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The two phase transitions for the contact process on trees [1]

Rinaldo B. Schinazi

Abstract: The contact process is a spatial stochastic process which has been used to model biological phenomena. Each particle can give birth to a new particle on a neighboring empty site with rate λ or die with rate 1. We consider a contact process on a homogeneous tree where each site has $d \geq 3$ neighbors. Let λ_1 (respectively, λ_2) be the infimum of λ such that the process starting with one particle has positive probability of surviving forever (respectively, of having a fixed site occupied at arbitrarily large times). It is known that for $d \geq 4$, $\lambda_1 < \lambda_2$. The exact critical values λ_1 and λ_2 are not known. But we show that it is possible to characterize λ_1 in a way that allows the analysis of the contact process at λ_1. We also discuss the characterizations of λ_1 and λ_2 in terms of the invariant distributions of the contact process.

Key words: contact process, homogeneous tree, phase transition.

1. Starting the contact process with a single particle on the tree

Let $\mathcal{T}$ be a homogeneous tree in which d branches emanate from each vertex of $\mathcal{T}$. Thus $\mathcal{T}$ is an infinite connected graph without cycles in which each vertex (also called site) has d neighbors for some integer $d \geq 3$.

We consider the contact process on $\mathcal{T}$ whose state at time t is denoted by η_t and which evolves according to the following rules. The contact process is a Markov process such that if there is a particle at site $x \in \mathcal{T}$ then this particle gives birth to a new particle on a neighboring site at rate λ, where $\lambda > 0$ is a parameter, for each of its d neighboring sites. A particle dies at rate 1. If there is a birth in an already occupied site then the two particles coalesce to one. So there is at most one particle per site.

For standard facts about the contact process on Z^d see Liggett (1985) or Durrett (1988).

Let O be a distinguished vertex of the tree that we call the root. Let η_t^x be the contact process with only one particle at time 0 located at site $x \in \mathcal{T}$. Let $\eta_t^x(y)$ be the number of particles at site y and let $|\eta_t^x| = \sum_{y\in\mathcal{T}} \eta_t^x(y)$ be the total number of particles. We define the following critical values

$$\lambda_1 = \inf\{\lambda : P_\lambda(|\eta_t^O| \geq 1, \forall t > 0) > 0\}$$

$$\lambda_2 = \inf\{\lambda : P_\lambda(\limsup_{t\to\infty} \eta_t^O(O) = 1) > 0\}.$$

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In words, λ_1 is the critical value corresponding to the global survival of the contact process and λ_2 corresponds to the local survival. Note that λ_1 is always smaller than or equal to λ_2.

On Z^d, the two critical values coincide: $\lambda_1 = \lambda_2$ (see Bezuidenhout and Grimmett (1990)). What makes the tree interesting is the following result due to Pemantle (1992).

Theorem 1. *If $d \geq 4$ then $\lambda_1 < \lambda_2$.*

Proof: We give a proof which is different and much more elementary than Pemantle's proof but which works for $d \geq 7$ only. Observe that $d = 3$ is still open (the conjecture is that $\lambda_1 < \lambda_2$ in this case too).

To prove Theorem 1 we will find an upper bound for λ_1 and a lower bound for λ_2.

To get a lower bound for λ_2, consider a process with the same birth and death rates than for the contact process but for which we have no bound on the number of particles per site (if there is a birth in an already occupied site the particles do not coalesce for this process). Such a process is called a branching random walk and exact computations of the critical values can be performed. The total number of particles on the tree is a Galton-Watson process for the branching random walk and it is easy to see that the first critical value is $1/d$. The second critical value gives more work but can be computed and is equal to $\frac{1}{2\sqrt{d-1}}$ (see Madras and Schinazi (1992) and Schinazi (1993)). It is easy to see that the contact process and the branching random walk can be constructed in the same probability space in such a way that the branching random walk has more particles than the contact process on each site of the tree, therefore

$$\lambda_2 \geq \frac{1}{2\sqrt{d-1}}. \tag{1}$$

To get an upper bound for λ_1, consider a process with the following rules. Start the process with a single particle at the root, pick $d-1$ sites among the d nearest neighbors. The particle at the root gives birth to a new particle at rate λ on each of the $d-1$ sites previously picked. Each new particle can give birth on neighboring sites but not on the site of the parent. Once a site has been occupied by a particle and this particle dies, the site remains empty forever. The death rate for each particle is 1 and there is at most one particle per site. Since a tree has no cycles the total number of particles for this process is a Galton-Watson process and it is supercritical if

$$(d-1)\frac{\lambda}{\lambda+1} > 1.$$

This implies that the first critical value for this process is $\frac{1}{d-2}$ and since the contact process has more particles than this process we get

$$\lambda_1 \leq \frac{1}{d-2}. \tag{2}$$

The bounds in (1) and (2) separate λ_1 from λ_2 when $d \geq 7$. This concludes the proof of Theorem 1.

