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Shocks in asymmetric one-dimensional exclusion processes

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Abstract: We review recent results concerning the local structure of the shocks in the one dimensional nearest neighbors totally asymmetric simple exclusion process. A microscopic shock is a random position X_t such that the system as seen from this position at time t has a stationary distribution which is equivalent to the product measure with densities ρ and λ to the left and right of the origin respectively. The diffusion coefficient of the shock $D = lim_{t\to\infty} t^{-1}(E(X_t)^2 - (EX_t)^2)$ has been found to be $D = (\lambda-\rho)^{-1}(\rho(1-\rho)+\lambda(1-\lambda))$. In the scale $\sqrt{t}$ the position of X_t is determined by the initial distribution of particles in a region of lenght proportional to t. The distribution of the process at the average position of the shock converges to a fair mixture of the product measures with densities ρ and λ. This is the so called dynamical phase transition. Under shock initial conditions the density fluctuation fields depend on the initial configuration. The results are a little weaker in the asymmetric case, when jumps to the left are also allowed.

Key words: Asymmetric simple exclusion. Shock fluctuations. Central limit theorem. Dynamical phase transition. Density fluctuation fields.

1. Introduction.

We study one dimensional lattice gas type systems. These systems are conservative (particles do not die or are created). We concentrate on the simple exclusion process but the results hold for the analogous cellular automata models, in particular for the so called Boghosian Levermore (1987) cellular automaton and some cases of the sand piles introduced by Bak et al. (1988).

In this paper we review work related to the microscopic formation of shock waves, the diffusive behavior of the shocks, the existence of a dynamical phase transition at the average position of the shock and the behavior of the fluctuation fields.

The simple exclusion process can be described in the following way. At most one particle is allowed at each site $x \in \mathbb{Z}$. Each particle has an internal clock that rings after a random time with exponential distribution of rate 1. As the clock rings for the particle sitting at x and if $x+1$ is empty, then the particle jumps from x to $x+1$. Then the internal clock is reset for the next jump. All particles do the same independently.

The generator of the asymmetric simple exclusion process is given by

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \eta(x)(1-\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)],$$

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where f is a continuous function on $\mathbf{X} = \{0,1\}^{\mathbb{Z}}$, the configuration $\eta^{x,y}(z)$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ \eta(x) & \text{if } z = y \\ \eta(y) & \text{if } z = x \end{cases}$$

Let $S(t)$ denote the corresponding semigroup. The set of extremal invariant measures is given by the union of two families: $\{\nu_\rho : 0 \le \rho \le 1\} \cup \{\nu^{(n)}, n \in \mathbb{Z}\}$, where ν_ρ is the product measures with parameter ρ and $\nu^{(n)}$ is a shock measure giving mass one to the configuration $\eta^{(n)}(x) = 1\{x \ge n\}$.

We fix $0 < \rho < \lambda < 1$ and consider as initial measure $\nu_{\rho,\lambda}$, the product measure with densities ρ and λ to the left and right of the origin respectively.

This model is related to the Burger's equation for $u(r,t) \in [0,1]$:

$$\frac{\partial u}{\partial t} + \frac{\partial[u(1-u)]}{\partial r} = 0 \tag{1.1}$$

We restrict ourselves to the case of non decreasing initial conditions that present only one shock: the initial condition $u(r,0) = u_0(r)$ is λ to the right of the origin and ρ to its left. The (weak) solution of this equation with this initial condition is $u(r,t) = u_0(r - vt)$, where $v = 1 - \rho - \lambda$ is the velocity of the shock. The characteristics of this equation are given by $(1-2u)$. Since for increasing initial conditions the characteristics to the right of the origin are smaller than the characteristics to the left of it, they conflict and give rise to the shock that is travelling at velocity $(1 - \lambda - \rho)$.

Theorem 1 (Hydrodinamical limit): Let $u_0(r)$ be a piecewise continuous function, and let $\nu^\varepsilon_{u_0}$ be a family of product measures with marginals $\nu^\varepsilon_{u_0}(\eta(\varepsilon^{-1}r)) = u_0(r)$. Then

$$\lim_{\varepsilon \to 0} \nu^\varepsilon_{u_0} S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} = \nu_{u(r,t)} \tag{1.2}$$

in the continuity points of $u(r,t)$, the solution of (1.1) with initial condition $u(r,0) = u_0(r)$.

