 2, \ldots, k)$

Variação devida a	G. L.	Variação	F
Adatação de $m, E^{(2)}_{i_2}, \ldots, E^{(\beta-1)}_{i_{\beta-1}}$, $E^{(\beta+1)}_{i_{\beta+1}}, \ldots, E^{(k)}_{i_k}$	$\sum_{\gamma=\beta=1}^{k} r_\gamma - (k-2)$	$\sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} + \sum_{\delta=2}^{k} \sum_{\gamma=2}^{k} \sum_{i_\delta=1}^{r_\delta-1} \sum_{i_\gamma=1}^{r_\gamma-1} Q^{(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta} D_{(r_2+\ldots+r_{\beta-1}+r_{\beta+1}+\ldots+r_{\gamma-1}-(\gamma-2-\theta_{\gamma\beta})+i_\gamma),(r_2+\ldots+r_{\beta-1}+r_{\beta+1}+\ldots+r_{\delta-1}-(\delta-2-\theta_{\delta\beta})+i_\delta)}$	
$E^{(\beta)}_{i_\beta}$	$r_\beta - 1$	diferença $= R_\beta$	$\dfrac{\dfrac{R_\beta}{r_\beta - 1}}{\dfrac{R}{n - \sum_{\gamma=1}^{k} r_\gamma + (k-1)}}$
Adatação de $m, E^{(1)}_{i_1}, \ldots, E^{(k)}_{i_k}$	$\sum_{\gamma=1}^{k} r_\gamma - (k-1)$	$\sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} + \sum_{\gamma=2}^{k} \sum_{\delta=2}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \sum_{i_\delta=1}^{r_\delta-1} Q^{(\delta)}_{i_\gamma} Q^{(\delta)}_{i_\delta} D_{(r_2+\ldots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\ldots r_{\delta-1}-(\delta-2)+i_\delta)}$	
Resíduo	$n - \sum_{\gamma=1}^{k} r_\gamma + (k-1)$	diferença $= R$	
Total	n	$\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \ldots \sum_{i_k=1}^{r_k} \sum_{i_{k+1}=1}^{n_{i_1 \ldots i_k}} y^2_{i_1 \ldots i_{k+1}}$	

Teste de $E^{(1)}_{i_1} = 0$ $(i_1 = 1,2,\ldots, r_1)$

Variação devida a	G. L.	Variação	F
Adatação de $m, E^{(2)}_{i_2},\ldots, E^{(k)}_{i_k}$	$\sum_{\gamma=2}^{k} r_\gamma - (k-2)$	$\sum_{i_2=1}^{r_2} \frac{\left[Y^{(2)}_{i_2}\right]}{n^{(2)}_{i_2}} + \sum_{\delta=3}^{k}\sum_{\gamma=3}^{k}\sum_{i_\delta=1}^{r_\delta-1}\sum_{i_\gamma=1}^{r_\gamma-1} Q'^{(\gamma)}_{i_\gamma} Q'^{(\delta)}_{i_\delta} D_1^{(r_3+\ldots+r_\gamma-1-(\gamma-3)+i_\gamma),(r_3+\ldots+r_{\delta-1}-(\delta-3)+i_\delta)}$	
$E^{(1)}_{i_1}$	$r_1 - 1$	diferença $= R_1$	$\dfrac{\frac{R_1}{r_1-1}}{\frac{R}{n-\sum_{\gamma=1}^{k} r_\gamma + (k-1)}}$
Adatação de $m, E^{(1)}_{i_1}, E_{i_2},\ldots,E^{(k)}_{i_k}$	$\sum_{\gamma=1}^{k} r_\gamma - (k-1)$	$\sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} + \sum_{\gamma=2}^{k}\sum_{\delta=2}^{k}\sum_{i_\gamma=1}^{r_\gamma-1}\sum_{i_\delta=1}^{r_\delta-1} Q^{m(\gamma)}_{i_\gamma} Q^{(\delta)}_{i_\delta} D^{(r_2+\ldots+r_{\gamma-1}-(\gamma-2)+i_\gamma),(r_2+\ldots+r_{\delta-1}-(\delta-2)+i_\delta)}$	
Resíduo	$n-\sum_{\gamma=1}^{k} r_\gamma + k-1$	diferença $= R$	
Total	n	$\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{i_3=1}^{r_3}\cdots\sum_{i_{k+1}=1}^{n_{i_1\ldots i_k}} Y^2_{i_1\ldots i_{k+1}}$	

3 — *1.º caso particular*: números proporcionais de observações por cela, isto é,

$$n_{i_1 i_2 \ldots i_k} = \frac{n^{(1)}_{i_1} \; n^{(2)}_{i_2} \; \ldots \; n^{(k)}_{i_k}}{n^{k-1}} \tag{3.1}$$

Vejamos, em primeiro lugar a expressão que $n\,\hat{\sigma}^2_{\Omega}$ assume neste caso.

Para tanto, comecemos pela determinação de $m^{[\beta,\gamma]}_{i_\beta \, i_\gamma}$ dado pela (2.14) sob a condição (3.1). Subsiste:

$$\sum_{i_1=1}^{r_1} \frac{n^{(1,\beta)}_{i_1 \, i_\beta} \; n^{(1,\gamma)}_{i_1 \, i_\gamma}}{n^{(1)}_{i_1}} = \sum_{i_1=1}^{r_1} \frac{\dfrac{n^{(1)}_{i_1} \; n^{(\beta)}_{i_\beta}}{n} \; \dfrac{n^{(1)}_{i_1} \; n^{(\gamma)}_{i_\gamma}}{n}}{n^{(1)}_{i_1}}$$

$$= \frac{n^{(\beta)}_{i_\beta} \; n^{(\gamma)}_{i_\gamma}}{n}$$

e $$n^{(\beta,\gamma)}_{i_\beta \, i_\gamma} = \frac{n^{(\beta)}_{i_\beta} \; n^{(\gamma)}_{i_\gamma}}{n}, \text{ então:}$$

$$m^{[\beta,\gamma]}_{i_\beta \, i_\gamma} = \begin{cases} \text{se } \gamma \neq \beta : \text{ zero} \\ \text{se } \beta = \gamma \text{ e } \begin{cases} \text{se } i_\gamma = i_\beta : \; n^{(\gamma)}_{i_\gamma} - \dfrac{n^{(\beta)}_{i_\beta} \; n^{(\gamma)}_{i_\gamma}}{n} = \\ \qquad = \dfrac{n^{(\gamma)}_{i_\gamma}}{n} \left(n - n^{(\beta)}_{i_\beta}\right) \\ \text{se } i_\gamma \neq i_\beta : \; - \dfrac{n^{(\gamma)}_{i_\gamma} \; n^{(\beta)}_{i_\beta}}{n} \end{cases} \\ \qquad \gamma = \beta, \; \beta + 1, \; \ldots, \; k \\ \qquad i_\gamma = 1, 2, \ldots, r_\gamma \end{cases} \tag{3.2}$$