Observe that as a byproduct of the preceding proof we get the following bounds
$$\frac{1}{d} \leq \lambda_1 \leq \frac{1}{d-2}.$$
To get bounds for λ_1 on Z^d is far more difficult, to prove that $\lambda_1 < \infty$ requires already a non trivial renormalization construction (see Durrett (1992)).

But even on the tree there is no hope to find the exact values of λ_1 and λ_2. So to study the phase transitions one has to find new characterizations of λ_1 and λ_2 which are more amenable to analysis. The first step in characterizing λ_1 was made by Madras and Schinazi (1992):

Theorem 2. *There exist constants c_λ and $C(d)$ such that*
$$e^{c_\lambda t} \leq E(|\eta_t^O|) \leq C(d)e^{c_\lambda t}.$$
Moreover c_λ is a continuous function of λ.

We now use c_λ to characterize λ_1:

Theorem 3. *We have $c_{\lambda_1} = 0$ and so at the critical value λ_1 we have*
$$1 \leq E(|\eta_t^O|) \leq C(d)$$
where $C(d)$ is a constant depending on d only.

Theorem 3 is proved in Morrow, Schinazi and Zhang (1994). While the lower bound is well known and easy to prove directly on any graph, the upper bound may be more surprising. Many people working in particle systems seem to believe that there are graphs for which the expected number of particles of the critical contact process is not bounded above. Theorem 3 is the first analysis of the expected number of particles of the critical contact process. There are no other graphs for which the behavior of $E(|\eta_t^O|)$ is known at the critical value.

Once Theorem 3 is known it is easy to analyse the first phase transition:

Theorem 4 . *The survival probability*
$$\lambda \rightarrow P_\lambda(|\eta_t^O| \geq 1, \forall t > 0)$$
is continuous at λ_1, i.e., the critical contact process dies out.

Pemantle (1992) proved Theorem 4 for $d \geq 4$. The following proof works for $d \geq 3$.

Proof. One of the keys to our analysis is the following well known fact about Markov chains with absorbing states. See for instance Durrett (1991) p.226 exercise 5.6.

Lemma 1. *On $\Omega_\infty = \{|\eta_t^O| \geq 1, \forall t > 0\}$ we have almost surely that*

$$\lim_{t\to\infty} |\eta_t^O| \to \infty.$$

First observe that

$$P_\lambda(\Omega_\infty) = \inf_{t>0} P_\lambda(|\eta_s^O| \geq 1, \forall s \leq t). \tag{3}$$

But $P_\lambda(|\eta_s^O| \geq 1, \forall s \leq t)$ is a continuous function of λ (it depends on a finite time only) and therefore the r.h.s. of (3) is upper semicontinuous. Using this fact together with the fact that $P_\lambda(\Omega_\infty)$ is an increasing function of λ gives the right continuity of this function.

Now to get the left continuity at λ_1 we observe that if at λ_1 we had $P_{\lambda_1}(\Omega_\infty) > 0$ then by Lemma 1 we would have that $E_{\lambda_1}(|\eta_t^O|)$ is unbounded. But this contradicts Theorem 3. Therefore $P_{\lambda_1}(\Omega_\infty) = 0$ and the survival probability is continuous at λ_1. This finishes the proof of Theorem 4.

The proof that the survival probability is continuous on Z^d is much more involved (see Bezuidenhout and Grimmett (1990)). Here we are able to take advantage of the nice structure (no cycles) of the tree.

Madras and Schinazi (1992) have proved that the second phase transition is discontinuous in the following sense:

Theorem 5. *If $\lambda_1 < \lambda_2$ then the function*

$$\lambda \to P_\lambda(\limsup_{t\to\infty} \eta_t^O(O) = 1)$$

is not continuous at λ_2.

So the second phase transition is discontinuous for all $d \geq 4$ and the behavior for $d = 3$ depends on the number of phase transitions we have there.

2. The stationary distributions of the contact process

We will now investigate the stationary distributions for the contact process on a tree. First, note that δ_0, the measure concentrating on the configuration with no particles, is always a stationary distribution. It is also known that if we start the contact process with one particle on each site of $\mathcal{T}$ then the law of the contact process converges weakly to a stationary distribution, ν, which is called the upper invariant measure since it is the largest stationary distribution in the natural partial ordering.

Using the self-duality of the contact process it is easy to see that

$$\nu(\eta : \eta(O) = 1) = P(|\eta_t^O| \geq 1 \text{ for all } t \geq 0)$$

so $\nu \neq \delta_0$ for $\lambda > \lambda_1$ and $\nu = \delta_0$ for $\lambda < \lambda_1$. Using the continuity of the survival probability of the contact process (Theorem 4) we also get $\nu = \delta_0$ for $\lambda = \lambda_1$.