Let $X(t)$ be the position of a "second class particle". Its motion is determined by the following rules: the second class particle also has an internal exponential clock of rate 1 and jumps to empty sites as the other particles do, but when one of the other particles attempts to jump over the second class particle, the jump is realized so that the second class particle and the other particle interchange positions. Let τ_y be translation by y: $\tau_y\eta(x) = \eta(x+y)$. The process $\eta'_t = \tau_{X(t)}\eta_t$ is Markovian and has generator given by

$$\begin{aligned} L'f(\eta) = \sum_{x \neq 0} & \eta(x)(1 - \eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \\ & + \eta(-1)[f(\tau_{-1}\eta^{0,-1}) - f(\eta)] \\ & + (1 - \eta(1))[f(\tau_1\eta^{0,1}) - f(\eta)] \end{aligned}$$

Theorem 2 (Microscopic interface): The process as seen from the second class particle –with generator L'– has an invariant measure μ' which is equivalent to the product measure $\nu_{\rho,\lambda}$.

Remarks: The measure μ' is explicitly described in Section 3.

The following heuristics justify the choice of a second class particle for a microscopic shock. If we start with a measure with densities ρ and λ to the left and right of the second class particle respectively and this densities stay thru time, the velocity of this particle equals the rate of jumping to the right $(1-\lambda)$ minus the rate of jumping to the left (ρ). This gives $(1-\rho-\lambda)$ that is the right macroscopic velocity of the shock in the Burger's equation. The next theorem says that indeed this is the average and asymptotic behavior of the second class particle. We use P and E for the probability and expectation related to the process with initial distribution either $\nu_{\rho,\lambda}$ or μ'. The statements below are limit statements and hold for both initial distributions.

Theorem 3: Assume that the process η_t has initial distribution μ'. Let X_t be the position of the shock given by a second class particle initially put at the origin. Then

$$EX_t = (1-\lambda-\rho)t. \tag{1.3}$$

Law of large numbers:

$$\lim_{t\to\infty} \frac{X_t}{t} = (1-\lambda-\rho)t \quad P\ a.s. \tag{1.4}$$

The next theorem gives the asymptotic variance of the second class particle. Moreover it establishes that its fluctuation is given by the initial configuration.

Theorem 4: Diffusion coefficient:

$$D := \lim_{t\to\infty} \frac{E(X_t)^2 - (EX_t)^2}{t} = \frac{\rho(1-\rho)+\lambda(1-\lambda)}{\lambda-\rho} \tag{1.5}$$

(Dependence on the initial configuration.) Let

$$N_t(\eta) = \sum_{x=0}^{(\lambda-\rho)t} (1-\eta(x)) - \sum_{x=-(\lambda-\rho)t}^{0} \eta(x).$$

Then

$$\lim_{t\to\infty} \frac{1}{t} E[X_t - (\lambda-\rho)^{-1} N_t(\eta_0)]^2 = 0. \tag{1.6}$$

The following Theorem is a corollary to (1.6).

Theorem 5: Convergence to the finite dimensional distributions of Brownian motion. Let $W(t)$ be Brownian motion with diffusion coefficient D. Then under the conditions of Theorem 1.1

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}(Z_{\varepsilon^{-1}.} - EZ_{\varepsilon^{-1}.}) = W(.) \tag{1.7}$$

weakly, in the sense of the finite dimensional distributions.

In the shock case the hydrodynamical limit (1.2) means that under initial distribution $\nu_{\rho,\lambda}$, a traveller moving at deterministic velocity r observes asymptotically that the particles are distributed as ν_ρ for $r > v$ and ν_λ for $r < v$, where $v = (1 - \lambda - \rho)$. Indeed $u(r,t) = \rho 1\{r < vt\} + \lambda 1\{r > vt\}$ is the solution of the Burgers equation when $u_0(r) = \lambda$ for $r > 0$ and ρ for $r \le 0$. When $r = v$ the system converges to a fair mixture of ν_ρ and ν_λ. The next result is based on the central limit theorem for X_t established in Theorem 5. Let $w(r,t) = P(W(t) \le r) = (1/\sqrt{2\pi Dt})\int_{-\infty}^r \exp(-s^2/2Dt)ds$, the normal distribution with variance Dt. Theorem 2 and (1.7) suggest that at the average velocity of the shock one would see a mixture of ν_ρ and ν_λ. This is one of the consequences of the next result.

