Finite Sample Properties of the Partially Restricted Reduced Form Estimator

Hassan Arvin-Rad

Department of Economics, Florida International University

RESUMO

O artigo examina a densidade exata de uma função linear do estimator PRRF sob o mesmo conjunto de suposições de Nagar e Sahay (1978). Obtemos a expressão geral dos momentos exatos dessa função, os quais podem ser utilizados para se chegar às expressões para momentos de ordem arbitrária. Usando este resultado, determinamos as fórmulas explícitas dos quatro primeiros momentos. Em seguida estendemos os resultados para o caso no qual apenas se assume que a matriz de variáveis exógenas tem posto igual ao número de colunas e a matriz de covariância das variáveis endógenas é positiva definida. A ferramenta analítica utilizada para obter esses resultados é a técnica de cálculo fracionário aplicada originalmente à econometria por Phillips (1984).

PALAVRAS-CHAVE

estimator de forma reduzido parcialmente restringido, densidade exata, momentos exatos, erro de previsão quadrático médio, cálculo fracionário

ABSTRACT

The paper considers the exact density of a linear function of the PRRF estimator under the same set of assumptions as in Nagar and Sahay (1978). We obtain the general expression for the exact moments of such a linear function, which can be used to obtain expressions for moments of arbitrary order. Using this result the explicit formulae for the first four integer moments are given. We will then extend the results to the case where the matrix of exogenous variables is only assumed to have full column rank, and the covariance matrix of the endogenous variables has to be only positive definite. The analytical tool used to work out these results is the technique of fractional calculus first applied to econometrics by Phillips (1984).

KEY WORDS

partially restricted reduced form estimator, exact density, exact moments, mean squared prediction error, fractional calculus

INTRODUCTION

There are several approaches to the estimation of the reduced form coefficients of a linear simultaneous system of equations. The traditional one is based on applying the ordinary least squares method to each reduced form equation. The estimator obtained is known as the unrestricted reduced form (URRF) estimator. This estimator is a linear function of the disturbance terms and will have moments to the order that disturbances have moments. Asymptotic properties of forecasts derived using the URRF have been investigated by Hooper and Zellner (1961). When the model is overidentified there will be some restrictions on the reduced form parameters. The URRF approach ignores these restrictions and is then inefficient.

An alternative approach is based on first estimating the structural coefficients using some consistent estimation procedure and then, by employing the well known relationship between reduced form and structural form coefficients, obtaining estimates of the reduced form parameters. This approach has come to be known as the restricted reduced form (RRF) estimation. Asymptotic properties of the RRF estimator has been examined by Dhrymes (1973). He shows that the RRF estimator, when three stage least squares estimates of structural coefficients are used, is asymptotically more efficient than the URRF estimator. This result, however, does not necessarily hold when two stage least squares estimates of structural coefficients are used. Asymptotic properties of forecasts using the RRF estimators have been considered by Goldberger, Nagar, and Odeh (1961). Finite sample properties of RRF estimators were analyzed by McCarthy (1972). He demonstrated that if there are overidentifying restrictions on the structural equations the RRF estimator, using 2SLS, will have no moments. Sargan (1976) has shown this to be true for a wide class of estimators that includes 2SLS and 3SLS. However he also shows that the RRF estimator, based on the FIML estimators of the structural coefficients will possess moments of certain order. The order of moments depends on the number of observations, and on the number of endogenous and exogenous variables in the model.

Obtaining the RRF estimator requires estimating the whole system of structural equations. computational considerations aside, misspecification in any structural equation will adversely affect the RRF estimator. The approach proposed by Amemiya (1966) and Kakwani and Court (1972) alleviates this problem. They suggest estimating parameters of a reduced form equation by utilizing overidentifying restrictions on the corresponding structural equation alone. This approach is known as the partially restricted reduced form (PRRF) estimator. The existence of moments of the PRRF estimator has been analyzed by Knight (1977), Swamy and Mehta (1980, 1981) and McCarthy (1981). It was shown by McCarthy that the PRRF estimator possesses moments to the order that dependent variables possess moments. For this result to hold the k-class estimators of structural coefficients should have $0 \leq k \leq 1$. Nagar and Sahay (1978) derived the exact and asymptotic bias and mean squared errors of the PRRF estimator and the forecasts obtained using this estimator, in the special case where there are only two endogenous variables in the structural equation under consideration and the matrix of exogenous variables is assumed to be orthogonal. In addition, the covariance matrix of the endogenous variables was assumed to be unity. Assuming orthogonal exogenous variable matrix, Knight and Kinal (1994) examined the finite sample properties of the PRRF estimator in a general $(n+1)$ endogenous variable model. Asymptotic properties of the PRRF estimator has been investigated by Dhrymes (1983). It is shown that the PRRF estimator is not necessarily efficient (asymptotically) relative to the URRF estimator.

In this paper we first derive the exact density of a linear function of the PRRF estimator under the same set of assumptions as in Nagar and Sahay (1978). We will also obtain the general expression for the exact moments of such a linear function, which can be used to obtain expressions for moments of arbitrary order. Using this result the explicit formulae for the first four integer moments are given. We will then extend the results to the case where the matrix of exogenous variables is only assumed to have full column rank, and the covariance matrix of the endogenous variables has to be only positive definite. It is worth pointing out that unlike the case of structural equation estimators, where the results obtained for moments of estimators of coefficients of transformed structural equation can be easily extended to obtain moments of estimators of the coefficients of the original structural equation, the moments of the PRRF estimators of the original reduced form equation cannot be derived in a straightforward manner from the corresponding expressions for estimators of the coefficients of the transformed reduced form equation (except for the first moment).

The analytical tool used to work out these results is the technique of fractional calculus first applied to econometrics by Phillips (1984). Further applications of fractional calculus could be found in Phillips (1985, 1986), Knight (1986a, 1986b), and Knight and Kinal (1994). For a detailed survey of fractional calculus see Ross (1974), Oldham and Spanier (1974), and Miller and Ross (1993).

The plan of this paper is as follows. In section 1 the linear model is specified. In section 2 the PRRF estimator is defined and a simple proof of existence of its moments is given. Section 3 discusses the standardizing transformations of the model. In section 4 we derive the exact density of the PRRF estimator. Expressions for the exact moments are given in section 5. In section 6 the exact bias and mean squared prediction error of forecasts using the PRRF estimator are derived. For the sake of brevity, proofs are not given in the paper. They can be obtained from the author upon request.

1. THE MODEL

Consider the simultaneous system of G linear structural equations

$$Y\Gamma + X\,B = U \tag{1.1}$$

where Y is a T x G matrix of T observations on G endogenous variables; X is a T x K matrix of T observations on K exogenous variables; U is a T x G matrix of disturbances; Γ and B are respectively G x G and K x G matrices of structural coefficients.

From (1.l) we obtain the reduced form

$$Y = X\,\Pi + V, \qquad \Pi = -B\Gamma^{-1}, \quad V = U\,\Gamma^{-1}$$

We assume

(A.l) X is nonstochastic with rank K < T

(A.2) Each row of U is independently and identically distributed according to a G-variate normal with mean zero and covariance matrix Σ, a positive definite matrix.

2. THE PARTIALLY RESTRICTED REDUCED FORM (PRRF) ESTIMATOR

The first equation of (1.1) may be written as

$$y_1 = Y_2\gamma + X_1\beta + u \tag{2.1}$$

where y_1 is a T x 1 vector and Y_2 is a T x G_1 matrix of observations on the endogenous variables appearing in equation; X_1 is a T x K_1 matrix of observations on the included exogenous variables; (γ and β are vectors of G_1 and K_1 components respectively, and u is the first column of U. The matrix X is partitioned as $X = (X_1 \mid X_2)$

where X_2 is a T x K_2 matrix of observations on the excluded exogenous variables (K_2 = K - K_1).