The results above easily imply that the stationary distribution is unique for $\lambda \leq \lambda_1$. Our next result implies that for $\lambda > \lambda_1$ the only translation invariant stationary distributions are δ_0 and ν

Theorem 6. *If the initial configuration is translation invariant and assigns 0 probability to the empty configuration then the distribution of the contact process on the tree converges weakly to the upper invariant measure ν as $t \to \infty$.*

Theorem 6 is essentially due to Harris (1976) since his argument for Z^d, as explained for instance in Durrett (1993), extends with minor modifications to trees.

Bezuidenhout and Grimmett (1990) have proved that δ_0 and ν are the only stationary distributions for the contact process on Z^d. Durrett and Schinazi (1994) have shown that the situation is quite different on the tree:

Theorem 7. *For $\lambda \in (\lambda_1, \lambda_2)$ there are infinitely many extremal nontranslation invariant stationary distributions for the contact process on the tree.*

Theorem 7 is reminiscent of results of Grimmett and Newman (1990) for percolation on the product of a tree with the integers where there is an intermediate phase with infinitely many infinite clusters.

The reader should observe that the graph and the flip rates are translation invariant and that it is rather surprising to get nontranslation invariant stationary distributions in this context (but there are other examples of interacting particle systems with the same behavior, see Liggett (1985)). Rather than reproduce the proof of Theorem 7 which is in Durrett and Schinazi (1994) we will try to explain why for $\lambda \in (\lambda_1, \lambda_2)$ nontranslation invariant stationary measures appear.

Define for $x \neq O$, the cone generated by x, $\Gamma(x)$, to be the set of all y for which the self-avoiding path from O to y contains x. If we let $\eta_t^{\Gamma(x)}$ denote the contact process with 1's on $\Gamma(x)$ and let $t \to \infty$ then one can prove that this process converges in distribution to a measure μ^x which is stationary for the contact process. We now show that μ^x is nontranslation invariant. To do so we introduce the contact process restricted to the cone $\Gamma(x)$, ξ_t^x, i.e. no births from outside $\Gamma(x)$ into $\Gamma(x)$ are allowed; ξ_t^x starts with a single particle located at x. Morrow, Schinazi and Zhang (1994) have proved that if the contact process on the whole tree survives so does the contact process on a cone i.e. if $\lambda > \lambda_1$ there is an $\alpha > 0$ such that

$$P(|\xi_t^x| \geq 1, \text{ for all } t) = \alpha > 0.$$

By self duality, monotonicity, and translation invariance if $y \in \Gamma(x)$ then we have

$$\begin{aligned} P(\eta_t^{\Gamma(x)}(y) = 1) &= P(\eta_t^y(z) = 1, \text{ for a } z \in \Gamma(x)) \\ &\geq P(\eta_t^y(z) = 1, \text{ for a } z \in \Gamma(y)) \geq P(|\xi_t^y| \geq 1, \text{ for all } t) = \alpha > 0 \end{aligned} \tag{4}$$

if $\lambda > \lambda_1$. We let $t \to \infty$ in (4) to get the following inequality uniformly in $y \in \Gamma(x)$

$$\mu^x(\eta : \eta(y) = 1) \geq \alpha. \tag{5}$$

On the other hand using self duality again, we have that if $y \notin \Gamma(x)$

$$P(\eta_t^{\Gamma(x)}(y) = 1) = P(\eta_t^y(z) = 1, \text{ for a } z \in \Gamma(x)) \leq P(\eta_s^y(x) = 1 \text{ for some } s). \tag{6}$$

But if $\lambda < \lambda_2$ it is easy to see that the r.h.s. (6) goes to zero as the distance between x and y increases to infinity (see Lemma 6.4 in Pemantle (1992)). So from (6) we get that

$$\lim_{|y-x|\to\infty, y\notin\Gamma(x)} \mu^x(\eta : \eta(y) = 1) = 0 \tag{7}$$

where $|y - x|$ is the distance between y and x. From (5) and (7) we see that μ^x is not tranlation invariant for $\lambda \in (\lambda_1, \lambda_2)$. Durrett and Schinazi (1994) go on and prove that the μ^x are all distinct and extremal. The question that remains open is: are there other extremal stationary distributions distinct from the ones we have just constructed?

It was conjectured by Pemantle (1992) that for $\lambda > \lambda_2$ the following complete convergence theorem should hold: for any initial configuration η_0, the contact process η_t converges in distribution to a convex combination of ν (the upper invariant measure) and δ_0. Zhang (1994) has recently announced a proof of this conjecture. Observe that in particular it implies that the only stationary distributions for the contact process when $\lambda > \lambda_2$ are ν and δ_0.

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Rinaldo B. Schinazi
Department of Mathematics
University of Colorado at Colorado Springs
Colorado Springs CO 80933-7150
e-mail:schinazi@vision.uccs.edu
USA