Theorem 6: Dynamical phase transition. Let $v = (1 - \lambda - \rho)$. Then

$$\lim_{\varepsilon\to 0} \nu_{\rho,\lambda} S(t)\tau_{vt+at^{1/2}} = (1 - w(a,1))\nu_\rho + w(a,1)\nu_\lambda \tag{1.8}$$

Let Υ_t^ε be the fluctuations fields defined by

$$\Upsilon_t^\varepsilon(\Phi) = \varepsilon^{1/2} \sum_{x\in\mathbb{Z}} \Phi(\varepsilon x)[\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)], \tag{1.9}$$

for smooth integrable test functions Φ. For $t = 0$, if η_0 is distributed according to $\nu_{\rho,\lambda}$,

$$\lim_{\varepsilon\to 0} \Upsilon^\varepsilon(\Phi) = \Upsilon(\Phi), \tag{1.10}$$

where $\Upsilon(\Phi)$ is Gaussian white noise with mean zero and covariance

$$E(\Upsilon(\Psi)\Upsilon(\Phi)) = \int u_0(r)(1 - u_0(r))\Psi(r)\Phi(r)dr. \tag{1.11}$$

where $u_0(r) = \lambda 1\{r \ge 0\} + \rho 1\{r < 0\}$.

Theorem 7: Convergence of the fluctuation fields. Assume that the initial distribution of the process is $\nu_{\rho,\lambda}$. Let $v = (1-\rho-\lambda)$. Let $u(r,t) = \lambda 1\{r > vt\} + \rho 1\{r < vt\} + \frac{1}{2}(\lambda+\rho)1\{r = vt\}$. As $\varepsilon \to 0$, the fluctuation fields Υ_t^ε defined in (1.9) converge in a weak sense to the conservative solution Υ_t of the nonhomogeneous linear equation

$$\frac{\partial}{\partial t}\Upsilon_t(r) = \frac{\partial}{\partial r}(1 - 2u(r,t))\Upsilon_t(r), \tag{1.12}$$

with initial condition Υ, *the Gaussian field with zero mean and covariance given by (1.11).*

Theorem 7 is a consequence of the L_2 convergence of the fluctuation fields established in the next theorem. The weak solutions of (1.12) present a singularity at the point (vt, t) due to the discontinuity of $u(r,t)$ at $r = vt$. For this reason there is no unique solution. However there is only one conservative solution. To better describe it let us introduce some notation. Assume that Φ is the indicator of the interval (a_1, a_2). For $i = 1, 2$ let

$$b_i(t) = \begin{cases} a_i - (1-2\rho)t & \text{if } a_i < vt \\ a_i - (1-2\lambda)t & \text{if } a_i > vt \end{cases}$$

Then Υ_t, the solution of (1.12) is given by the following.

$$\int \Upsilon_t(r)\Phi(r)dr = \int_{a_1}^{a_2} \Upsilon_t(r)dr = \int_{b_1(t)}^{b_2(t)} \Upsilon_0(r)dr$$

We can interpret this by saying that if $vt \in (a_1, a_2)$ then, the fluctuations present in the interval $(-(\lambda-\rho)t, (\lambda-\rho)t)$ at time zero concentrate in the point vt at time t. Formula (1.14) below says that these fluctuations are present in the scale $\sqrt{t}$. Indeed they reflect the shock fluctuations that occur in this scale.

Theorem 8: Let $A_\varepsilon = \mathbb{Z} \cap (\varepsilon^{-1}a_1, \varepsilon^{-1}a_2)$, $B_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}b_1(t), \varepsilon^{-1}b_2(t))$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon E\left(\sum_{x\in A_\varepsilon} [\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)] - \sum_{x\in B_\varepsilon(t)} (\eta_0(x) - E\eta_0(x))\right)^2 = 0. \quad (1.13)$$