Com isto, D, dado pela (2.17), torna-se:

$$
D=\left|\begin{array}{ccccccccccc}
\frac{n_1^{(2)}}{n}\left(n-n_1^{(2)}\right) & -\frac{n_1^{(2)}n_2^{(2)}}{n} & \cdots & -\frac{n_1^{(2)}n_{n_2-1}^{(2)}}{n} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \;\cdots\; 0 \\
-\frac{n_1^{(2)}n_2^{(2)}}{n} & \frac{n_2^{(2)}}{n}\left(n-n_2^{(2)}\right) & \cdots & -\frac{n_2^{(2)}n_{n_2-1}^{(2)}}{n} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \;\cdots\; 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \;\ddots\; \vdots \\
-\frac{n_1^{(2)}n_{n_2-1}^{(2)}}{n} & -\frac{n_2^{(2)}n_{n_2-1}^{(2)}}{n} & \cdots & \frac{n_{n_2-1}^{(2)}}{n}\left(n-n_{n_2-1}^{(2)}\right) & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \;\cdots\; 0 \\
0 & 0 & \cdots & 0 & \frac{n_1^{(3)}}{n}\left(n-n_1^{(3)}\right) & -\frac{n_1^{(3)}n_2^{(3)}}{n} & \cdots & \frac{n_1^{(3)}n_{n_3-1}^{(3)}}{n} & \cdots & 0 & 0 \;\cdots\; 0 \\
0 & 0 & \cdots & 0 & -\frac{n_1^{(3)}n_2^{(3)}}{n} & \frac{n_2^{(3)}}{n}\left(n-n_2^{(3)}\right) & \cdots & -\frac{n_2^{(3)}n_{n_3-1}^{(3)}}{n} & \cdots & 0 & 0 \;\cdots\; 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \;\ddots\; \vdots \\
0 & 0 & \cdots & 0 & -\frac{n_1^{(3)}n_{n_3-1}^{(3)}}{n} & -\frac{n_2^{(3)}n_{n_3-1}^{(3)}}{n} & \cdots & \frac{n_{n_3-1}^{(3)}}{n}\left(n-n_{n_3-1}^{(3)}\right) & \cdots & 0 & 0 \;\cdots\; 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \frac{n_1^{(k)}}{n}\left(n-n_1^{(k)}\right) & -\frac{n_1^{(k)}n_2^{(k)}}{n} \;\cdots\; -\frac{n_1^{(k)}n_{n_k-1}^{(k)}}{n} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -\frac{n_1^{(k)}n_2^{(k)}}{n} & \frac{n_2^{(k)}}{n}\left(n-n_2^{(k)}\right) \;\cdots\; -\frac{n_2^{(k)}n_{n_k-1}^{(k)}}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \;\ddots\; \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -\frac{n_1^{(k)}n_{n_k-1}^{(k)}}{n} & -\frac{n_2^{(k)}n_{n_k-1}^{(k)}}{n} \;\cdots\; \frac{n_{n_k-1}^{(k)}}{n}\left(n-n_{n_k-1}^{(k)}\right)
\end{array}\right| \tag{3.3}
$$

Refere-se a um resultado como o obtido, dizendo que estamos diante de um caso em que há ortogonalidade.

O resultado mostra ainda que a obtenção da matriz inversa da matriz tendo D por determinante depende exclusivamente de se encontrar a matriz inversa da matriz:

$$[d_\gamma] = \begin{bmatrix} \frac{n_1^{(\gamma)}}{n}\left(n - n_1^{(\gamma)}\right) & -\frac{n_1^{(\gamma)}\, n_2^{(\gamma)}}{n} & \dots & -\frac{n_1^{(\gamma)}\, n_{r_\gamma - 1}^{(\gamma)}}{n} \\ -\frac{n_1^{(\gamma)}\, n_2^{(\gamma)}}{n} & \frac{n_2^{(\gamma)}}{n}\left(n - n_2^{(\gamma)}\right) & \dots & -\frac{n_2^{(\gamma)}\, n_{r_\gamma - 1}^{(\gamma)}}{n} \\ \vdots & \vdots & & \vdots \\ -\frac{n_1^{(\gamma)}\, n_{r_\gamma - 1}^{(\gamma)}}{n} & -\frac{n_2^{(\gamma)}\, n_{r_\gamma - 1}^{(\gamma)}}{n} & \dots & \frac{n_{r_\gamma - 1}^{(\gamma)}}{n}\left(n - n_{r_\gamma - 1}^{(\gamma)}\right) \end{bmatrix} \tag{3.4}$$

$$(\gamma = 2, 3, \dots, k)$$

de que nos ocuparemos daqui a um instante.

Antes, porém, notemos que a expressão:

$$\sum_{\gamma=2}^{k} \sum_{\delta=2}^{k} \sum_{i_\gamma=1}^{r_\gamma - 1} \sum_{i_\delta=1}^{r_\delta - 1} Q_{i_\gamma}^{(\gamma)} Q_{i_\delta}^{(\delta)} D^{(r_2 + \dots + r_{\gamma-1} - (\gamma-2) + i_\gamma)\ (r_2 + \dots + r_{\delta-1} - (\delta-2) + i_\delta)}$$

entrante em $n\ \hat{\sigma}_\Omega^2$ dado pela (2.17), torna-se, no caso em aprêço:

$$\sum_{\gamma=2}^{k} \sum_{i_\gamma, j_\gamma=1}^{r_\gamma - 1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} d_\gamma^{i_\gamma\, j_\gamma}$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma - 1} \left[Q_{i_\gamma}^{(\gamma)} \right]^2 d_\gamma^{i_\gamma\, i_\gamma} + \sum_{\gamma=2}^{k} \sum_{i_\gamma \neq j_\gamma = 1}^{r_\gamma - 1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} d_\gamma^{i_\gamma\, j_\gamma} \tag{3.5}$$

Posto isto, voltemos à determinação da matriz inversa de $[d_\gamma]$, começando por calcular o valor do determinante d_γ.

Colocando em evidência:

na 1.ª linha, $\dfrac{n_1^{(\gamma)}}{n}$

na 2.ª linha $\dfrac{n_2^{(\gamma)}}{n}$

$\vdots$

na $(r_\gamma - 1)^{ma}$ linha, $\dfrac{n_{r_\gamma - 1}^{(\gamma)}}{n}$

vem:

$$d_\gamma = \frac{n_1^{(\gamma)}\, n_2^{(\gamma)} \ldots n_{r_\gamma-1}^{(\gamma)}}{n^{r_\gamma - 1}} \begin{vmatrix} (n - n_1^{(\gamma)}) & -n_2^{(\gamma)} & \ldots\ldots & -n_{r_\gamma-2}^{(\gamma)} & -n_{r_\gamma-1}^{(\gamma)} \\ -n_1^{(\gamma)} & (n - n_2^{(\gamma)}) & \ldots\ldots & -n_{r_\gamma-2}^{(\gamma)} & -n_{r_\gamma-1}^{(\gamma)} \\ \vdots & \vdots & & \vdots & \vdots \\ -n_1^{(\gamma)} & -n_2^{(\gamma)} & \ldots & (n - n_{r_\gamma-2}^{(\gamma)}) & -n_{r_\gamma-1}^{(\gamma)} \\ -n_1^{(\gamma)} & -n_2^{(\gamma)} & \ldots\ldots & -n_{r_\gamma-2}^{(\gamma)} & (n - n_{r_\gamma-2}^{(\gamma)}) \end{vmatrix} \tag{3.6}$$