The reduced forms corresponding to y_1 and Υ_2 can be written as

$$y_1 = X\pi_1 + v_1 = X_1\pi_{11} + X_2\pi_{12} + v_1 = X\begin{bmatrix}\pi_{11}\\ \pi_{12}\end{bmatrix} + v_1 \tag{2.2}$$

and

$$Y_2 = X\pi_2 + V_2 = X_1\pi_{21} + X_2\pi_{22} + V_2 = X\begin{bmatrix}\pi_{21}\\ \pi_{22}\end{bmatrix} + V_2 \tag{2.3}$$

where each row of (v_1 V_2) is normally distributed with covariance matrix Ω, a positive definite matrix:

$$\Omega = \begin{bmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \Omega_{22}\end{bmatrix}$$

Substituting (2.3) into (2.1) we get

$$\begin{aligned} y_1 &= X\begin{bmatrix}\pi_{21}\\ \pi_{22}\end{bmatrix}\gamma + X_1\beta + u + V_2\gamma \\ &= X\begin{bmatrix}\pi_{21} & I\\ \pi_{22} & O\end{bmatrix}\begin{bmatrix}\gamma\\ \beta\end{bmatrix} + u + V_2\gamma \end{aligned} \tag{2.4}$$

Comparison of (2.4) and (2.2) suggests estimating the coefficient vector $\pi_1 = (\pi'_{11} \quad \pi'_{12})'$ using the relationship

$$\begin{bmatrix}\pi_{11}\\ \pi_{12}\end{bmatrix} = \begin{bmatrix}\pi_{21} & I\\ \pi_{22} & O\end{bmatrix}\begin{bmatrix}\gamma\\ \beta\end{bmatrix} \tag{2.5}$$

The PRRF estimator is then given by

$$\begin{bmatrix} \hat{\pi}_{11} \\ \hat{\pi}_{12} \end{bmatrix} = \begin{bmatrix} \tilde{\pi}_{21} & I \\ \tilde{\pi}_{22} & O \end{bmatrix} \begin{bmatrix} \hat{\gamma} \\ \hat{\beta} \end{bmatrix}$$
$$= \begin{bmatrix} (X'X)^{-1}X'Y_2 & \vdots & \begin{matrix} I \\ O \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{\gamma} \\ \hat{\beta} \end{bmatrix} \tag{2.6}$$

where $\tilde{\pi}_2 = [\tilde{\pi}'_{21} \ \tilde{\pi}'_{22}]' = (X'X)^{-1}X'Y_2$ is the URRF estimator of $\pi_2 = [\pi'_{21} \ \pi'_{22}]'$ in (2.3), and $\hat{\gamma}$ and $\hat{\beta}$ are consistent estimators of the structural coefficients γ and β respectively. I is a K_1 x K_1 identity matrix and O is a K_2 x K_1 null matrix.

An alternative approach to deriving the PRRF estimator lies in a suggestion by Amemiya (1966) for predicting y_1 using equation (2.1). Amemiya suggested predicting y_1 by

$$\hat{y}_1 = \tilde{Y}_2\hat{\gamma} + X_1\hat{\beta} \ , \tag{2.7}$$

where $\tilde{Y}_2 = X\tilde{\pi}_2 = X(X'X)^{-1}X'Y_2$. Substituting for $\tilde{Y}_2$ in (2.7) we get

$$\hat{y}_1 = X(X'X)^{-1}X'Y_2\hat{\gamma} + X_1\hat{\beta}$$
$$= X\begin{bmatrix} (X'X)^{-1}X'Y_2 & \vdots & \begin{matrix} I \\ O \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{\gamma} \\ \hat{\beta} \end{bmatrix}, \tag{2.8}$$

which when compared to (2.2) suggests estimating the reduced form coefficients by the PRRF estimators (2.6).

We now give a simple proof of the existence of moments of the PRRF estimator for the case where $\hat{\gamma}$ and $\hat{\beta}$ are the 2SLS estimators of γ and β.[1] Write (2.6) as

$$\hat{\pi}_1 = \hat{J}\hat{\delta} \ ,$$

1 Several authors have proved the existence of moments of arbitrary order for the PRRF estimator. However, some of these proofs are erroneous and some are rather lengthy (see McCARTHY, 1972, for a discussion of this point).

where

$$\hat{\pi}_1 = \begin{bmatrix} \hat{\pi}_{11} \\ \hat{\pi}_{12} \end{bmatrix},$$

$$\hat{J} = \begin{bmatrix} (X'X)^{-1}X'Y_2 & \vdots & \begin{matrix} I \\ O \end{matrix} \end{bmatrix},$$

and

$$\hat{\delta} = \begin{bmatrix} \hat{\gamma} \\ \hat{\beta} \end{bmatrix} = (W_1'P_xW_1)^{-1}W_1'P_xy_1 \text{ , the 2SLS estimator of } \delta = \begin{bmatrix} \gamma \\ \beta \end{bmatrix}.$$

The matrix W_1 is defined as $W_1 = (Y_2 \mid X_1)$, and the idempotent matrix P_x is defined as

$$P_x = X(X'X)^{-1}X'.$$

Now

$$\begin{aligned} \hat{J} &= (X'X)^{-1}X'(Y_2 \vdots X_1) \\ &= (X'X)^{-1}X'W_1, \end{aligned}$$

hence

$$\hat{\pi}_1 = (X'X)^{-1}X'W_1(W_1'P_xW_1)^{-1}W_1'P_xy_1,$$

and

$$\hat{y}_1 = X\hat{\pi}_1 = P_x W_1(W_1'P_xW_1)^{-1}W_1'P_xy_1 \ ,$$

from which we get

$$\begin{aligned}\hat{y}_1'\hat{y}_1 &= [\, y_1'P_x W_1(W_1'P_xW_1\,)^{-1}W_1'P_x\,][\,P_x W_1(W_1'P_xW_1\,)^{-1}W_1'P_xy_1\,]\\ &= y_1'P_x W_1(W_1'P_xW_1\,)^{-1}W_1'P_xy_1\\ &= y_1'y_1 - y_1'[\,I - P_x W_1(W_1'P_xW_1\,)^{-1}W_1'P_x\,]\,y_1,\\ &\le y_1'y_1,\end{aligned}$$

where the inequality follows since $I - P_x W_1(W_1'P_xW_1\,)^{-1}W_1'P_x$ is an idempotent matrix. Therefore, $\hat{y}_1'\hat{y}_1$ will have moments to the order that $y_1' y_1$ possesses moments. Since $\hat{y}_1'\hat{y}_1 = \hat{\pi}_1'\,X'\,X\hat{\pi}_1$, it is easy to show that $\hat{\pi}_1$ will have moments to the order that y_1 has moments.

3. THE EFFECTS OF STANDARDIZED TRANSFORMATIONS ON THE PRRF ESTIMATOR

We assume there are only two endogenous variables in the structural equation (2.1). Hence Υ_2 is a T x 1 vector and we denote it by y_2. We apply the usual standardizing transformations which reduce Ω, the covariance matrix of the endogenous variables, and $X'X$ to identity matrices. The transformed structural equation for y_1 is

$$y_1^* = y_2^*\gamma^* + \overline{X}_1\overline{\beta}^* + u^* \quad , \tag{3.1}$$

and the transformed reduced form equations are

$$y_1^* = \overline{X}\overline{\pi}_1^* + v_1^* = \overline{X}_1\overline{\pi}_{11}^* + \overline{X}_2\overline{\pi}_{12}^* + v_1^* \tag{3.2}$$

$$y_2^* = \overline{X}\overline{\pi}_2^* + v_2^* = \overline{X}_1\overline{\pi}_{21}^* + \overline{X}_2\overline{\pi}_{22}^* + v_2^*, \tag{3.3}$$

where $*$ indicates reduction to canonical form and indicates orthogonalization of $X'X$. Consequently y_1^* and y_2^* are now independently normally distributed with identity covariance matrix for each row of $(\,y_1^* \mid y_2^*\,)$, and $\overline{X}'\,\overline{X} = I$.

The relationship between estimators of coefficients of the original equations with estimators of coefficients of the transformed equations are given by

$$\hat{\gamma}^{*}_{2SLS} = \frac{l_{22}}{l_{11}} \hat{\gamma}_{2SLS} - \frac{l_{21}}{l_{11}}, \tag{3.4}$$

$$\hat{\beta}^{*}_{2SLS} = \frac{1}{l_{11}} J_{11} \hat{\beta}_{2SLS}, \tag{3.5}$$

$$\hat{\pi}^{*}_{11} = \frac{1}{l_{11}} J_{11} \hat{\pi}_{11} + \frac{1}{l_{11}} J_{12} \hat{\pi}_{12} - \frac{l_{21}}{l_{11} l_{22}} (J_{11} \tilde{\pi}_{21} + J_{12} \tilde{\pi}_{22}), \tag{3.6}$$

$$\hat{\pi}^{*}_{12} = \frac{1}{l_{11}} J_{22} \hat{\pi}_{12} - \frac{l_{21}}{l_{11} l_{22}} J_{22} \tilde{\pi}_{22}, \tag{3.7}$$

where ~ and ^ stand for URRF and PRRF estimators respectively; $\hat{\pi}_{11}$ and $\hat{\pi}_{12}$ are the PRRF estimators of the coefficients of the original reduced form equation (2.2); $\hat{\pi}^{*}_{11}$ and $\hat{\pi}^{*}_{12}$ are the PRRF estimators of the coefficients of the transformed equation (3.2); and $\tilde{\pi}_{21}$ and $\tilde{\pi}_{22}$ are the PRRF estimators of coefficients of the original reduced form equation (2.3); J_{11}, J_{12}, and J_{22} are elements of a nonsingular upper triangular matrix J such that $J'J = X'X$, and l_{11}, l_{21}, and l_{22} are elements of a non singular lower triangular matrix L such that $L'L$ = W. Explicitly

$$J_{11} = (X_1' X_1)^{1/2}, \qquad l_{11} = (\omega_{11} - \omega_{12}^2 / \omega_{22})^{1/2},$$

$$J_{12} = (X_1' X_1)^{-1/2} (X_1' X_2), \qquad l_{21} = \omega_{12} / (\omega_{22})^{1/2},$$

$$J_{22} = (X_2' \bar{P}_{x_1} X_2)^{1/2}, \qquad l_{22} = (\omega_{22})^{1/2},$$

where $\bar{P}_{x_1} = I - P_{x_1} = I - X_1 (X_1' X_1)^{-1} X_1$.