Let $c > 0$, $C_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}vt - \varepsilon^{-1/2}c, \varepsilon^{-1}vt + \varepsilon^{-1/2}c)$ *and* $K_\varepsilon(t) = \mathbb{Z} \cap (-\varepsilon^{-1}t(\lambda-\rho), \varepsilon^{-1}t(\lambda-\rho))$. *Then*

$$\lim_{\varepsilon\to 0} \varepsilon E\Big(\sum_{x\in C_\varepsilon(t)} [\eta_{\varepsilon^{-1}t}(x) - E\eta_{\varepsilon^{-1}t}(x)]$$

$$-T_{\varepsilon^{-1/2}c} \sum_{x\in K_\varepsilon(t)} (\eta_0(x) - E\eta_0(x))\Big)^2 = 0, \quad (1.14)$$

where T_c *is truncation by* c:

$$T_cF(.) = \begin{cases} F(.) & \text{if } |F(.)| \le c \\ c & \text{if } F(.) > c \\ -c & \text{if } F(.) < -c. \end{cases}$$

Note that $C_\varepsilon(t)$ is an interval of length proportional to $\varepsilon^{-1/2}$ around the macroscopic point vt. When $c \to \infty$, (1.14) says that the fluctuations at time t in a region of length proportional to $\sqrt{t}$ around vt are given by the fluctuations at time 0 in a region of length proportional to t.

2. Graphical construction and coupling.

The main tool to show the above results is to couple the joint realization of two versions of the process with different initial configurations. One way to define a coupling is via the joint generator (Liggett (1976), (1985)). Another way is by graphically constructing the process. This is something like to use the same random numbers for different initial configurations. In order to describe the invariant measure μ' for the process as seen from the second class particle it is useful to describe the graphical construction and the coupling.

To describe the graphical construction attach a rate 1 Poisson processes to each pair of sites $(x, x+1)$. A Poisson process is a sequence of random times. To each of these times draw an arrow from x to $x+1$. The product of these Poisson processes induces a probability space $(\Omega,\ \mathcal{F},\ P)$. We discard the null event "two arrows occur at the same time". Given an initial configuration η and a set of arrows ω, the configuration at time t starting from η is denoted $\eta_t^{\eta,\omega}$ and is constructed in the following way. When an arrow appears from site x to y, if there is a particle at x and no particle at y then, after the arrow the particle will be at y and x will be empty. We denote η_t^η the random process defined on $(\Omega,\ \mathcal{F},\ P)$ with initial configuration η.

Consider now two initial configurations η^0 and η^1 and write $\eta_t^i = \eta_t^{\eta^i}$, for the configurations at time t. Use the same structure of arrows for η_t^0 and η_t^1. In this case (η_t^0, η_t^1) is the "basic coupling"(Liggett (1985)). If $\eta^0(x) \le \eta^1(x)$ for all $x \in \mathbb{Z}$ (in this case we say $\eta^0 \le \eta^1$) then for all times $\eta_t^0 \le \eta_t^1$. This property is called attractivity. Let ν_ρ be the product measure with density ρ. Take $\rho < \lambda$ and realize jointly the measures ν_ρ and ν_λ in the following way. Let $U(x) \in [0,1]$ be i.i.d. uniformly distributed random variables. Then define $\eta^0(x) = 1\{U(x) \le \rho\}$, $\eta^1(x) = 1\{U(x) \le \lambda\}$. Hence, η^0 is distributed according to ν_ρ, η^1 is distributed according to ν_λ and $\eta^0 \le \eta^1$. Define $\sigma(x) = \eta^0(x)$ and $\xi(x) = \eta^1(x) - \eta^0(x)$. We say that the distribution of (σ, ξ) has the good marginals if the σ marginal is ν_ρ and the $\sigma + \xi$ marginal is ν_λ. Calling π_2 the distribution of (σ, ξ), we have that

$$\pi_2 \text{ is a product measure with the good marginals.} \tag{2.1}$$

Define $\sigma_t(x) = \eta_t^0(x)$ and $\xi_t(x) = \eta_t^1(x) - \eta_t^0(x)$. The motion of (σ_t, ξ_t) obeys the following rule. The σ particles have priority over the ξ particles: when an arrow from a σ particle to a ξ particle appears, then after the arrow the particles interchange positions. Otherwise the particles interact by exclusion. We say that the ξ particles behave as "second class particles". If the distribution of (σ_0, ξ_0) has the good marginals, the same is true for the distribution of (σ_t, ξ_t). We call $S_2(t)$ the corresponding semigroup.

Let ν_2 be a translation invariant measure with the good marginals and $\nu_2' = \nu_2(.|\xi(0) = 1)$. Let X_t be the position of the ξ particle initially at the origin.