De cada uma das $(r_\gamma - 2)$ primeiras linhas subtraindo-se a $(r_\gamma - 1)^{ma}$, vem:

$$d_\gamma = \frac{n_1^{(\gamma)}\, n_2^{(\gamma)} \ldots n_{r_\gamma-1}^{(\gamma)}}{n^{r_\gamma - 1}} \begin{vmatrix} n & 0 & \ldots\ldots & 0 & -n \\ 0 & n & \ldots\ldots & 0 & -n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots\ldots & n & -n \\ -n_1^{(\gamma)} & -n_2^{(\gamma)} & \ldots\ldots & -n_{r_\gamma-2}^{(\gamma)} & (n - n_{r_\gamma-1}^{(\gamma)}) \end{vmatrix} \tag{3.7}$$

Acrescentando à $(r_\gamma - 1)^{ma}$ coluna, a soma das $(r_\gamma - 2)$ primeiras colunas, vem:

$$d_\gamma = \frac{n_1^{(\gamma)} \, n_2^{(\gamma)} \ldots n_{r_\gamma - 1}^{(\gamma)}}{n^{r_\gamma - 1}} \begin{vmatrix} n & 0 & \ldots & 0 & 0 \\ 0 & n & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & n & 0 \\ -n_1^{(\gamma)} & -n_2^{(\gamma)} & \ldots & -n_{r_\gamma - 2}^{(\gamma)} & n_{r_\gamma}^{(\gamma)} \end{vmatrix} \tag{3.8}$$

donde:

$$d = \frac{n_1^{(\gamma)} \, n_2^{(\gamma)} \ldots n_{r_\gamma - 1}^{(\gamma)}}{n^{r_\gamma - 1}} \, n_\gamma^{(\gamma)} \, n^{r_\gamma - 2}$$

ou, finalmente:

$$d_\gamma = \frac{n_1^{(\gamma)} \, n_2^{(\gamma)} \ldots n_{r_\gamma}^{(\gamma)}}{n} \qquad (\gamma = 2, \ldots, k) \tag{3.9}$$

Procedendo-se de forma absolutamente análoga, obtém-se:

$$d^*_{i_\gamma \, i_\gamma} = \frac{n_1^{(\gamma)} \ldots n_{i_\gamma - 1}^{(\gamma)} \, n_{i_\gamma + 1}^{(\gamma)} \ldots n_{r_\gamma - 1}^{(\gamma)}}{n} \left(n_{i_\gamma}^{(\gamma)} + n_{r_\gamma}^{(\gamma)}\right) \tag{3.10}$$

$$d^*_{i_\gamma \, j_\gamma} = \frac{n_1^{(\gamma)} \, n_2^{(\gamma)} \ldots n_{i_\gamma}^{(\gamma)} \ldots n_{j_\gamma}^{(\gamma)} \ldots n_{r_\gamma - 1}^{(\gamma)}}{n}$$

$$(i_\gamma \neq j_\gamma = 1, 2, \ldots, r_\gamma - 1) \tag{3.11}$$

Daqui segue que os elementos da matriz inversa de $[\, d_\gamma \,]$, isto é, de $[\, \mathrm{d}_\gamma \,]^{-1}$ são dados por:

$$d^{i_\gamma \, i_\gamma} = \frac{1}{n_{i_\gamma}^{(\gamma)}} + \frac{1}{n_{r_\gamma}^{(\gamma)}} \quad \left| \begin{array}{l} i = 1, \ldots, r_\gamma - 1 \\ \gamma = 2, \ldots, k \end{array} \right. \tag{3.12}$$

$$d^{i_\gamma \, j_\gamma} = \frac{1}{n_{r_\gamma}^{(\gamma)}} \qquad (\gamma = 2, \ldots, k) \tag{3.13}$$

Substituindo-se êstes resultados em (3.5) esta torna-se:

$$\sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \left[Q_{i_\gamma}^{(\gamma)} \right]^2 \left(\frac{1}{n_{i_\gamma}^{(\gamma)}} + \frac{1}{n_{r_\gamma}^{(\gamma)}} \right) + \sum_{\gamma=2}^{k} \sum_{i_\gamma \neq j_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} \frac{1}{n_{r_\gamma}^{(\gamma)}}$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \frac{\left[Q_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} + \sum_{\gamma=2}^{k} \frac{1}{n_{r_\gamma}^{(\gamma)}} \left\{ \sum_{i_\gamma=1}^{r_\gamma-1} \left[Q_{i_\gamma}^{(\gamma)} \right]^2 + \sum_{i_\gamma \neq j_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} \right\}$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \frac{\left[Q_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} + \sum_{\gamma=2}^{k} \frac{\left\{ \sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} \right\}^2}{n_{r_\gamma}^{(\gamma)}} \qquad (3.14)$$

Mas, desde que, pela (2.16),

$$\sum_{i_\gamma=1}^{r_\gamma} Q_{i_\gamma}^{(\gamma)} = 0,$$

então, segue que:

$$\sum_{i_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} = - Q_{r_\gamma}^{(\gamma)}$$

Com isto a (3.14), torna-se:

$$\sum_{\gamma=2}^{k} \sum_{i_\gamma, j_\gamma=1}^{r_\gamma-1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} d_\gamma^{i_\gamma j_\gamma} =$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma-1} \frac{\left[Q_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} + \sum_{\gamma=2}^{k} \frac{\left[Q_{r_\gamma}^{(\gamma)} \right]^2}{n_{r_\gamma}^{(\gamma)}} =$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Q_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} \qquad (3.15)$$

Notando que, de acôrdo com a (3.1), $Q_{i_\gamma}^{(\gamma)}$ é dada por:

$$Q_{i_\gamma}^{(\gamma)} = Y_{i_\gamma}^{(\gamma)} - \sum_{i_1=1}^{r_1} \frac{\dfrac{n_{i_\gamma}^{(1)} \; n_{i_1}^{(\gamma)}}{n}}{n_{i_1}^{(1)}} \; Y_{i_1}^{(1)}$$

$$= Y_{i_\gamma}^{(\gamma)} - \frac{n_{i_\gamma}^{(\gamma)}}{n} \; Y \qquad (3.16)$$

pode-se escrever:

$$\sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y_{i_\gamma}^{(\gamma)} - \frac{n_{i_\gamma}^{(\gamma)}}{n} Y \right]^2}{n_{i_\gamma}^{(\gamma)}} =$$

$$= \sum_{\gamma=2}^{k} \left\{ \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} - \frac{Y^2}{n} \right\} \qquad (3.17)$$

$$= \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} - (k-1) \frac{Y^2}{n}$$

Substituindo a (3.17) em $n\,\hat{\sigma}_\Omega^2$, tem-se:

$$n\,\hat{\sigma}_\Omega^2 = \Sigma\, y_{i_1 \ldots i_{k+1}}^2 - \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)} \right]^2}{n_{i_1}^{(1)}} -$$

$$- \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} + (k-1) \frac{Y^2}{n}$$

ou, finalmente:

$$n\,\hat{\sigma}_\Omega^2 = \Sigma\, y_{i_1 \ldots i_{k+1}}^2 - \sum_{\gamma=1}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y_{i_\gamma}^{(\gamma)} \right]^2}{n_{i_\gamma}^{(\gamma)}} + (k-1) \frac{Y^2}{n}$$

(3.18)

3a — Vejamos, agora, $n\,\hat{\sigma}_{\omega_\beta}^2$ dado pela (2.18) para $\beta = 2, 3, \ldots, k$.