Alternatively, we may write (3.6) and (3.7) as

$$\hat{\pi}_{11} = l_{11} J_{11}^{-1} \hat{\pi}^{*}_{11} - l_{11} J_{11}^{-1} J_{12} J_{22}^{-1} \hat{\pi}^{*}_{12} + l_{21} J_{11}^{-1} \tilde{\pi}^{*}_{21} - l_{21} J_{11}^{-1} J_{12} J_{22}^{-1} \tilde{\pi}^{*}_{22}, \tag{3.8}$$

$$\hat{\pi}_{12} = l_{11} J_{22}^{-1} \hat{\bar{\pi}}_{12}^{*} + l_{21} J_{22}^{-1} \tilde{\bar{\pi}}_{22}^{*}, \tag{3.9}$$

where $\tilde{\bar{\pi}}_{21}^{*}$ and $\tilde{\bar{\pi}}_{22}^{*}$ are the URRF estimators of the transformed reduced form equation (3.3). Combining (3.8) and (3.9) we get

$$\hat{\pi}_{1} = l_{11} J^{-1} \hat{\bar{\pi}}_{1}^{*} + l_{21} J^{-1} \tilde{\bar{\pi}}_{2}^{*}. \tag{3.10}$$

The PRRF estimator of in (3.2) is defined as

$$\hat{\bar{\pi}}_{1}^{*} = \begin{bmatrix} \hat{\bar{\pi}}_{11}^{*} \\ \hat{\bar{\pi}}_{12}^{*} \end{bmatrix} = \begin{bmatrix} \tilde{\bar{\pi}}_{21}^{*} & I \\ \tilde{\bar{\pi}}_{22}^{*} & O \end{bmatrix} \begin{bmatrix} \hat{\gamma}_{2SLS}^{*} \\ \hat{\bar{\beta}}_{2SLS}^{*} \end{bmatrix}, \tag{3.11}$$

where

$$\begin{aligned} \tilde{\bar{\pi}}_{2}^{*} &= \begin{bmatrix} \tilde{\bar{\pi}}_{21}^{*} \\ \tilde{\bar{\pi}}_{22}^{*} \end{bmatrix} = (\bar{X}' \bar{X})^{-1} \bar{X}' y_{2}^{*} \\ &= \bar{X}' y_{2}^{*} = \begin{bmatrix} \bar{X}_{1}' y_{2}^{*} \\ \bar{X}_{2}' y_{2}^{*} \end{bmatrix}, \end{aligned} \tag{3.12}$$

and the 2SLS estimators of the coefficients of the transformed structural equation (3.1) are given by

$$\hat{\gamma}_{2SLS}^{*} = (y_{2}'^{*} \bar{X}_{2} \bar{X}_{2}' y_{2}^{*})^{-1} y_{2}'^{*} \bar{X}_{2} \bar{X}_{2}' y_{1}^{*}, \tag{3.13}$$

$$\hat{\bar{\beta}}_{2SLS}^{*} = \bar{X}_{1}' (y_{1}^{*} - y_{2}^{*} \hat{\gamma}_{2SLS}^{*}). \tag{3.14}$$

Inserting (3.12), (3.13) and (3.14) in (3.11) we obtain

$$\hat{\bar{\pi}}_{1}^{*} = \begin{bmatrix} \hat{\bar{\pi}}_{11}^{*} \\ \hat{\bar{\pi}}_{12}^{*} \end{bmatrix} = \begin{bmatrix} \bar{X}' y_{1}^{*} \\ \bar{X}_{2}' y_{2}^{*} (y_{2}'^{*} \bar{X}_{2} \bar{X}_{2}' y_{2}^{*})^{-1} y_{2}'^{*} \bar{X}_{2} \bar{X}_{2}' y_{1}^{*} \end{bmatrix}. \tag{3.15}$$

Define z_i, $i = 1,2,3,4$, as follows

$$z_1 = \bar{X}_2' y_1^*, \qquad z_2 = \bar{X}_2' y_2^*,$$
$$z_3 = \bar{X}_1' y_1^*, \qquad z_4 = \bar{X}_1' y_2^*. \tag{3.16}$$

We then have

$$\hat{\bar{\pi}}_1^* = \begin{bmatrix} \hat{\bar{\pi}}_{11}^* \\ \hat{\bar{\pi}}_{12}^* \end{bmatrix} = \begin{bmatrix} z_3 \\ z_2(z_2'z_2)^{-1} z_2' z_1 \end{bmatrix} = \begin{bmatrix} z_3 \\ P_{z_2} z_1 \end{bmatrix}, \tag{3.17}$$

$$\tilde{\bar{\pi}}_2^* = \begin{bmatrix} \tilde{\bar{\pi}}_{21}^* \\ \tilde{\bar{\pi}}_{22}^* \end{bmatrix} = \begin{bmatrix} z_4 \\ z_2 \end{bmatrix}, \tag{3.18}$$

$$\hat{\gamma}_{2SLS}^* = (z_2' z_2)^{-1} z_2' z_1, \tag{3.19}$$

and

$$\hat{\bar{\beta}}_{2SLS}^* = z_3 - z_4 (z_2' z_2)^{-1} z_2' z_1 . \tag{3.20}$$

From (3.16) it is obvious that z_i, $i = 1,2,3,4$, are independently normally distributed with

$$\bar{z}_1 = E(z_1) = \bar{X}_2' E(\bar{X}_1 \bar{\pi}_{11}^* + \bar{X}_2 \bar{\pi}_{12}^* + v_1^*) = \bar{\pi}_{12}^*,$$

$$\bar{z}_2 = E(z_2) = \bar{X}_2' E(\bar{X}_1 \bar{\pi}_{21}^* + \bar{X}_2 \bar{\pi}_{22}^* + v_2^*) = \bar{\pi}_{22}^*, \tag{3.21}$$

$$\bar{z}_3 = E(z_3) = \bar{X}_1' E(\bar{X}_1 \bar{\pi}_{11}^* + \bar{X}_2 \bar{\pi}_{12}^* + v_1^*) = \bar{\pi}_{11}^*,$$

$$\bar{z}_4 = E(z_4) = \bar{X}_1' E(\bar{X}_1 \bar{\pi}_{21}^* + \bar{X}_2 \bar{\pi}_{22}^* + v_2^*) = \bar{\pi}_{21}^*$$

and

$$\begin{aligned} E(z_i - \bar{z}_i)(z_j - \bar{z}_j)' &= I \qquad i = j \\ &= 0 \qquad i \neq j \end{aligned} \tag{3.22}$$

where I is a K_2 x K_2 unit matrix and O is a K_2 x K_2 null matrix.

Nagar and Sahay (1978) have obtained the first two moments of $\hat{\pi}^*_{11}$ and $\hat{\pi}^*_{12}$. However as is seen from (3.8) and (3.9) to derive the exact moments of $\hat{\pi}_{11}$ and $\hat{\pi}_{12}$ we need to work out the covariances between the right hand terms.

To obtain the exact density and exact moments of $\hat{\pi}_1$ we first derive the corresponding results for the PRRF estimator of coefficients of the transformed reduced form equation (3.2). The next section is devoted to this task.