Let $S_2'(t)$ be the semigroup of the process as seen from the second class particle $(\tau_{X_t}\sigma_t, \tau_{X_t}\xi_t)$. The key tool in Ferrari, Kipnis and Saada (1991) to show that X_t is a microscopic shock is the following. If ν_2 is translation invariant and has the good marginals, then

$$(\nu_2 S_2(t))' = \nu_2' S_2'(t) \tag{2.2}$$

In words, the law of the process as seen from the tagged second class particle looks as the law of the process seen from the origin conditioned to have a second class particle at the origin. Let ν_2 have the good marginals, then under initial measure ν_2',

$$\lim_{t\to\infty} \frac{X_t}{t} = v \quad \text{almost surely.} \tag{2.3}$$

Using the same arrows there is a natural coupling between (σ_t, ξ_t) with initial measure π_2' and η_t with initial measure $\nu_{\rho,\lambda}$. To describe it take (σ, ξ) from the distribution π_2'. Now mark independently the i-th ξ particle as γ with probability $(p/q)^i/(1+(p/q)^i)$, otherwise as ζ. Then consider the process $(\sigma_t, \gamma_t, \zeta_t)$ with priorities σ over γ over ζ. In this way σ_t has distribution ν_ρ for all t, $\eta_t = \sigma_t + \gamma_t$ has distribution $\nu_{\rho,\lambda}S(t)$ and $\sigma_t + \gamma_t + \zeta_t$ has distribution ν_λ.

3. Description of the invariant shock measure as seen from the second class particle.

To describe the measure μ' one needs to use the joint process (σ_t, ξ_t). Let μ_2' be the measure on $\mathbf{X}^2$ described as follows. Let $\mathbf{Y}$ be the space of finite configurations of 0's and 1's, i.e.

$$\mathbf{Y} = \cup_{n\geq 0}\{0,1\}^n = \{\emptyset, 0, 1, 00, 01, 10, 11, 000, \ldots\}$$

Let the number of sites of $\zeta \in Y$ be $N(\zeta) = n$ if and only if $\zeta \in \{0,1\}^n$ and the number of particles of ζ be $K(\zeta) = \sum_{x=1}^{N(\zeta)} \zeta(x)$. Let $M(\zeta)$ = number of different configurations that can be obtained from ζ by translating ones to the right (including ζ). Examples: $M(100) = 3, M(1010) = 5, M(000111) = 1$, etc. Let $\{\zeta_i\}_{i\in\mathbf{Z}} \subset \mathbf{Y}$ be a double infinite iid sequence of finite configurations with distribution

$$P(\zeta_i = \zeta) = \lambda(1-\rho)M(\zeta)(\lambda\rho)^{K(\zeta)}((1-\lambda)(1-\rho))^{N(\zeta)-K(\zeta)}. \tag{3.1}$$

Indeed, it holds

$$\sum M(\zeta) = \frac{1}{k+1}\binom{n}{k}\binom{n+1}{k},$$

where the sum is running on $\{\zeta : N(\zeta) = n, K(\zeta) = k\}$. Notice that given the number of sites and the number of ones of a configuration, the relative weight of the configuration does not depend on λ and ρ.

To define the two classes measure, conditioned to have a second class particle at the origin, display the ζ_i's on the integers separated by the second class particles.

More rigorously, let $N_i = N(\zeta_i) + 1$ and $S_i = \sum_{j=0}^{i-1} N_j$. Let $I(x) = i$ if and only if $S_i \leq x < S_{i+1}$. Define $\sigma(0) = 0, \xi(0) = 1$ and for $x > 0$,

$$\sigma(x) = \begin{cases} \zeta_{I(x)}(x - S_{I(x)}) & \text{if } S_{I(x)} < x < S_{I(x)+1} \\ 0 & \text{if } x = S_{I(x)} \end{cases}$$

$$\xi(x) = \begin{cases} 1 & \text{if } x = S_{I(x)} \\ 0 & \text{otherwise} \end{cases}$$

Define in an analogous form $\sigma(x)$ and $\xi(x)$ for $x < 0$. Call μ_2' the resulting distribution.

Theorem: If $\{\zeta_i\}$ are chosen with distribution (3.1), then μ_2' is invariant for the process $(\tau_{X(t)}\sigma_t, \tau_{X(t)}\xi_t)$. Furthermore, defining μ_2 as the unique translation invariant measure satisfying $\mu_2(.|\xi(0) = 1) = \mu_2'(.)$, it holds that the σ marginal of μ_2 is ν_ρ while the $\sigma + \xi$ marginal of μ_2 is ν_λ.