O mesmo raciocínio que nos levou a escrever para a (2.17) a expressão (3.5) nos permite escrever para a (2.18):

$$n\,\hat{\sigma}_{\omega_\beta} = \sum_{i_1 \ldots i_{k+1}} y_{i_1 \ldots i_{k+1}}^2 - \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)} \right]^2}{n_{i_1}^{(1)}} -$$

$$- \sum_{\gamma \neq \beta = 2}^{k} \sum_{i_\gamma, j_\gamma=1}^{r_\gamma - 1} Q_{i_\gamma}^{(\gamma)} Q_{j_\gamma}^{(\gamma)} d^{i_\gamma j_\gamma}$$

$$(\beta = 2, 3, \ldots, k)$$

Pela (3.17), vem:

$$n\ \hat{\sigma}^2_{w_\beta} = \Sigma\ y^2_{i_1 \ldots i_{k+1}} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}}$$

$$- \sum_{\gamma \neq \beta = 2}^{k} \left\{ \sum_{i_\gamma=1}^{r} \frac{\left[Y^{(\gamma)}_{i_\gamma}\right]^2}{n^{(\gamma)}_{i_\gamma}} - \frac{Y^2}{n} \right\}$$

$$= \Sigma\ y^2_{i_1 \ldots i_{k+1}} - \sum_{\gamma \neq \beta = 1}^{k} \sum_{r_\gamma=1}^{r_\gamma} \frac{\left[Y^{(\gamma)}_{i_\gamma}\right]^2}{n^{(\gamma)}_{i_\gamma}} + (k-2)\frac{Y^2}{n} \qquad (3.19)$$

$$(\beta = 2, 3, \ldots, k)$$

Um desenvolvimento em tudo análogo nos conduz ao seguinte valor de $n\ \hat{\sigma}_{w_1}$:

$$n\ \hat{\sigma}^2_{w_1} = \Sigma\ Y^2_{i_1 \ldots i_{k+1}} - \sum_{\gamma=2}^{k} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y^{(\gamma)}_{i_\gamma}\right]^2}{n^{(\gamma)}_{i_\gamma}} + (k-2)\frac{Y^2}{n}$$

A comparação desta expressão com a (3.19) nos permite, imediatamente, escrever:

$$n\ \hat{\sigma}^2_{w_\beta} = \Sigma\ y^2_{i_1 \ldots i_{k+1}} - \sum_{\gamma \neq \beta} \sum_{i_\gamma=1}^{r_\gamma} \frac{\left[Y^{(\gamma)}_{i_\gamma}\right]^2}{n^{(\gamma)}_{i_\gamma}} + (k-2)\frac{Y^2}{n} \qquad (3.20)$$

para $\beta = 1, 2, \ldots, k$.

Daqui segue:

$$n\ \hat{\sigma}^2_{\omega_\beta} - n\ \hat{\sigma}^2_{\Omega} = \sum_{i_\beta=1}^{r_\beta} \frac{\left[Y^{(\beta)}_{i_\beta}\right]^2}{n^{(\beta)}_{i_\beta}} - \frac{Y^2}{n} \qquad (3.21)$$

$$(\beta = 1, 2, \ldots, k)$$

Portanto, o teste da hipótese: $H_{\beta} : E_{i_\beta}^{(\beta)} = 0$

$(i = 1, 2, \ldots, r_\beta)$

$(\beta = 1, 2, \ldots, k)$

será feito através de:

$$F = \frac{\dfrac{1}{r_\beta - 1}\left\{\sum_{i_\beta=1}^{r_\beta} \dfrac{\left[Y_{i_\beta}^{(\beta)}\right]^2}{n_{i_\beta}^{(\beta)}} - \dfrac{Y^2}{n}\right\}}{\dfrac{1}{n - r_1 - \ldots - r_k + k - 1}\left\{\Sigma\, y_{i_1 \ldots i_{k+1}}^2 - \sum_{\gamma=1}^{k} \sum_{i_\gamma=1}^{r_\gamma} \dfrac{\left[Y_{i_\gamma}^{(\gamma)}\right]^2}{n_{i_\gamma}^{(\gamma)}} + (k-1)\dfrac{Y^2}{n}\right\}}$$

a qual terá distribuição F com $(r_\beta - 1)$ e $(n - r_1 - \ldots - r_k + k - 1$ gráus de liberdade para $\beta = 1, 2, \ldots, k$.

Uma visão sintética dos diversos testes pode ser dada através do único quadro pelo qual se aprecia, como era aliás de se esperar, que no caso vertente há aditividade.

QUADRO (3a. 1)

Variação devida a	G.L.	Variação	F.
$E_{i_1}^{(1)}$	$r_1 - 1$	$\sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)}\right]^2}{n_{i_1}^{(1)}} - \frac{Y^2}{n}$	$\frac{1}{r_1 - 1}\left\{\sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)}\right]^2}{n_{i_1}^{(1)}} - \frac{Y^2}{n}\right\} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
$E_{i_2}^{(2)}$	$r_2 - 1$	$\sum_{i_2=1}^{r_2} \frac{\left[Y_{i_2}^{(2)}\right]^2}{n_{i_2}^{(2)}} - \frac{Y^2}{n}$	$\frac{1}{r_2 - 1}\left\{\sum_{i_2=1}^{r_2} \frac{\left[Y_{i_2}^{(2)}\right]^2}{n_{i_2}^{(2)}} - \frac{Y^2}{n}\right\} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
......	......	......	......
$E_{i_k}^{(k)}$	$r_k - 1$	$\sum_{i_k-1}^{r_k} \frac{\left[Y_{i_k}^{(k)}\right]^2}{n_{i_k}^{(k)}} - \frac{Y^2}{n}$	$\frac{1}{r_k - 1}\left\{\sum_{i_k=1}^{r_k} \frac{\left[Y_{i_k}^{(k)}\right]^2}{n_{i_k}^{(k)}} - \frac{Y^2}{n}\right\} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
Resíduo	$n - \sum_{\beta=1}^{k} r_\beta + (k-1)$	diferença $= R$	
Total	$n - 1$	$\Sigma\, y^2_{i_1 \ldots i_{k+1}} - \frac{Y^2}{n}$	

3b — *Números iguais de observações por cela, isto é,*

$$n_{i_1 i_2 \ldots i_k} = m \qquad \begin{cases} i_\gamma = 1, \ldots, r_\gamma \\ \gamma = 1, \ldots, k \end{cases}$$