4. THE EXACT DENSITY OF THE PRRF ESTIMATOR

We consider deriving the exact density of a linear function, $r = h'\hat{\pi}^*_1$, of $\hat{\pi}^*_1$ by first conditioning on z_2 and then finding the unconditional density of r. We partition h conformably with $\hat{\pi}^*_1$ and write r as

$$\begin{aligned} r &= h'\hat{\pi}^*_1 = h_1'\hat{\pi}^*_{11} + h_2'\hat{\pi}^*_{12} \\ &= h_1'z_3 + h_2'P_{z_2}z_1 . \end{aligned}$$

Clearly

$$r \mid z_2 \sim N(h_1'\bar{z}_3 + h_2'P_{z_2}\bar{z}_1 \quad , \quad h_1'h_1 + h_2'P_{z_2}h_2),$$

hence

$$pdf(r \mid z_2) = \frac{1}{\sqrt{2\pi}}(h_1'h_1 + h_2'P_{z_2}h_2)^{-1/2} exp\left[-\frac{1}{2}\frac{[r-(h_1'\bar{z}_3 + h_2'P_{z_2}\bar{z}_1)]^2}{(h_1'h_1 + h_2'P_{z_2}h_2)}\right]. \qquad (4.1)$$

Following Phillips (1984) we can write (4.1) as

$$\begin{aligned} pdf(r \mid z_2) = {} & \frac{1}{\sqrt{2\pi}}(h_1'h_1 + h_2'\partial_x\Delta_x^{-1}\partial_x'h_2)^{-1/2} \\ & \cdot exp\left[-\frac{1}{2}\frac{[r-(h_1'\bar{z}_3 + h_2'\partial_x\Delta_x^{-1}\partial_x'\bar{z}_1)]^2}{(h_1'h_1 + h_2'\partial_x\Delta_x^{-1}\partial_x'h_2)}\right] e^{x'z_2}\Big|_{x=0} , \end{aligned}$$

where ∂_x denotes the vector operator ∂ / ∂_x and Δ_x is the Laplacian operator $\partial'_x \partial_x$. Now

$$\begin{aligned} pdf(r) &= \int_{z_2} pdf(r \mid z_2) pdf(z_2) dz_2 \\ &= \int_{z_2} \frac{1}{\sqrt{2\pi}} \phi_x^{-1/2} exp[-\frac{1}{2\phi_x}(r-\eta_x)^2] e^{x' z_2} pdf(z_2) dz_2 \big|_{x=0} \\ &= \frac{1}{\sqrt{2\pi}} \phi_x^{-1/2} exp[-\frac{1}{2\phi_x}(r-\eta_x)^2] \int_{z_2} e^{x' z_2} pdf(z_2) dz_2 \big|_{x=0} \\ &= \frac{1}{\sqrt{2\pi}} \phi_x^{-1/2} exp[-\frac{1}{2\phi_x}(r-\eta_x)^2] exp(x' \bar{z}_2 + x'x/2) \big|_{x=0}, \end{aligned} \tag{4.2}$$

where

$$\phi_x = h_1' h_1 + h_2' \partial_x \Delta_x^{-1} \partial_x' h_2, \tag{4.3}$$

and

$$\eta_x = h_1' \bar{z}_3 + h_2' \partial_x \Delta_x^{-1} \partial_x' \bar{z}_1. \tag{4.4}$$

To derive the exact density of a linear function, $r_1 = h_1' \hat{\pi}_{11}^*$, of $\hat{\pi}_{11}^*$ alone we set $h_2 = 0$ in (4.3) and (4.4), and from (4.2) we get

$$pdf(r_1) = \frac{1}{\sqrt{2\pi}} (h_1' h_1)^{-1/2} exp\left[-\frac{1}{2} \frac{(r_1 - h_1' \bar{z}_3)^2}{h_1' h_1} \right],$$

which implies that $r_1 = h_1' \hat{\pi}_{11}^*$ has a normal distribution with mean $h_1' \bar{z}_3$ and variance $h_1' h_1$. This is not surprising since from (3.17) we have

$$\hat{\pi}_{11}^* = z_3 \sim N(\bar{z}_3, I),$$

which implies

$$r_1 = h_1'\hat{\bar{\pi}}_{11}^* \sim N(h_1'\bar{z}_3 \;,\; h_1'h_1\,).$$

To obtain the exact density of a linear function, $r_2 = h_2'\hat{\bar{\pi}}_{12}^*$, of $\hat{\bar{\pi}}_{12}^*$ alone we set $h_1 = 0$ in (4.3) and (4.4), and from (4.2) we get

$$pdf(\,r_2\,) = \frac{1}{\sqrt{2\pi}}(\,h_2'\partial_x\Delta_x^{-1}\partial_x'h_2\,)^{-1/2} \cdot exp\left[-\frac{1}{2}\frac{(\,r_2 - h_2'\,\partial_x\,\Delta_x^{-1}\,\partial_x'\,\bar{z}_1\,)^2}{(\,h_2'\,\partial_x\,\Delta_x^{-1}\,\partial_x'\,h_2\,)}\right]exp(\,x'\,\bar{z}_2 + x'x/2\,)\Big|_{x=0}\,. \tag{4.5}$$

Also, notice that when the structural equation is just identified $\hat{\bar{\pi}}_{12}^*$ will be scalar. Hence ∂_x will be the scalar differential operator, and $\Delta_x = \partial_x^2$. Assuming $h_2 = 1$, (4.5) reduces to

$$pdf(\,\hat{\bar{\pi}}_{12}^*\,) = \frac{1}{\sqrt{2\pi}}exp[-\frac{1}{2}(\hat{\bar{\pi}}_{12}^* - \bar{z}_1\,)^2\,]\,,$$

which shows that $\hat{\bar{\pi}}_{12}^*$ has a normal distribution with mean $\bar{z}_1$ and variance one. This is to be expected since in the just identified case z_2 will be scalar and from (3.16) we get

$$\hat{\bar{\pi}}_{12}^* = z_2(\,z_2'z_2\,)^{-1}z_2'z_1 = z_1 \sim N(\,\bar{z}_1\;,1\,).$$

5. THE EXACT MOMENTS OF THE PRRF ESTIMATOR

To derive the exact moments of $r = h'\hat{\bar{\pi}}_1^*$ we use (4.2)

$$
\begin{aligned}
E(r^n) &= \int_{-\infty}^{+\infty} r^n \, pdf(r)\, dr \\
&= \int_{-\infty}^{+\infty} r^n \frac{1}{\sqrt{2\pi}} \phi_x^{-1/2} \, exp[-\frac{1}{2\phi_x}(r-\eta_x)^2] \, exp(x'\bar{z}_2 + x'x/2)\, dr \Big|_{x=0} \\
&= n! \sum_{k=0}^{[n/2]} \frac{(1/2)^k}{(n-2k)!\,k!} \phi_x^k \eta_x^{n-2k} \, exp(x'\bar{z}_2 + x'x/2) \Big|_{x=0} \\
&= e^{-\bar{z}_2'\bar{z}_2/2} n! \sum_{k=0}^{[n/2]} \frac{(1/2)^k}{(n-2k)!\,k!} \phi_w^k \eta_w^{n-2k} \, exp(w'w/2) \Big|_{w=\bar{z}_2},
\end{aligned}
\tag{5.1}
$$

where [n/2] is the integral part of n/2. For the special case of n = 1, 2, we have

$$
\begin{aligned}
E(r) &= e^{-\mu} \eta_w \, exp(w'w/2) \Big|_{w=\bar{z}_2} \\
&= e^{-\mu} (h_1'\bar{z}_3 + h_2' \partial_w \Delta_w^{-1} \partial_w' \bar{z}_1) \, exp(w'w/2) \Big|_{w=\bar{z}_2},
\end{aligned}
\tag{5.2}
$$

and

$$
\begin{aligned}
E(r^2) &= e^{-\mu} (\phi_w + \eta_w^2) exp(w'w/2) \Big|_{w=\bar{z}_2} \\
&= e^{-\mu} (h_1'h_1 + h_2' \partial_w \Delta_w^{-1} \partial_w' h_2) \, exp(w'w/2) \Big|_{w=\bar{z}_2} \\
&\quad + e^{-\mu} (h_1'\bar{z}_3 + h_2' \partial_w \Delta_w^{-1} \partial_w' \bar{z}_1)^2 \, exp(w'w/2) \Big|_{w=\bar{z}_2},
\end{aligned}
\tag{5.3}
$$

where $\mu = \bar{z}_2'\bar{z}_2/2$. By setting $h_1 = 0$ in (5.1) we get

$$
\begin{aligned}
E(r_2^n) = E(h_2'\hat{\bar{\pi}}_{12}^*)^n &= e^{-\mu} n! \sum_{k=0}^{[n/2]} \frac{(1/2)^k}{(n-2k)!\,k!} \\
&. (h_2' \partial_w \Delta_w^{-1} \partial_w' h_2)^k (h_2' \partial_w \Delta_w^{-1} \partial_w' \bar{z}_1)^{n-2k} \, exp(w'w/2) \Big|_{w=\bar{z}_2}.
\end{aligned}
\tag{5.4}
$$