Let T be the following operator on $\mathbf{X}$: $T\eta(x) = \eta(x)1\{x \geq 0\}$. Define

$$\int d\mu'(\eta)f(\eta) = \int d\mu_2'(\sigma, \xi)(f(\sigma + T\xi))$$

Theorem: The measure μ' is equivalent to $\nu_{\rho,\lambda}$. Furthermore μ' is invariant for η_t', the process as seen from the second class particle.

4. Tagged second class particles and currents.

In this section we give the key results for the proof of Theorem 4. The main point is the relationship between the variance of the current of second class particles and the variance of the tagged second class particle. Consider the joint process (σ_t, ξ_t) described in the previous section. Define the current of ξ particles as $J_{2,t} :=$ number of ξ particles to the left of the origin at time 0 and to the right of the origin at time t minus number of ξ particles to the right of the origin at time 0 and to the left of the origin at time t. Analogously define $J_{0,t}$ for the current of σ particles and and $J_{1,t}$ for the total current of $\sigma + \xi$ particles.

Consider a configuration (σ, ξ) taken from π_2', the measure π_2 conditioned to have a ξ particle at the origin. This configuration has $\xi(0) = 1$ and $\sigma(0) = 0$, *i.e.*, it has a ξ particle at the origin. Let $\sigma^*(x) = 1\{x \neq 0\}\sigma(x) + 1\{x = 0\}(1 - \sigma(x))$ and analogously ξ^*. Now, using the same arrows, couple (σ_t, ξ_t) with (σ_t, ξ_t^*). At time t the two processes will differ at only one site whose position is called R_t. Similarly, coupling (σ_t, ξ_t) with (σ_t^*, ξ_t^*) we get only one discrepancy located at a position denoted $\bar{R}_t$. In words, R_t is like a third class particle, while $\bar{R}_t$ is a second class particle with respect to σ_t but has priority over ξ_t.

Theorem 3.1: Let (σ_t, ξ_t) be the joint process of first and second class particles with initial product measure π_2 defined in (2.1). Let X_t be the position of the tagged second class particle put initially at the origin. Then it holds that

$$EJ_{2,t} = (\lambda - \rho)EX_t \tag{3.1}$$

where the expected values are taken with respect to the process with initial distribution π_2. Furthermore, denoting the variance by V,

$$\begin{aligned} VJ_{2,t} = (\lambda-\rho)^2 VX_t - (\lambda-\rho)(1-(\lambda-\rho))E(X_t) \\ + 2(\lambda-\rho)(1-\lambda)(E(R_t)^+ - E(R_t - X_t)^+) \\ + 2(\lambda-\rho)\rho(E(\bar{R}_t)^+ - E(\bar{R}_t - X_t)^+). \end{aligned} \tag{3.2}$$

Theorem 3.2: Under the conditions of Theorem 3.1, it holds that

$$\lim_{t\to\infty} \frac{1}{t} E(J_{2,t} - N_{2,t}(\sigma_0, \xi_0) - (\lambda^2 - \rho^2)t)^2 = 0, \tag{3.3}$$

where $N_{2,t}(\sigma, \xi)$ is a random variable that does not depend on ω. It depends only on the initial configurations σ and ξ and it is given below by (3.13).

The proof of Theorem 3.2 is based on the computation of the variance of the current.

Notes and references.

The simple exclusion process has been introduced by Spitzer (1970). The set of invariant measures was described by Liggett (1976), (1985).

The hydrodynamical limit of Theorem 1 has been proven first by Liggett (1975, 1977) for the case $r = 0$, then by Rost (1982) and Benassi and Fouque (1987) for decreasing one step profiles and by Andjel and Vares (1987) for increasing ones. Benassi, Fouque, Saada and Vares (1991) computed the limit for monotone initial profiles. Fouque (1991) reviews these approachs. For general initial conditions (1.2) is a consequence of the law of large numbers of Rezakhanlou (1990) and the proof of local equilibrium of Landim (1992).