Com isto,

$$n_{i_\gamma}^{(\gamma)} = r_1 \, r_2 \, \ldots \, r_{\gamma-1} \, r_{\gamma+1} \, \ldots \, r_k \, m$$

$$n = r_1 \, r_2 \, \ldots \, r_k \, m$$

resultados que substituídos nas (3.18) e (3.21) nos levam, após simplificações evidentes:

$$n \, \hat{\sigma}_\Omega^2 = \Sigma^2_{i_1 \ldots i_{k+1}} - \frac{1}{r_1 \, r_2 \, \ldots \, r_k \, m} \left\{ \sum_{\gamma=1}^{k} \sum_{i_\gamma=1}^{r_\gamma} \left[Y_{i_\gamma}^{(\gamma)} \right]^2 r_\gamma \right.$$

$$\left. - (k-1) \, Y^2 \right\}$$

$$n \, \hat{\sigma}^2_{w_\beta} - n \, \hat{\sigma}_\Omega^2 = \frac{1}{r_1 \, \ldots \, r_k \, m} \left\{ \sum_{i_\beta=1}^{r_\beta} r_\beta \left[Y_{i_\beta}^{(\beta)} \right]^2 - Y^2 \right\}$$

e, portanto, para testar a hipótese $E_{i_\beta}^{(\beta)} = 0$, vem após simplificação:

$$F = \frac{\dfrac{1}{r_\beta - 1} \left\{ \displaystyle\sum_{i_\beta=1}^{r} r_\beta \left[Y_{i_\beta}^{(\beta)} \right]^2 - Y^2 \right\}}{\dfrac{1}{r-r_1-\ldots-r_k+k-1} \left\{ m_1 \, r \ldots r_k \, \Sigma \, y^2_{i_1 \ldots i_{k+1}} - \displaystyle\sum_{\gamma=1}^{k} \sum_{i_\gamma=1}^{r_\gamma} r_\gamma \left[Y_{i_\gamma}^{(\gamma)} \right]^2 + (k-1) \, Y^2 \right\}}$$

O quadro (3a.1) assume a forma:

QUADRO (3a. I)

Variação devida a	G. L.	Variação	F.
$E_{i_1}^{(1)}$	$r_1 - 1$	$\frac{1}{r_1 \ldots r_k m} \left\{ \sum_{i_1=1}^{r_1} r_1 \left[Y_{i_1}^{(1)} \right]^2 - Y^2 \right\} = R_1$	$\frac{R_1}{r_1 - 1} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
$E_{i_2}^{(2)}$	$r_2 - 1$	$\frac{1}{r_1 \ldots r_k m} \left\{ \sum_{i_2=1}^{r_2} r_2 \left[Y_{i_2}^{(2)} \right]^2 - Y^2 \right\} = R_2$	$\frac{R_2}{r_2 - 1} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
.....	.....	.....	.....
$E_{i_k}^{(k)}$	$r_k - 1$	$\frac{1}{r_1 \ldots r_k m} \left\{ \sum_{i_k=1}^{r_k} r_k \left[Y_{i_k}^{(k)} \right]^2 - Y^2 \right\} = R_k$	$\frac{R_k}{r_k - 1} \Big/ \frac{R}{n - \sum_{\beta=1}^{k} r_\beta + (k-1)}$
Resíduo	$n - \sum_{\beta=1}^{k} r_\beta + (k-1)$	diferença $= R$	
Total	$n - 1$	$\frac{1}{r_1 \ldots r_k m} \left\{ r_1 \ldots r_k m \sum y_{i_1 \ldots i_{k+1}}^2 - Y^2 \right\}$	

4 — *O caso* $k = 3$

Da (2.18) segue:

$$n\,\hat{\sigma}^2_{\Omega} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} \sum_{i_4=1}^{n_{i_1 i_2 i_3}} y^2_{i_1 i_2 i_3 i_4} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} -$$

$$- \sum_{i_2=1}^{r_2-1} \sum_{j_2=1}^{r_2-1} Q^{(2)}_{i_2} \, Q^{(2)}_{j_2} \, D^{i_2 j_2} -$$

(4.1)

$$- 2 \sum_{i_2=1}^{r_2-1} \sum_{i_3=1}^{r_3-1} Q^{(2)}_{i} \, Q^{(3)}_{i_3} \, D^{i_2 (r_2 - 1 + i_3)}$$

$$- \sum_{i_3=1}^{r_3-1} \sum_{j_3=1}^{r_3-1} Q^{(3)}_{i_3} \, Q^{(3)}_{j_3} \, D^{(r_2 - 1 + i_3)(r_2 - 1 + j_3)}$$

onde D é dado por:

$$
D=\left|\begin{array}{cccccccc}
n^{(2)}_{1}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,2)}_{i_1 1}\right]^2}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{11}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{12}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{1 n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} \\
-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & n^{(2)}_{2}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,2)}_{i_1 2}\right]^2}{n^{(1)}_{i_1}} & \cdots & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 2}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{21}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & n^{(2,3)}_{22}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{2 n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,2)}_{i_1 2}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & \cdots & n^{(2)}_{n_2-1}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,2)}_{i_1 n_2-1}\right]^2}{n^{(1)}_{i_1}} & n^{(2,3)}_{n_2-1,1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{n_2-1,2}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{n_2-1,n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} \\
n^{(2,3)}_{11}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{21}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{n_2-1,1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & n^{(3)}_{1}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,3)}_{i_1 1}\right]^2}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,3)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,3)}_{i_1 n_3-1}}{n^{(1)}_{i_1}} \\
n^{(2,3)}_{12}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{22}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{n_2-1,2}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,3)}_{i_1 2}}{n^{(1)}_{i_1}} & n^{(3)}_{2}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,3)}_{i_1 2}\right]^2}{n^{(1)}_{i_1}} & \cdots & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,3)}_{i_1 n_3-1}}{n^{(1)}_{i_1}} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
n^{(2,3)}_{1 n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 1}}{n^{(1)}_{i_1}} & n^{(2,3)}_{2 n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & \cdots & n^{(2,3)}_{n_2-1,n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 n_3-1}\,n^{(1,2)}_{i_1 n_2-1}}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 1}\,n^{(1,3)}_{i_1 n_3-1}}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{n_1}\frac{n^{(1,3)}_{i_1 2}\,n^{(1,3)}_{i_1 n_3-1}}{n^{(1)}_{i_1}} & \cdots & n^{(3)}_{n_3-1}-\sum\limits_{i_1=1}^{n_1}\frac{\left[n^{(1,3)}_{i_1 n_3-1}\right]^2}{n^{(1)}_{i_1}}
\end{array}\right|
$$