Similarly, setting $h_2 = 0$ in (5.1) will result in the general expression for the moments of $r_1 = h_1'\hat{\bar{\pi}}_{11}^*$:

$$E(r_1^n) = n! \sum_{k=0}^{[n/2]} \frac{(1/2)^k}{(n-2k)!\,k!} (h_1' h_1)^k (h_1' \bar{z}_3)^{n-2k}. \tag{5.5}$$

In what follows we focus our attention on r_2, since r_1 is normally distributed. To make (5.4) computationally operational we apply the technique of fractional calculus. By extending one form of the Weyl fractional integral to Δ_w operator, Phillips (1984) gives the following definition for negative powers of Δ_w:

$$\Delta_w^{-\alpha} f(w) = \frac{1}{\Gamma(\alpha)} \int_0^\infty [exp(-\Delta_w s) f(w)] s^{\alpha-1} ds, \qquad \alpha > 0 \tag{5.6}$$

provided the integral converges. Applying the above definition to (5.4) yields

$$E(r_2^n) = e^{-\mu} \sum_{j=0}^{[n/2]} \sum_{u=0}^{[n/2]-j} \sum_{k=0}^{n-2j-2u} \sum_{v=0}^{[n/2-k/2]} \binom{n}{2j} \binom{n}{k}$$

$$. \frac{(2j)!(n-2j-2u+1)_{2u}(n-k-2v+1)_{2v}(n-2j-2u-k+1)_k}{j!\,u!\,v!\,2^{n+2u+2v}} \tag{5.7}$$

$$.(h_2' \bar{z}_1)^k (\bar{z}_1' \bar{z}_2)^{n-2j-2u-k} (2_1 \bar{z}_1' \bar{z}_1)^u (2h_2' h_2)^v (h_2' \bar{z}_2)^{n-k-2v}$$

$$.f(n-j-k-u-v\,;\,2n-2j-k-u-v),$$

where

$$\binom{x}{q} = \frac{x!}{q!(x-q)!},$$

$$\begin{aligned} (a)_t &= (a)(a+1)(a+2)......(a+t-1) && \text{for } t>0 \\ &= 1, && \text{for } t=0 \end{aligned}$$

and

$$f(i;j) = \frac{\Gamma(m/2+i)}{\Gamma(m/2+j)} {}_1F_1(m/2+i\,;\,m/2+j\,;\,\mu), \tag{5.8}$$

where $m = k_2$, and ${}_1F_1(a;b;x)$ is the confluent hypergeometric function defined by

$$ {}_1F_1(a;b;x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j\, j!} x^j . \tag{5.9} $$

For the transformed structural equation (3.1) to be identified it requires the following relationships to hold

$$ \bar{\pi}_{11}^* = \bar{\pi}_{21}^* \gamma^* + \bar{\beta}^* $$

$$ \bar{\pi}_{12}^* = \bar{\pi}_{22}^* \gamma^* , $$

from which it follows that $\bar{z}_1 = \gamma^* \bar{z}_2$. Inserting this result in (5.7) we get

$$ E(r_2^n) = e^{-\mu} \sum_{j=0}^{[n/2]} \sum_{u=0}^{[n/2]-j} \sum_{k=0}^{n-2j-2u} \sum_{v=0}^{[n/2-k/2]} \binom{n}{2j} \binom{n}{k} $$

$$ \cdot \frac{(2j)!(n-2j-2u+1)_{2u}(n-k-2v+1)_{2v}(n-2j-2u-k+1)_k}{j!\,u!\,v!\,2^{2u+2j+k+v}} \tag{5.10} $$

$$.(h_2'\bar{\pi}_{12}^*)^{n-2v}(h_2' h_2)^v (\gamma^*)^{2v-2j}(\mu)^{n-2j-u-k} $$

$$ .f(n-j-k-u-v\,;\,2n-2j-k-u-v). $$

Let n = 1, 2, 3, 4. Then from (5.10) we get the following expressions for the first four integer moments of r_2

$$ E(r_2) = e^{-\mu}\left[h_2'\mu\, f(1;2) + 1/2 h_2' f(0;1)\right]\bar{\pi}_{12}^* , \tag{5.11} $$

$$E(r_2^2) = e^{-\mu}\Big\{(h_2'\bar{\pi}_{12}^*)^2\mu^2 f(2;4) + \big[h_2'h_2\gamma^{*2}\mu^2 + 5/2(h_2'\bar{\pi}_{12}^*)^2\mu\big] f(1;3)$$

$$+1/2\big[(h_2'\bar{\pi}_{12}^*)^2 + h_2'h_2\gamma^{*2}\mu\big] f(0;2) \qquad (5.12)$$

$$+1/2(h_2'\bar{\pi}_{12}^*)^2\gamma^{*-2} f(1;2) + 1/2h_2'h_2 f(0;1)\Big\},$$

$$E(r_2^3) = e^{-\mu}[\ \eta_{36} f(3;6) + \eta_{25} f(2;5) + \eta_{14} f(1;4) + \eta_{03} f(0;3) \qquad (5.13)$$

$$+\eta_{24} f(2;4) + \eta_{13} f(1;3) + \eta_{02} f(0;2)\ ],$$

and

$$E(r_2^4) = e^{-\mu}[\ a_{48} f(4;8) + a_{37} f(3;7) + a_{26} f(2;6) + a_{15} f(1;5)$$

$$+a_{04} f(0;4) + a_{36} f(3;6) + a_{25} f(2;5) + a_{14} f(1;4) \qquad (5.14)$$

$$+a_{03} f(0;3) + a_{24} f(2;4) + a_{13} f(1;3) + a_{02} f(0;2)\ ],$$

where

$$\eta_{36} = (h_2'\bar{\pi}_{12}^*)^3\mu^3,$$

$$\eta_{25} = 3(h_2'\bar{\pi}_{12}^*)(h_2'h_2)\gamma^{*2}\mu^3 + 6(h_2'\bar{\pi}_{12}^*)^3\mu^2,$$

$$\eta_{14} = 18(h_2'\bar{\pi}_{12}^*)(h_2'h_2)\gamma^{*2}\mu + 27/4(h_2'\bar{\pi}_{12}^*)^3\mu,$$

$$\eta_{04} = 9/4(h_2'\bar{\pi}_{12}^*)(h_2'h_2)\gamma^{*2}\mu + 3/4(h_2'\bar{\pi}_{12}^*)^3,$$

$$\eta_{24} = 3/2(h_2'\bar{\pi}_{12}^*)^3\gamma^{*-2}\mu,$$

$$\eta_{13} = 9/2(h_2'\bar{\pi}_{12}^*)(h_2'h_2)\mu + 9/4(h_2'\bar{\pi}_{12}^*)^3\gamma^{*-2},$$

$$\eta_{02} = 9/8(h_2'\bar{\pi}_{12}^*)(h_2'h_2),$$

and

$$a_{48} = (h_2'\bar{\pi}_{12}^*)^4\mu^4,$$

$$a_{37} = 6(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\gamma^{*2}\mu^4 + 11(h_2'\bar{\pi}_{12}^*)^4\mu^3,$$

$$a_{26} = 3(h_2'h_2)^2\gamma^{*4}\mu^4 + 51/2(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\gamma^{*2}\mu^3 + 75/4(h_2'\bar{\pi}_{12}^*)^4\mu^2,$$

$$a_{15} = 9(h_2'h_2)^2\gamma^{*4}\mu^3 + 117/2(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\gamma^{*2}\mu^2 + 21(h_2'\bar{\pi}_{12}^*)^4\mu,$$

$$a_{04} = 9/4(h_2'h_2)^2\gamma^{*4}\mu^2 + 3/2(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\gamma^{*2}\mu + 3/2(h_2'\bar{\pi}_{12}^*)^4,$$

$$a_{36} = 3(h_2'\bar{\pi}_{12}^*)^4\gamma^{*-2}\mu^2,$$

$$a_{25} = 18(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\mu^2 + 27/2(h_2'\bar{\pi}_{12}^*)^4\gamma^{*-2}\mu,$$

$$a_{14} = 9(h_2'h_2)^2\gamma^{*2}\mu^2 + 45(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\mu + 9(h_2'\bar{\pi}_{12}^*)^4\gamma^{*-2},$$

$$a_{03} = 9/2(h_2'h_2)^2\gamma^{*2}\mu + 9(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2),$$

$$a_{24} = 3/4(h_2'\bar{\pi}_{12}^*)^4\gamma^{*-4},$$

$$a_{13} = 9/2(h_2'\bar{\pi}_{12}^*)^2(h_2'h_2)\gamma^{*-2},$$

$$a_{02} = -144(h_2'h_2)^2.$$

Since $r_2 = h_2'\hat{\pi}_{12}^*$, it follows from (5.ll) and (5.l2) that

$$E(\hat{\pi}_{12}^*) = e^{-\mu}\left[\mu f(1;2) + 1/2 f(0;1)\right]\bar{\pi}_{12}^*, \tag{5.15}$$