A microscopic shock as the one described in Theorem 2 appears first in Liggett's (1976) blocking measures for the case $\rho = 0$ and $\lambda = 1$. The case $\rho = 0$, $\lambda < 1$ was studied by Ferrari (1986). In the general case, the existence of the microscopic shock was simulated by Boldrighini, Cosimi, Frigio and Nunes (1989) and proven by Ferrari, Kipnis and Saada (1991) in a weaker form, as the process as seen from the shock is not Markovian and the invariant measure μ' is proven to behave asymptotically as $\nu_{\rho,\lambda}$. Ferrari (1992) shows that a isolated second class particle describes the microscopic shock. Derrida, Janowsky, Lebowitz and Speer (1993) compute the invariant measure μ'. Based on this computation Ferrari, Fontes and Kohayakawa (1993) described μ' as is given in Section 3 of this paper. From this description the equivalence between μ' and $\nu_{\rho,\lambda}$ and the properties of μ' follow.

In chapter 5 of Spohn (1991) (1.3) was proven and (1.2) conjectured. Boldrighini et al. (1989) performed computer simulations confirming (1.5). Gärtner and Presutti (1989) showed (1.6) for $\rho = 0$ and $p = 1$. Ferrari (1992) showed the law of large numbers (1.4), the equivalence between (1.5) and (1.6) and that the right hand side of (1.5) is a lowerbound for D. Ferrari and Fontes (1993b) show

(1.3) and (1.5) using a relationship between the expected value and the variance of a tagged particle with the variance of the current of particles through a fixed or travelling position established in Ferrari and Fontes (1993a).

Bramson (1988), Lebowitz, Presutti and Spohn (1988) and Spohn (1991) reviewed some of the results.

Related results concerning the behavior of a tagged particle for an equilibrium system starting with the invariant measure ν'_ρ are the following. Kipnis (1986) proved a central limit theorem and law of large numbers for the position of the tagged particle. De Masi and Ferrari (1985) computed the variance of the limiting Gaussian distribution. Ferrari and Fontes (1993c) showed that the position of the tagged particle is given by a Poisson process or rate $(1-\rho)$ plus a perturbation of order 1.

Theorems 3 to 8 are proven by Ferrari and Fontes (1993b). Theorem 4 was conjectured by Spohn and proven in a weaker form by Gärtner and Presutti (1989) for $\rho = 0$.

The dynamical phase transition of Theorem 6 was proven by Wick (1985) and De Masi et al. (1988) for $\rho = 0$ and by Andjel, Bramson and Liggett (1988) for $\lambda + \rho = 1$. Theorem 7 was proven by Benassi and Fouque (1992) for functions depending on regions away from the shock.

References

abl E. D. Andjel, M. Bramson, T. M. Liggett , *Shocks in the asymmetric simple exclusion process*, Probab. Theor. Rel. Field, **78** (1988), 231-247.

av E. D. Andjel, M. E. Vares *Hydrodynamic equations for attractive particle systems on* $\mathbb{Z}$. J. Stat. Phys. **47**(1987) 265-288.

btw P. Bak, Ch. Tang, K. Wiesenfeld *.Self-Organized criticallity.* Phys. Rev. **A, 38 (1)** 364-373, (1988)

bf1 A. Benassi, J-P. Fouque . *Hydrodynamical limit for the asymmetric simple exclusion process.* Ann. Probab. **15** 546-560,(1987).

bf2 A. Benassi, J-P. Fouque *Fluctuation field for the asymmetric simple exclusion process.* Proceedings of Oberwolfach Conference in SPDE, Nov 89, Birkhauser,(1992).

bfsv A. Benassi, J-P. Fouque, E. Saada, M. E. Vares *Asymmetric attractive particle systems on Z: hydrodynamical limit for monotone initial profiles.* J. Stat. Phys., (1991).

bl B. M. Boghosian, C. D. Levermore *.A cellular automaton for Burgers' equation.* Complex Systems **1** 17-30,(1987).

bcfg C. Boldrighini, C. Cosimi, A. Frigio, M. Grasso-Nunes . *Computer simulations of shock waves in completely asymmetric simple exclusion process.* J. Stat. Phys. **55,** 611-623,(1989).

br M. Bramson *Front propagation in certain one dimensional exclusion models.* J. Stat. Phys. **51,** 863-869,(1988).

cls Z. Cheng, J. L. Lebowitz, E. R. Speer .*Microscopic shock structure in model particle systems: the Boghosian Levermore revisited.* Preprint,(1990).

df A. De Masi, P. A. Ferrari *Self diffusion in one dimensional lattice gases in the presence of an external field.* J. Stat. Phys. **38,** 603-613,(1985).