Das (2.18) e (2.20) segue:

$$n\,\hat{\sigma}^2_{\omega_2} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} \sum_{i_4=1}^{n_{i_1 i_2 i_3}} y^2_{i_1 i_2 i_3 i_4} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} - \sum_{i_3=1}^{r_3-1} \sum_{j_3=1}^{r_3-1} Q^{(3)}_{i_3} \, Q^{(3)}_{j_3} \, D_2^{i_3 j_3} \tag{4.2}$$

$$n\,\hat{\sigma}^2_{\omega_3} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} \sum_{i_4=1}^{n_{i_1 i_2 i_3}} y^2_{i_1 i_2 i_3 i_4} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} - \sum_{i_2=1}^{r_2-1} \sum_{j_2=1}^{r_2-1} Q^{(2)}_{i_2} \, Q^{(2)}_{j_2} \, D_3^{i_2 j_2} \tag{4.3}$$

$$n\,\hat{\sigma}^2_{\omega_1} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} \sum_{i_4=1}^{n_{i_1 i_2 i_3}} y^2_{i_1 i_2 i_3 i_4} - \sum_{i_2-1}^{r_2} \frac{\left[Y^{(2)}_{i_2} \right]^2}{n^{(2)}_{i_2}} - \sum_{i_3=1}^{r_3-1} \sum_{i_3=1}^{r_3-1} Q'^{(3)}_{i_3} \, Q'^{(3)}_{j_3} \, D_1^{i_3 j_3} \tag{4.4}$$

onde D_2 é obtido de D eliminando as $(r_2 - 1)$ primeiras linhas e colunas; D_3 é obtido de D eliminando as $(r_3 - 1)$ últimas linhas e colunas e D_1 é dado por:

$$D_1 = \begin{vmatrix} n_1^{(3)} - \sum_{i_2=1}^{r_2} \dfrac{\left[n_{i_2 1}^{(2,3)}\right]^2}{n_{i_2}^{(2)}} & -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 1}^{(2,3)} n_{i_2 2}^{(2,3)}}{n_{i_2}^{(2)}} & \cdots & -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 1}^{(2,3)} n_{i_2 r_3-1}^{(2,3)}}{n_{i_2}^{(2)}} \\ -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 1}^{(2,3)} n_{i_2 2}^{(2,3)}}{n_{i_2}^{(2)}} & n_2^{(3)} - \sum_{i_2=1}^{r_2} \dfrac{\left[n_{i_2 2}^{(2,3)}\right]^2}{n_{i_2}^{(2)}} & \cdots & -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 2}^{(2,3)} n_{i_2 r_3-1}^{(2,3)}}{n_{i_2}^{(2)}} \\ \vdots & \vdots & & \vdots \\ -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 1}^{(2,3)} n_{i_2 r_3-1}^{(2,3)}}{n_{i_2}^{(2)}} & -\sum_{i_2=1}^{r_2} \dfrac{n_{i_2 2}^{(2,3)} n_{i_2 r_3-1}^{(2,3)}}{n_{i_2}^{(2)}} & \cdots & n_{r_3-1}^{(3)} - \sum_{i_2=1}^{r_2} \dfrac{\left[n_{i_2 r_3-1}^{(2,3)}\right]^2}{n_{i_2}^{(2)}} \end{vmatrix} \qquad (4.5)$$

Dêstes resultados seguem, diretamente os de F destinados a pôr em prova as hipóteses $E_{i_2}^{(2)} = 0$, $E_{i_3}^{(3)} = 0$ e $E_{i_1}^{(1)} = 0$. O primeiro tem distribuição F com $(r_2 - 1)$ e $(n - r_1 - r_2 - r_3 + 2)$ gráus de liberdade e o último com $(r_1 - 1)$ e $(n - r_1 - r_2 - r_3 + 2)$ gráus de liberdade.

4a — *Números proporcionais.*

Mais, em particular, se

$$n_{i_1 i_2 i_3} = \frac{n_{i_1}^{(1)} \; n_{i_2}^{(2)} \; n_{i_3}^{(3)}}{n^2}$$

os testes das hipóteses $E_{i_1}^{(1)}$, $E_{i_2}^{(2)}$ e $E_{i_3}^{(3)}$ acham-se sintetizados no seguinte quadro:

QUADRO (4a. I)

Variação devida a	G. L.	Variação	F.
$E_{i_1}^{(1)}$	$r_1 - 1$	$\sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)} \right]^2}{n_{i_2}^{(1)}} - \frac{Y^2}{n}$	$\frac{1}{r_1 - 1} \left\{ \sum_{i_1=1}^{r_1} \frac{\left[Y_{i_1}^{(1)} \right]^2}{n_{i_1}^{(1)}} - \frac{Y^2}{n} \right\} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
$E_{i_2}^{(2)}$	$r_2 - 1$	$\sum_{i_2=1}^{r_2} \frac{\left[Y_{i_2}^{(2)} \right]^2}{n_{i_2}^{(2)}} - \frac{Y^2}{n}$	$\frac{1}{r_2 - 1} \left\{ \sum_{i_2=1}^{r_2} \frac{\left[Y_{i_2}^{(2)} \right]^2}{n_{i_2}^{(2)}} - \frac{Y^2}{n} \right\} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
$E_{i_3}^{(3)}$	$r_3 - 1$	$\sum_{i_3=1}^{r_3} \frac{\left[Y_{i_3}^{(3)} \right]^2}{n_{i_3}^{(3)}} - \frac{Y^2}{n}$	$\frac{1}{r_3 - 1} \left\{ \sum_{i_3=1}^{r_3} \frac{\left[Y_{i_3}^{(3)} \right]^2}{n_{i_3}^{(3)}} - \frac{Y^2}{n} \right\} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
Resíduo	$n - r_1 - r_2 - r_3 + 2$	Diferença $= R$	
Total	$n - 1$	$\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \sum_{i_3=1}^{r_3} \sum_{i_4=1}^{n_{i_1 i_2 i_3}}$	$y_{i_1 i_2 i_3 i_4}^2 - \frac{Y^2}{n}$

4b — Números iguais, isto é,

$$n_{i_1 i_2 i_3} = m .$$

Neste caso o quadro (4a.I) torna-se:

QUADRO (4b. I)

Variação devida a	G. L.	Variação	F.
$E_{i_1}^{(1)}$	$r_1 - 1$	$\frac{1}{m\, r_1\, r_2\, r_3} \left\{ \sum_{i_1 = 1}^{r_1} r_1 \left[Y_{i_1}^{(1)} \right]^2 - Y^2 \right\} = R_1$	$\frac{R_1}{r_1 - 1} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
$E_{i_2}^{(2)}$	$r_2 - 1$	$\frac{1}{m\, r_1\, r_2\, r_3} \left\{ \sum_{i_2 = 1}^{r_2} r_2 \left[Y_{i_2}^{(2)} \right]^2 - Y^2 \right\} = R_2$	$\frac{R_2}{r_2 - 1} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
$E_{i_3}^{(3)}$	$r_3 - 1$	$\frac{1}{m\, r_1\, r_2\, r_3} \left\{ \sum_{i_3 = 1}^{r_3} r_3 \left[Y_{i_3}^{(3)} \right]^2 - Y^2 \right\} = R_3$	$\frac{R_3}{r_3 - 1} \Big/ \frac{R}{n - r_1 - r_2 - r_3 + 2}$
Resíduo	$n - r_1 - r_2 - r_3 + 2$	Diferença $= R$	
Total	$n - 1$	$\frac{1}{m\, r_1\, r_2\, r_3} \left\{ r_1 r_2 r_3 m \sum_{i_1, i_2, i_3, i_4 = 1}^{r_1, r_2, r_3, n_{i_1 i_2 i_3}} y_{i_1 i_2 i_3 i_4} - Y^2 \right\}$	