and

$$\begin{aligned} E(\hat{\pi}_{12}^*\hat{\pi}_{12}^{*\prime}) = e^{-\mu}\Big\{ & \bar{\pi}_{12}^*\bar{\pi}_{12}^{*\prime}\mu^2 f(2;4) + \left[\gamma^{*2}\mu^2 I + 5/2\bar{\pi}_{12}^*\bar{\pi}_{12}^{*\prime}\mu\right] f(1;3) \\ & + 1/2\left[\bar{\pi}_{12}^*\bar{\pi}_{12}^{*\prime} + \gamma^{*2}\mu I\right] f(0;2) \\ & + 1/2\bar{\pi}_{12}^*\bar{\pi}_{12}^{*\prime}\gamma^{*-2} f(1;2) + 1/2 I f(0;1)\Big\}, \end{aligned} \tag{5.16}$$

where I is a mxm unit matrix. The expression for the bias of $\hat{\bar{\pi}}^*_{12}$ is

$$E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12}) = \left\{e^{-\mu}\left[\mu\, f(1;2) + 1/2\, f(0;1)\right] - 1\right\}\bar{\pi}^*_{12}\,. \tag{5.17}$$

To get the expression for the mean squared error of $\hat{\bar{\pi}}^*_{12}$ we first write it as

$$E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' = E(\hat{\bar{\pi}}^*_{12}\,\hat{\bar{\pi}}^{*\prime}_{12}) - E(\hat{\bar{\pi}}^*_{12})\bar{\pi}^{*\prime}_{12} - \bar{\pi}^*_{12}E(\hat{\bar{\pi}}^{*\prime}_{12}) + \bar{\pi}^*_{12}\,\bar{\pi}^{*\prime}_{12}\,.$$

Hence, using (5.15) and (5.16) in the above relation, we get

$$\begin{aligned} &E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' \\ &\quad = e^{-\mu}\Big\{\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12}\mu^2\, f(2;4) + \left[\gamma^{*2}\mu^2 I + 5/2\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12}\mu\right] f(1;3) \\ &\quad + 1/2\left[\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12} + \gamma^{*2}\mu I\right] f(0;2) + \left[1/2\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12}\gamma^{*-2} - 2\mu\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12}\right] f(1;2) \\ &\quad + \left[1/2I - \bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12}\right] f(0;1)\Big\} + \bar{\pi}^*_{12}\,\bar{\pi}^{*\prime}_{12}\,. \end{aligned} \tag{5.18}$$

Equations (5.15), (5.16), and (5.18), upon translation of notation and application of the relations between associate confluent hypergeometric functions, are identical to equations (3.12), (4.38), and (4.39) in Nagar and Sahay (1978).

It is worth pointing out that while the frequency of appearance of the function $f(i;j)$ increases as we increase the order of moments, this will cause no undue difficulty for computational purposes. This is due to the fact that any confluent hypergeometric function can be expressed in terms of its two associate functions. Clearly, similar relations exist between associate $f(i;j)$ functions.Repeated application of these relations will reduce the computational burden drastically. In fact, using these relations, (5.18) may be written as

$$\begin{aligned} E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' &= 1/2(m+1)\gamma^{*2}I + 1/4\left[2\gamma^{*-2} - (m+1)\right]\mu^{-1}\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12} \\ &+ \Big\{1/8\left[(m^3 + m^2 - 2\gamma^{*-2})\mu^{-1} + (2m^2 - 6m + 4)\right]\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{12} \\ &+ \left[-1/4(m^2 + m - 2m\mu + 2\mu)\gamma^{*2} + 1/2\right]\; I\Big\} e^{-\mu} f(0;1). \end{aligned} \tag{5.19}$$

It is a well-known result that when the structural equation is just identified, the PRRF estimator of the corresponding reduced form coefficients is identical to the URRF estimator. Hence, the PRRF estimator is unbiased in the just identified case. To examine the dependence of the bias in the PRRF estimator on the degree of over identification, *m*-1, we rewrite (5.17) as

$$E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12}) = \left\{ e^{-\mu} \left[\frac{2}{m} \mu \, {}_1F_1(m/2+1\,;\, m/2+2;\mu) \right.\right.$$
$$\left.\left. + \frac{1}{m+2} \, {}_1F_1(m/2\,;\, m/2+1;\mu) \right] - 1 \right\} \bar{\pi}^*_{12} \qquad (5.20)$$
$$= \frac{1-m}{m} e^{-\mu} \, {}_1F_1(m/2\,;\, m/2+1;\mu) \bar{\pi}^*_{12},$$

where the second equality is obtained by using one of the relations between associate confluent hypergeometric functions.[2]

It is clear from (5.20) that unless the structural equation for y_1 is just identified ($m=1$), $\hat{\bar{\pi}}^*_{12}$ will be biased. However, the relative bias is bounded. To verify this let $\bar{\pi}^*_{12,i}$ and $\hat{\bar{\pi}}^*_{12,i}$ be the ith elements of $\bar{\pi}^*_{12}$ and $\hat{\bar{\pi}}^*_{12}$, respectively. We then have from (5.20)

$$\left| \frac{E(\hat{\bar{\pi}}^*_{12,i} - \bar{\pi}^*_{12,i})}{\bar{\pi}^*_{12,i}} \right| = \frac{m-1}{m} e^{-\mu} \, {}_1F_1(m/2\,;\, m/2+1;\mu), \qquad (5.21)$$

where $| \;\; |$ denotes absolute value. Provided: is not affected by the changes in *m*, the confluent hypergeometric function ${}_1F_1(m/2\,;\, m/2+1;\mu)$ is an increasing function of *m*. Thus, as *m* increases so does the size of the relative bias of the PRRF estimator. However, since[3]

$$\lim_{m \to \infty} e^{-\mu} \, {}_1F_1(m/2\,;\, m/2+1;\mu) = 1,$$

2 See SLATER (1960, p. 19).

3 *Op. Cit.*, p. 65.

we have

$$\lim_{m\to\infty} \left| \frac{E(\hat{\bar{\pi}}^*_{12,i} - \bar{\pi}^*_{12,i})}{\bar{\pi}^*_{12,i}} \right| = 1.$$

From (5.20) it is also easy to see that $\lim_{m\to\infty} E(\hat{\bar{\pi}}^*_{12,i}) = 0$. Hence, the mean of the PRRF estimator shrinks to zero as the degree of overidentification increases.

We now turn our attention to the moments of the PRRF estimator of coefficients of the original reduced form equation (2.2). From (B.12) and (B.13) in appendix B we get

$$\hat{\pi}_1 - \pi_1 = l_{11} J^{-1}(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1) + l_{21} J^{-1}(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2). \tag{5.22}$$

Hence

$$E(\hat{\pi}_1 - \pi_1) = l_{11} J^{-1} E(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1), \tag{5.23}$$

since $\tilde{\bar{\pi}}^*_2 = \bar{X}' y_2$ is an unbiased estimator of $\bar{\pi}^*_2$. Rewrite (5.23) in partitioned form as

$$\begin{aligned} E\begin{bmatrix} \hat{\pi}_{11} - \pi_{11} \\ \hat{\pi}_{12} - \pi_{12} \end{bmatrix} &= l_{11} J^{-1} E\begin{bmatrix} \hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11} \\ \hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12} \end{bmatrix} = l_{11} J^{-1} \begin{bmatrix} 0 \\ E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12}) \end{bmatrix} \\ &= \begin{bmatrix} -l_{11} J^{-1}_{11} J_{12} J^{-1}_{22} E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12}) \\ l_{11} J^{-1}_{22} E(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12}) \end{bmatrix}, \end{aligned} \tag{5.24}$$

where we have used the result that $\hat{\bar{\pi}}^*_{11}$ is an unbiased estimator of $\bar{\pi}^*_{11}$. From the above relation we get

$$E(\hat{\pi}_{11} - \pi_{11}) = -J^{-1}_{11} J_{12} E(\hat{\pi}_{12} - \pi_{12}), \tag{5.25}$$

which allows us to write (5.23) as

$$E(\hat{\pi}_1 - \pi_1) = \begin{bmatrix} -J_{11}^{-1}J_{12} \\ I \end{bmatrix} E(\hat{\pi}_{12} - \pi_{12}). \tag{5.26}$$