dfv A. De Masi, P. A. Ferrari, M. E. Vares *A microscopic model of interface related to the Burgers equation.* J. Stat. Phys. **55**, 3/4 601-609,(1989)

dkps A. De Masi, C. Kipnis, E. Presutti, E. Saada . *Microscopic structure at the shock in the asymmetric simple exclusion.* Stochastics **27**, 151-165,(1988).

djls B. Derrida, S. Janowsky, J.L. Lebowitz, E. Speer *Exact solution of the totally asummetric simple exclusion process: shock profiles.* To appear J. Stat. Phys. (1993).

f1 P. A. Ferrari.*The simple exclusion process as seen from a tagged particle.* Ann. Probab. **14** 1277-1290,(1986).

f2 P. A. Ferrari *Shock fluctuations in asymmetric simple exclusion.* Probab. Theor. Related Fields. **91**, 81–101,(1992).

ff1 P. A. Ferrari, L. R. G. Fontes .*Current fluctuations for the asymmetric simple exclusion process.* To appear in Ann. Probab.(1993a)

ff2 P. A. Ferrari, L. R. G. Fontes *Current fluctuations in the asymmetric simple exclusion process.* IME-USP preprint,(1993b).

ff3 P. A. Ferrari, L. R. G. Fontes (1993c) In preparation.

ffk P. A. Ferrari, L. R. G. Fontes, Y. Kohayakawa *Invariant measure for a two species asymmetric process* (1993) (In preparation)

fks P. A. Ferrari, C. Kipnis, E. Saada *Microscopic structure of travelling waves for asymmetric simple exclusion process.* Ann. Probab. **19** 226-244,(1991).

fo J-P. Fouque . *Hydrodynamical behavior of asymmetric attractive particle systems. One example: one dimensional nearest neighbors asymmetric simple exclusion process.* Preprint,(1989).

gp J. Gärtner, E. Presutti. *Shock fluctuations in a particle system.* Ann. Inst. H. Poincaré, Sect B **53**, 1-14,(1989).

go E. Goles .*Sand Piles, combinatorial games and cellular automata.* Preprint,(1990).

k C. Kipnis . *Central limit theorems for infinite series of queues and applications to simple exclusion.* Ann. Probab. **14** 397-408,(1986).

la1 C. Landim . *Conservation of local equilibrium for attractive particle systems on Z^d.* To appear Ann. Probab..

lax P. D. Lax . *The formation and decay of shock waves.* Amer. Math. Monthly, (March),(1972).

lps J. L. Lebowitz, E. Presutti, H. Spohn.*Microscopic models of hydrodynamical behavior.* J. Stat. Phys. **51,** 841-862,(1988).

L T. M. Liggett,*Ergodic theorems for the asymmetric simple exclusion process.* Trans. Amer.Math. Soc. **213,** 237-261,(1975).

L T. M. Liggett ,*Ergodic theorems for the asymmetric simple exclusion process, II.* Ann. Probab., **4,** 339-356,(1977).

L T. M. Liggett . *Coupling the simple exclusion process.* Ann. Probab. **4** 339-356,(1976).

L T. M. Liggett . *Interacting Particle Systems.* Springer, Berlin,(1985).

lps J. L. Lebowitz, E. Presutti, H. Spohn. *Microscopic models of hydrodynamical behavior.* J. Stat. Phys. **51,** 841-862,(1988).

r H. Rezakhanlou ,*Hydrodynamic limit for attractive particle systems on Z^d.* Comm. Math. Phys. **140** 417-448,(1990).

r H. Rost ,*Nonequilibrium behavior of a many particle process: density profile and local equilibrium.* Z. Wahrsch. verw. Gebiete, **58** 41-53,(1982).

spi F. Spitzer .*Interaction of Markov processes.* Adv. Math., **5** 246-290,(1970).

S H. Spohn . *Large Scale Dynamics of Interacting Particles.* Springer,(1991).

vb H. Van Beijeren *Fluctuations in the motions of mass and of patterns in one-dimensional driven diffusive systems.* J. Stat. Phys,(1991).

w D.Wick . *A dynamical phase transition in an infinite particle system.* J. Stat. Phys. **38** 1015-1025,(1985).

wa J. Walker .*How to analyze the shock waves that sweep through expressway traffic.* Scientific America, August 1989:84-87,(1989).

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