5 — *O caso* $k = 2$

Da (2.18) segue:

$$n\,\hat{\sigma}^2_{\Omega} = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{i_3=1}^{n_{i_1 i_2}} y^2_{i_1 i_2 i_3} - \sum_{i_1=1}^{r_1}\frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} - \sum_{i_2=1}^{r_1-1}\sum_{j_2=1}^{r_2-1} Q^{(2)}_{i_2}\,Q^{(2)}_{j_2}\,D^{i_2 j_2} \tag{5.1}$$

onde D é dado por:

$$D = \begin{vmatrix} n^{(2)}_1 - \sum\limits_{i_1=1}^{r_1}\dfrac{\left[n^{(1,2)}_{i_1 1}\right]^2}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{r_1}\dfrac{n_{i_1 1}\,n_{i_1 2}}{n^{(1)}_{i_1}} & \cdots\cdots & -\sum\limits_{i_1=1}^{r_1}\dfrac{n^{(1,2)}_{i_1 i}\,n^{(1,2)}_{i_1 r_2-1}}{n^{(1)}_{i_1}} \\ -\sum\limits_{i_1=1}^{r_1}\dfrac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 2}}{n^{(1)}_{i_1}} & n^{(2)}_2 - \sum\limits_{i_1=1}^{r_1}\dfrac{\left[n^{(1,2)}_{i_1 2}\right]^2}{n^{(1)}_{i_1}} & \cdots\cdots & -\sum\limits_{i_1=1}^{r_1}\dfrac{n^{(1,2)}_{i_1}\,n^{(1,2)}_{i_1 r_2-1}}{n^{(1)}_{i_1}} \\ \vdots & \vdots & & \vdots \\ -\sum\limits_{i_1=1}^{r_1}\dfrac{n^{(1,2)}_{i_1 1}\,n^{(1,2)}_{i_1 r_2-1}}{n^{(1)}_{i_1}} & -\sum\limits_{i_1=1}^{r_1}\dfrac{n^{(1,2)}_{i_1 2}\,n^{(1,2)}_{i_1 r_2-1}}{n^{(1)}_{i_1}} & \cdots\cdots & n^{(2)}_{r_2-1} - \sum\limits_{i_1=1}^{r_1}\dfrac{\left[n^{(1,2)}_{i_1 r_2-1}\right]^2}{n^{(1)}_{i_1}} \end{vmatrix}$$

Das (2c.19) e (2c.21) segue:

$$n\,\hat{\sigma}^2_{w_2} = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{i_3=1}^{n_{i_1 i_2}} y^2_{i_1 i_2 i_3} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} \tag{5.2}$$

$$n\,\hat{\sigma}^2_{w_1} = \sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\sum_{i_3=1}^{n_{i_1 i_2}} y^2_{i_1 i_2 i_3} - \sum_{i_2=1}^{r_2} \frac{\left[Y^{(2)}_{i_2}\right]^2}{n^{(2)}_{i_2}} \tag{5.3}$$

Êstes resultados seguem, diretamente os de F para pôr em prova as hipóteses $E^{(1)}_{i_1} = 0$ e $E^{(2)}_{i_2} = 0$. Para hipótese $E^{(1)}_{i_1} = 0$, temos:

$$F = \frac{\dfrac{1}{r_1 - 1}\left\{\displaystyle\sum_{i_2=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} - \sum_{i_2=1}^{r_2} \frac{\left[Y^{(2)}_{i_2}\right]^2}{n^{(2)}_{i_1}} + \sum_{i_2, j_2=1}^{r_2-1} Q^{(2)}_{i_2} Q^{(2)}_{j_2} D^{i_2 j_2}\right\}}{\dfrac{1}{n - r_1 - r_2 + 1}\left\{\displaystyle\sum y^2_{i_1 i_2 i_3} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1}\right]^2}{n^{(1)}_{i_1}} - \sum_{i_2, j_2=1}^{r_2-1} Q^{(2)}_{i_2} Q^{(2)}_{j_2} D^{i_2 j_2}\right\}} \tag{5.4}$$

a qual tem distribuição F com $(r_1 - 1)$ e $(n - r_1 - r_2 + 1)$ gráus de liberdade.

Para a hipótese $E^{(2)}_{i_2} = 0$, temos:

$$F = \frac{\dfrac{1}{r_2 - 1}\left\{\displaystyle\sum_{i_2=1}^{r_2-3}\sum_{j_2=1}^{r_2-1} Q^{(2)}_{i_2} Q^{(2)}_{j_2} D^{i_2 j_2}\right\}}{\dfrac{1}{n - r_1 - r_2 + 1}\left\{\displaystyle\sum y^2_{i_1 i_2 i_3} - \sum_{i_1=1}^{r_1} \frac{Y^{(1)}_{i_1}}{n^{(1)}_{i_1}} - \sum_{i_2, j_2=1}^{r_2-1} Q^{(2)}_{i_2} Q^{(2)}_{j_2} D^{i_2 j_2}\right\}} \tag{5.5}$$

que tem distribuição F com $(r_2 - 1)$ e $(n - r_1 - r_2 + 1)$ gráus de liberdade.

Mais em particular, se $r_2 = 2$, temos para a expressão:

$$\sum_{i_2, j_2 = 1}^{r_2} Q_{i_2}^{(2)} \; Q_{j_2}^{(2)} \; D^{i_2 i_2} = \left[Q_1^{(2)} \right]^2 D^{11} \tag{5.6}$$

Mas:

$$Q_1^{(2)} = Y_1^{(2)} - \sum_{i_1 = 1}^{r_1} Y_{i_1}^{(1)} \frac{n_{i_1}^{(1,2)}}{n_{i_1}^{(1)}}$$

$$= Y_{i_1}^{(2)} - \sum_{i_1 = 1}^{r_1} Y_{i_1}^{(1)} \frac{n_{i_1 1}}{n_{i_1 1} + n_{i_1 2}}$$

$$= \sum_{i_1 = 1}^{r_2} \sum_{i_3 = 1}^{n_{i_1 1}} y_{i_1 1 i_2} - \sum_{i_1 = 1}^{r_1} \sum_{i_2 = 1}^{2} \sum_{i_3 = 1}^{r_{i_1 i_2}} y_{i_1 i_2 i_3} \frac{n_{i_1 1}}{n_{i_1} + n_{i_1 2}}$$

$$= \sum_{i_1 = 1}^{r_1} \sum_{i_3 = 1}^{n_{i_1 1}} y_{i_1 1 i_3} - \sum_{i_1 = 1}^{r_1} \sum_{i_3 = 1}^{n_{i_1 1}} y_{i_1 i i_3} \frac{n_{i_1 1}}{n_{i_1 1} + n_{i_1 2}} -$$

$$- \sum_{i_1 = 1}^{r_1} \sum_{i_3 = 1}^{n_{i_1 2}} y_{i_1 2 i_3} \frac{n_{i_1 1}}{n_{i_1 1} + n_{i_1 2}}$$

$$= \sum_{i_1 = 1}^{r_1} \sum_{i_3 = 1}^{n_{i_1 1}} \frac{y_{i_1 1 i_3} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}} - \sum_{i_1 = 1}^{r_1} \sum_{i_3 = 1}^{n_{i_1 2}} \frac{y_{i_1 2 i_3} \cdot n_{i_1 1}}{n_{i_1 1} + n_{i_1 2}}$$

$$= \sum_{i_1 = 1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}} \left(\frac{\sum\limits_{i_3 = 1}^{n_{i_1 1}} y_{i_1 1 i_3}}{n_{i_1 1}} - \frac{\sum\limits_{i_3 = 1}^{n_{j_1 2}} y_{j_1 2 i_3}}{n_{i_1 2}} \right)$$

$$= \sum_{i_1 = 1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_2 1} + n_{i_1 2}} \left(\bar{y}_{i_1 1} - \bar{y}_{i_1 2} \right)$$