Since from (5.24) we get $E(\hat{\pi}_{12} - \pi_{12}) = l_{11}J_{22}^{-1}E(\hat{\bar{\pi}}_{12}^{*} - \bar{\pi}_{12}^{*})$, by substituting for $l_{11}, J_{11}, J_{12}, J_{22}$, and $\bar{\pi}_{12}^{*}$ from their definitions given in section 3 and appendix B, and using (5.17) and (5.20) we derive

$$\begin{aligned} E(\hat{\pi}_{12} - \pi_{12}) &= l_{11}J_{22}^{-1}E(\hat{\bar{\pi}}_{12}^{*} - \bar{\pi}_{12}^{*}) \\ &= (\omega_{11} - \omega_{12}^2/\omega_{22})^{1/2}(X_2'\bar{P}_{x_1}X_2)^{-1/2}E(\hat{\bar{\pi}}_{12}^{*} - \bar{\pi}_{12}^{*}) \\ &= (\omega_{11} - \omega_{12}^2/\omega_{22})^{1/2}(X_2'\bar{P}_{x_1}X_2)^{-1/2}\left\{e^{-\mu}\left[\mu f(1;2) + 1/2 f(0;1)\right] - 1\right\}\bar{\pi}_{12}^{*} \\ &= (\gamma - \omega_{12}/\omega_{22})\gamma^{-1}\left\{e^{-\mu}\left[\mu f(1;2) + 1/2 f(0;1)\right] - 1\right\}\pi_{12} \\ &= (\gamma - \omega_{12}/\omega_{22})\gamma^{-1}\left(\frac{1-m}{m}\right)e^{-\mu}\,{}_1F_1(m/2\,;\,m/2+1;\mu)\pi_{12}, \end{aligned} \tag{5.27}$$

and

$$\begin{aligned} E(\hat{\pi}_{11} - \pi_{11}) &= -J_{11}^{-1}J_{12}\,E(\hat{\pi}_{12} - \pi_{12}) \\ &= -(\omega_{11} - \omega_{12}^2/\omega_{22})^{1/2}(X_1'X_1)^{-1}(X_1'X_2)(X_2'\bar{P}_{x_1}X_2)^{-1/2} \\ &\quad \cdot\left\{e^{-\mu}\left[\mu f(1;2) + 1/2 f(0;1)\right] - 1\right\}\bar{\pi}_{12}^{*} \\ &= -(\gamma - \omega_{12}/\omega_{22})\gamma^{-1}\left\{e^{-\mu}\left[\mu f(1;2) + 1/2 f(0;1)\right] - 1\right\}(X_1'X_1)^{-1}(X_1'X_2)\pi_{12} \\ &= -(\gamma - \omega_{12}/\omega_{22})\gamma^{-1}\left(\frac{1-m}{m}\right)e^{-\mu}\,{}_1F_1(m/2\,;\,m/2+1;\mu)(X_1'X_1)^{-1}(X_1'X_2)\pi_{12}. \end{aligned} \tag{5.28}$$

So, while $\hat{\bar{\pi}}^*_{11}$ is an unbiased estimator of $\bar{\pi}^*_{11}$, as is evident from the above equation $\hat{\pi}_{11}$ is in general a biased estimator for π_{11}.

Relations (5.27) and (5.28) show that when the structural equation for y_1 is just identified, or when $\gamma = \omega_{12} / \omega_{22}$, the PRRF estimators $\hat{\pi}_{11}$ and $\hat{\pi}_{12}$ are unbiased. The latter condition also guarantees the unbiasedness of the two stage least squares estimator of y.[4] In addition, when X_1 and X_2 are orthogonal the PRRF estimator $\hat{\pi}_{11}$ will be unbiased.

To derive the exact mean squared error of the PRRF estimator $\hat{\pi}_1$, we see from (5.22) that

$$
\begin{aligned}
(\hat{\pi}_1 - \pi_1)(\hat{\pi}_1 - \pi_1)' &= J^{-1}\left[l_{11}(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1) + l_{21}(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)\right] \\
&\quad \cdot \left[l_{11}(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1) + l_{21}(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)\right]' J^{-1\prime} \qquad (5.29) \\
&= J^{-1}\left[l_{11}^2(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1)(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1)' + l_{11}l_{21}(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1)(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)'\right. \\
&\quad \left. + l_{11}l_{21}(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)(\hat{\bar{\pi}}^*_1 - \bar{\pi}^*_1)' + l_{21}^2(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)(\tilde{\bar{\pi}}^*_2 - \bar{\pi}^*_2)'\right] J^{-1\prime},
\end{aligned}
$$

hence

$$
\begin{aligned}
E(\hat{\pi}_1 - \pi_1)(\hat{\pi}_1 - \pi_1)' = J^{-1}\Bigg\{ & l_{11}^2 E\begin{bmatrix} (\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})(\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})' & (\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' \\ (\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})' & (\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' \end{bmatrix} \\
& + l_{11}l_{21} E\begin{bmatrix} (\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})(\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})' & (\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})(\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})' \\ (\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})' & (\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})(\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})' \end{bmatrix} \qquad (5.30) \\
& + l_{11}l_{21} E\begin{bmatrix} (\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})(\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})' & (\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' \\ (\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})(\hat{\bar{\pi}}^*_{11} - \bar{\pi}^*_{11})' & (\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})(\hat{\bar{\pi}}^*_{12} - \bar{\pi}^*_{12})' \end{bmatrix} \\
& + l_{21}^2 E\begin{bmatrix} (\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})(\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})' & (\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})(\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})' \\ (\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})(\tilde{\bar{\pi}}^*_{21} - \bar{\pi}^*_{21})' & (\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})(\tilde{\bar{\pi}}^*_{22} - \bar{\pi}^*_{22})' \end{bmatrix} \Bigg\} J^{-1\prime}.
\end{aligned}
$$

4 See, for example, SAWA (1972). This, of course, assumes the first four moments of the 2SLS estimator exists

It can be seen from (3.17), (3.18), (3.21), and (3.22) that

$$E(\hat{\pi}^*_{11}-\bar{\pi}^*_{11})(\hat{\pi}^*_{11}-\bar{\pi}^*_{11})'=I, \qquad E(\hat{\pi}^*_{11}-\bar{\pi}^*_{11})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})'=0,$$

$$E(\hat{\pi}^*_{11}-\bar{\pi}^*_{11})(\tilde{\bar{\pi}}^*_{21}-\bar{\pi}^*_{21})'=0, \qquad E(\hat{\pi}^*_{11}-\bar{\pi}^*_{11})(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})'=0,$$

$$E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\bar{\pi}}^*_{21}-\bar{\pi}^*_{21})'=0, \qquad E(\tilde{\bar{\pi}}^*_{21}-\bar{\pi}^*_{21})(\tilde{\bar{\pi}}^*_{21}-\bar{\pi}^*_{21})'=I,$$

$$E(\tilde{\bar{\pi}}^*_{21}-\bar{\pi}^*_{21})(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})'=0, \qquad E(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})'=I.$$

Substituting these results in to (5.30) yields

$$\begin{aligned} E(\hat{\pi}_1-\pi_1)(\hat{\pi}_1-\pi_1)' = J^{-1}\Bigg\{ & l_{11}^2\begin{bmatrix} I & 0 \\ 0 & E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})' \end{bmatrix} \\ & + l_{11}l_{21}\begin{bmatrix} 0 & 0 \\ 0 & E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})' \end{bmatrix} \\ & + l_{11}l_{21}\begin{bmatrix} 0 & 0 \\ 0 & E(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})' \end{bmatrix} + l_{21}^2\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\Bigg\} J^{-1\prime} \\ = {} & J^{-1}\begin{bmatrix} (l_{11}^2+l_{21}^2)I & 0 \\ 0 & G \end{bmatrix} J^{-1\prime}, \end{aligned} \tag{5.31}$$

where

$$\begin{aligned} G = {} & l_{11}^2 E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})' + l_{11}l_{21}E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})' \\ & + l_{11}l_{21}E(\tilde{\bar{\pi}}^*_{22}-\bar{\pi}^*_{22})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})' + l_{21}^2 I. \end{aligned} \tag{5.32}$$

The expression for the first term on the right side of the above relation is given in (5.19). Using the method of fractional calculus we obtain the following expression for the second term in (5.32)

$$E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\pi}^*_{22}-\bar{\pi}^*_{22})'=e^{-\mu}\left\{\left[\gamma^*\mu I+(1-\mu)\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{22}\right]f(1;2)\right.$$
$$\left.-1/2\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{22}f(0;1)+\mu\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{22}f(2;3)\right\} \qquad (5.33)$$
$$=\gamma^* I-m/2\mu^{-1}\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{22}-1/2\left\{\gamma^* I+(1-m-m/2\mu^{-1})\bar{\pi}^*_{12}\bar{\pi}^{*\prime}_{22}\right\}$$
$$\cdot e^{-\mu}f(0;1).$$