Por sua vez D torna-se:

$$D = n_1^{(2)} - \sum_{i_1=1}^{r_1} \frac{\left[n_{i_1 1}^{(1,2)} \right]^2}{n_{i_1}^{(1)}}$$

$$= \sum_{i_1=1}^{r_1} n_{i_1 1} - \sum_{i_1=1}^{r_1} \frac{n_{i_1 1}^2}{n_{i_1 1} + n_{i_1 2}}$$

$$= \sum_{i_1=1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}}$$

logo:

$$D^{11} = \frac{1}{\displaystyle\sum_{i_1=1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}}}$$

Com isto a (5.6) assume a seguinte explícita:

$$\sum_{i_2, j_2 = 1}^{r_2 - 1} Q_{i_2}^{(2)} \, Q_{j_2}^{(2)} \, D^{i_2 j_2} = \frac{\left\{ \displaystyle\sum_{i_1=1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}} \left(\bar{y}_{i_1 1} - \bar{y}_{i_1 2} \right) \right\}^2}{\displaystyle\sum_{i_1=1}^{r_1} \frac{n_{i_1 1} \; n_{i_1 2}}{n_{i_1 1} + n_{i_1 2}}} \tag{5.7}$$

que substituída em F nos dá os testes das hipóteses em questão.

5a — *Números proporcionais,* isto é,

$$n_{i_1 \, i_2} = \frac{n_{i_1}^{(1)} \; n_{i_2}^{(2)}}{n}$$

Neste caso os testes das hipóteses $E_{i_1}^{(1)} = 0$ e $E_{i_2}^{(2)} = 0$ estão sintetizados no quadro:

QUADRO (5a. I)

Variação devida a	G. L.	Variação	F.
$E_{i_1}^{(1)}$	$r_1 - 1$	$\sum_{i_1=1}^{r_1} \frac{\left\| Y_{i_1}^{(1)} \right\|}{n_{i_1}^{(1)}} - \frac{Y^2}{n} = R_1$	$\frac{R_1}{r_1 - 1} \Big/ \frac{R}{n - r_1 - r_2 + 1}$
$E_{i_2}^{(2)}$	$r_2 - 2$	$\sum_{i_2=1}^{r_2} \frac{\left\| Y_{i_2}^{(2)} \right\|}{n_{i_2}^{(2)}} - \frac{Y^2}{n} = R_2$	$\frac{R_2}{r_2 - 2} \Big/ \frac{R}{n - r_1 - r_2 + 1}$
Resíduo	$n - r_1 - r_2 + 1$	Diferença $= R$	
Total	$n - 1$	$\sum_{i_1, i_2, i_3 = 1}^{r_1, r_2, n_{i_1 i_2}} y_{i_1 i_2 i_3}^2 - \frac{Y^2}{n}$	

5b — *Números iguais*, isto é,

$$n_{i_1 i_2} = m$$

Neste caso o quadro (5a. I) torna-se:

QUADRO (5b. I)

Variação devida a	G.L.	Variação	F.
$E^{(1)}_{i_1}$	$r_1 - 1$	$\frac{1}{m\, r_1\, r_2} \left\{ \sum_{i_1 = 1}^{r_1} r_1 \left[Y^{(1)}_{i_1} \right]^2 - Y^2 \right\} = R_1$	$\frac{R_1}{r_1 - 1} \Big/ \frac{R}{n - r_1 - r_2 + 1}$
$E^{(2)}_{i_2}$	$r_2 - 1$	$\frac{1}{m\, r_1\, r_2} \left\{ \sum_{i_2 = 1}^{r_2} r_2 \left[Y^{(2)}_{i_2} \right]^2 - Y^2 \right\} = R_2$	$\frac{R_2}{r_2 - 1} \Big/ \frac{R}{n - r_1 - r_2 + 1}$
Resíduo	$n - r_1 - r_2 + 1$	Diferença $= R$	
Total	$n - 1$	$\frac{1}{m\, r_1\, r_2} \left\{ m\, r_1\, r_2 \sum_{i_1 i_2 i_3 = 1}^{r_1 r_2 n_{i_1 i_2}} y^2_{i_1 i_2 i_3} - Y^2 \right\}$	

6 — *O caso $k = 1$*

Da (2.17) segue:

$$n\,\hat{\sigma}^2_{\Omega} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} y^2_{i_1 i_2} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}}$$

Por sua vez, da (2c.21) vem:

$$n\,\hat{\sigma}^2_{\omega_1} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{n_{i_1}} y^2_{i_1 i_2}$$

Portanto, o teste de hipótese $E^{(1)}_{i_1} = 0$ será feito através de:

$$F = \frac{\dfrac{1}{r_1 - 1} \left[\displaystyle\sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} \right]}{\dfrac{1}{n - r_1} \left[\displaystyle\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{n_{i_1}} y^2_{i_1 i_2 - 1} - \sum_{i_1=1}^{r_1} \frac{\left[Y^{(1)}_{i_1} \right]^2}{n^{(1)}_{i_1}} \right]}$$

que terá distribuição F com $(r_1 - 1)$ e $(m - r_1)$ gráus de liberdade.

SUMMARY

As it is well known to all, the problems in testing of hypotheses, pertinent to the statistical chapter designated as analysis of variance, can be totally solved according to a certain theorem established in the study of the so-called general linear hypothesis.

The application of the mentioned theorem, however, usually leads us to determinants of so high an order that the search for a simpler expression is, at first sight, discouraging.

As a consequence of this fact, the authors, when setting up the tests, prefer, as a rule, doing it through the use of only a few of the demonstrated principles in the study of the linear hypothesis and not of the final results to which those principles convey. It is obvious that, with such

behavior, they must repeat, in each case, a series of intermediary steps adapted, naturally, to the matter in view.

The present work follows a different orientation, trying to attain the aimed objectives by means of a more immediate procedure.

Even in the case considered by us and which is sufficiently general, it is seen that — using devices which do not present difficulties like the ones to be anticipated — it is possible to obtain, for the originary determinants, expressions which will depend on deetrminants incomparably simpler to be handled.

As to the latter, it should be emphasized that their order is smaller than that of determinants given by other authors, in the particular case in which the subject is tackled in the way followed here. This fact, which deserves proper emphasis, has as corollary a much greater simplicity in obtaining certain expressions of highly practical interest.

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