The expression (5.33) is symmetric since $\bar{\pi}^*_{12}=\gamma^*\bar{\pi}^*_{22}$. Therefore

$$E(\tilde{\pi}^*_{22}-\bar{\pi}^*_{22})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})'=E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\pi}^*_{22}-\bar{\pi}^*_{22})'.$$

Consequently (5.32) may be written as

$$G=l_{11}^2E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})'+2l_{11}l_{21}E(\hat{\pi}^*_{12}-\bar{\pi}^*_{12})(\tilde{\pi}^*_{22}-\bar{\pi}^*_{22})'+l_{21}^2I. \qquad (5.34)$$

Finally, since $\pi_1=(\pi_{11}' \;\; \pi_{12}')'$, it follows from (5.31) that

$$E(\hat{\pi}_{12}-\pi_{12})(\hat{\pi}_{12}-\pi_{12})'=J^{22}GJ^{22}$$
$$=(X_2'\bar{P}_{x_1}X_2)^{-1/2}G(X_2'\bar{P}_{x_1}X_2)^{-1/2},$$

$$E(\hat{\pi}_{11}-\pi_{11})(\hat{\pi}_{11}-\pi_{11})'=(l_{11}^2+l_{21}^2)J^{11^2}+J^{12}GJ^{12\prime}$$
$$=(X_1'X_1)^{-1}(X_1'X_2)[E(\hat{\pi}_{12}-\pi_{12})(\hat{\pi}_{12}-\pi_{12})'](X_2'X_1)(X_1'X_1)^{-1} \qquad (5.35)$$
$$+\omega_{11}(X_1'X_1)^{-1},$$

$$E(\hat{\pi}_{11}-\pi_{11})(\hat{\pi}_{12}-\pi_{12})'=J^{12}GJ^{22}$$
$$=-(X_1'X_1)^{-1}(X_1'X_2)E(\hat{\pi}_{12}-\pi_{12})(\hat{\pi}_{12}-\pi_{12})'.$$

Examination of the above expressions shows that the results obtained by Nagar and Sahay (1978) are valid only for the special case of transformed reduced form equation (3.2), and can not be used in a straightforward manner to find expressions for the exact moments of the PRRF estimators of coefficients of the original reduced form equation (2.2). This is in contrast to the case of estimators of coefficients of structural equations where expressions for exact moments of estimators of coefficients of the original structural equation are obtained quite easily from the corresponding expressions for the transformed structural equation.

6. THE EXACT BIAS AND MEAN SQUARED PREDICTION ERROR OF PRRF-BASED FORECASTS

The PRRF forecast of y_{1f}, the outside the sample value of y_1, is given by

$$\hat{y}_{1f} = x'_f \hat{\pi}_1,$$

where x'_f is the 1 x K row vector of outside the sample values of the exogenous variables, and $\hat{\pi}_1$ is the PRRF estimator of π_1.

Since

$$y_{1f} = x'_f \pi_1 + v_{1f},$$

we have

$$\hat{y}_{1f} - y_{1f} = x'_f (\hat{\pi}_1 - \pi_1) - v_{1f},$$

and the prediction bias is

$$\begin{aligned} E(\hat{y}_{1f} - y_{1f}) &= x'_f E(\hat{\pi}_1 - \pi_1) \\ &= (x'_{1f} \quad x'_{2f}) \begin{bmatrix} -(X'_1 X_1)^{-1}(X'_1 X_2) \\ I \end{bmatrix} E(\hat{\pi}_{12} - \pi_{12}) \\ &= [-x'_{1f}(X'_1 X_1)^{-1}(X'_1 X_2) + x'_{2f}]\, E(\hat{\pi}_{12} - \pi_{12}) \\ &= (\gamma - \omega_{12}/\omega_{22})\gamma^{-1}\left\{e^{-\mu}\left[\mu\, f(1;2) + 1/2\, f(0;1)\right] - 1\right\} \\ &\quad \cdot [-x'_{1f}(X'_1 X_1)^{-1}(X'_1 X_2) + x'_{2f}]\, \pi_{12}, \end{aligned} \tag{6.1}$$

where $x_f' = (x_{1f}' \;\; x_{2f}')$ is partitioned conformably with $(X_1 \;\; X_2)$, and use is made of the results given in (5.26), (5.27). Equation (6.1) is in general nonzero. However, when the structural equation is just identified the PRRF and URRF estimators are identical which in turn implies identical forecasts. Since it is a well-known result that the URRF forecasts are unbiased, equation (6.1) will be zero. Park (1982) has shown that for the special case where the values of the exogenous variables in the forecast period are equal to their sample means, the PRRF-based and URRF-based forecasts are equal. This also implies a zero value for (6.1).

For the mean squared prediction error (MSPE) we notice that

$$(\hat{y}_{1f} - y_{1f})^2 = x_f'(\hat{\pi}_1 - \pi_1)(\hat{\pi}_1 - \pi_1)'x_f - 2v_{1f}(\hat{\pi}_1 - \pi_1)'x_f + v_{1f}^2,$$

which leads to

$$MSPE = E(\hat{y}_{1f} - y_{1f})^2 = x_f' E(\hat{\pi}_1 - \pi_1)(\hat{\pi}_1 - \pi_1)'x_f + \omega_{11}.$$

Considering the first term on the right side, it is easily verified using the result in (5.35) that

$$E(\hat{\pi}_1 - \pi_1)(\hat{\pi}_1 - \pi_1)' = E\begin{bmatrix} (\hat{\pi}_{11} - \pi_{11})(\hat{\pi}_{11} - \pi_{11})' & (\hat{\pi}_{11} - \pi_{11})(\hat{\pi}_{12} - \pi_{12})' \\ (\hat{\pi}_{12} - \pi_{12})(\hat{\pi}_{11} - \pi_{11})' & (\hat{\pi}_{12} - \pi_{12})(\hat{\pi}_{12} - \pi_{12})' \end{bmatrix}$$

$$= \begin{bmatrix} -(X_1'X_1)^{-1}(X_1'X_2) \\ I \end{bmatrix} E(\hat{\pi}_{12} - \pi_{12})(\hat{\pi}_{12} - \pi_{12})'$$

$$\cdot \begin{bmatrix} -(X_2'X_1)(X_1'X_1)^{-1} & I \end{bmatrix} + \begin{bmatrix} \omega_{11}(X_1'X_1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, we have

$$\begin{aligned} E(\hat{y}_{1f} - y_{1f})^2 = {} & [-x_{1f}'(X_1'X_1)^{-1}X_1'X_2 + x_{2f}'] \, [E(\hat{\pi}_{12} - \pi_{12})(\hat{\pi}_{12} - \pi_{12})'] \\ & \cdot [-X_2'X_1(X_1'X_1)^{-1}x_{1f} + x_{2f}] + \omega_{11}[1 + x_{1f}'(X_1'X_1)^{-1}x_{1f}]. \end{aligned} \tag{6.2}$$

The MSPE of $\tilde{y}_{1f} = x_f' \tilde{\pi}_1$, the forecast obtained using the unrestricted reduced form estimator $\tilde{\pi}_1$, is the well known result

$$E(\tilde{y}_{1f} - y_{1f})^2 = \omega_{11}[1 + x_f'(X'X)^{-1} x_f],$$

which can be easily shown to be equivalent to

$$\begin{aligned} E(\tilde{y}_{1f} - y_{1f})^2 = {} & [-x_{1f}'(X_1'X_1)^{-1}X_1'X_2 + x_{2f}']\,[E(\tilde{\pi}_{12} - \pi_{12})(\tilde{\pi}_{12} - \pi_{12})'] \\ & \cdot [-X_2'X_1(X_1'X_1)^{-1}x_{1f} + x_{2f}] + \omega_{11}[1 + x_{1f}'(X_1'X_1)^{-1}x_{1f}]. \end{aligned} \tag{6.3}$$

Subtracting (6.3) from (6.2) yields

$$\begin{aligned} E(\hat{y}_{1f} - y_{1f})^2 - E(\tilde{y}_{1f} - y_{1f})^2 = {} & [-x_{1f}'(X_1'X_1)^{-1}X_1'X_2 + x_{2f}'] \\ & \cdot [E(\hat{\pi}_{12} - \pi_{12})(\hat{\pi}_{12} - \pi_{12})' - E(\tilde{\pi}_{12} - \pi_{12})(\tilde{\pi}_{12} - \pi_{12})'] \\ & \cdot [-X_2'X_1(X_1'X_1)^{-1}x_{1f} + x_{2f}]. \end{aligned}$$

It is clear from the above equation that if the PRRF estimator of π_{12} has smaller MSE than its URRF estimator, the PRRF-based forecast will have smaller MSPE than the URRF-based forecast.

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