Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces¹

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Abstract: We prove infinite-dimensional versions of the shadowing lemma and Smale's theorem (for a transverse homoclinic orbit) of a $C^1$ map, not a diffeomorphism, using the notion of an exponential dichotomy.

Key words: Exponential dichotomy, homoclinic orbit, shadowing lemma.

Introduction

The title shows our indebtedness to Palmer's articles [12, 13]. Based on the notion of exponential dichotomies we prove infinite-dimensional versions of the shadowing lemma and Smale's theorem for a transverse homoclinic orbit of a $C^1$ map (not a diffeomorphism).

Blazquez [1] gives a shadowing lemma, Theorem 4.2, which would be interesting if it were proved. Chow, Lin and Palmer [3] prove an infinite dimensional shadowing lemma with a special notion of "hyperbolicity"; ours is a natural extension of that of Palmer [13].

We will need many results about exponential dichotomies, which are treated in Section 1. Some results are merely quoted from [7], but others -- some new, some appearing only as exercises in [7], along with versions of results of Palmer [12] and Lin [10] -- are completely proved. In fact, the treatment of dichotomies is more extensive than is strictly necessary here; I couldn't resist the temptation, and anyway I hope to extend also some results of Melnikov, Shilnikov and Deng in later publications.

1 Exponential dichotomies

Let $X$ be a Banach space, $J \subset \mathbb{R}$ an interval and $\{T(t,s); t \geq s \in J\} \subset \mathcal{L}(X)$ a family of evolution operators, i.e.,

$$T(s,s) = I, \quad T(t,s)T(s,r) = T(t,r) \quad \text{for } t \geq s \geq r \text{ in } J.$$  \hspace{1cm} (1)

Sometimes we assume $\sup\{\|T(t,s)\| : 0 \leq t - s \leq 1\} < \infty$ and sometimes we assume $(t,s) \mapsto T(t,s)$ is strongly continuous; any such assumption is explicitly stated when needed.

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Definition 1.1 A family of evolution operators \( \{T(t,s); t \geq s \text{ in } J \} \) has an exponential dichotomy (on \( J \), with exponent \( \beta \), bound \( M \) and projections \( P(t) \), \( t \in J \) ) if there are constants \( \beta > 0 \), \( M \geq 1 \) and projections \( P(t) = P(t)^2 \in \mathcal{L}(X) \) for \( t \in J \) such that:

(i) \( T(t,s)P(s) = P(t)T(t,s) \) for \( t \geq s \) in \( J \);

(ii) the restriction \( T(t,s)|\mathcal{R}(P(s)) \rightarrow \mathcal{R}(P(t)) \) is an isomorphism (bicontinuous bijection) for \( t \geq s \) in \( J \), and \( T(s,t) \) is defined as the inverse from \( \mathcal{R}(P(t)) \) onto \( \mathcal{R}(P(s)) \);

(iii) \( \|T(t,s)(I-P(s))\| \leq Me^{-\beta(t-s)} \) for \( t \geq s \) in \( J \);

(iv) \( \|T(t,s)P(s)\| \leq Me^{-\beta(s-t)} \) for \( t \leq s \) in \( J \), where \( T(t,s)P(s) \) is defined in (ii).

If \( \dim \mathcal{R}(P(t)) = m < \infty \) for some \( t \in J \), equality holds for all \( t \in J \), by (ii), and we say the dichotomy has rank \( m \). We sometimes call \( \mathcal{R}(P(t)) = U(t) \) the unstable space and \( \mathcal{N}(P(t)) = S(t) \) the stable space.

Remarks: We only deal with exponential dichotomies and often say merely dichotomy. Lin [10], among others authors, requires \( t \rightarrow P(t) \) to be strongly continuous; this follows from strong continuity of the evolution operators, as we show in 1.12. We have \( \|P(t)\| \leq M \). Defining the angle \( \langle (E, F) \rangle \) between nonzero subspaces \( E, F \), with \( E \cap F = \{0\} \) by

\[
\langle (E, F) \rangle = \inf\{|e - f| : e \in E, f \in F, |e| = 1 = |f|\},
\]

it is easy to see, for any non-trivial projection \( P \), that \( 2/\|P\| \geq \langle (\mathcal{R}(P), \mathcal{N}(P)) \rangle \geq 1/\|P\| \). Thus \( \langle (\mathcal{R}(P(t)), \mathcal{N}(P(t))) \rangle \geq 1/M \) for all \( t \in J \), and the assumption to this effect in [8] is unnecessary. (In a Hilbert space, there is a geometrically natural angle \( \theta(E, F) \), and \( \langle (E, F) \rangle = 2 \sin \frac{1}{2} \theta(E, F) \).)

In general, the projection of a dichotomy is not unique. If \( J \supset [\tau, \infty) \) for some \( \tau \), the stable subspace \( S(t) = \mathcal{N}(P(t)), t \geq \tau \), is unique: \( S(t) = \{x|T(\theta, t)x \rightarrow 0 \} \) or, is bounded] as \( \theta \rightarrow +\infty \). If \( J \supset (-\infty, \tau) \) for some \( \tau \), \( U(t) = \mathcal{R}(P(t)) \) is unique for \( t \leq \tau \):

\[
U(t) = \{x| \text{there is a bounded } \varphi : (-\infty, t) \rightarrow X \text{ with } \varphi(t) = x \text{ and } \varphi(s) = T(s,r)\varphi(r) \text{ when } r \leq s \leq t\}.
\]

In this case, the “backward continuation” \( \varphi \) is unique, \( \varphi(s) \in \mathcal{R}(P(s)) \), and \( \varphi(s) \rightarrow 0 \) as \( s \rightarrow -\infty \). If \( J = \mathbb{R} \), the projection is uniquely determined.

Hale and Lin [6] define a trichotomy for \( T \), which is equivalent to saying \( \{e^{\lambda(t-s)}T(t,s) : t \geq s \text{ in } J\} \) has a dichotomy for both \( \lambda = \pm \varepsilon \), some \( \varepsilon > 0 \), with different projections. (If the projections were equal, \( T \) would have a dichotomy.)

Examples
(1) If \( \{e^{At}, t \geq 0\} \subset \mathcal{L}(X) \) is a strongly continuous semigroup and we define 
\[ T(t, s) = e^{A(t-s)} \]
for \( t \geq s \), for any interval \( J \subset \mathbb{R} \), \( \{T(t, s), t \geq s \text{ in } J\} \) is a family of evolution operators. If also \( \sigma(e^{A_{t_0}}) \cap S^1 = \emptyset \) for some (hence, every) \( t_0 > 0 \), define the projection \( P \) by 
\[ I - P = \frac{1}{2\pi i} \int_{|\mu| = 1} (\mu - e^{A_{t_0}})^{-1} d\mu; \]
Then \( e^{At} P = P e^{At} \) and we have an exponential dichotomy in \( J \) with projection \( P(t) = P \) constant. If \( \beta > 0 \) and \( \sigma(e^{A_{t_0}}) \cap \{\mu : e^{-\beta t_0} \leq |\mu| \leq e^{\beta t_0}\} = \emptyset \), we may suppose the exponent is \( \beta \). If the essential spectrum of \( e^{A_{t_0}} \) is strictly inside the unit circle, \( r_{ess}(e^{A_{t_0}}) < 1 \), the dichotomy has finite rank.

(2) Suppose \( A \) is the generator of a strongly-continuous semigroup on \( X \), \( B : \mathbb{R} \rightarrow \mathcal{L}(X) \) is strongly continuous with 
\[ B(t+p) = B(t) \]
for all \( t \) and fixed \( p > 0 \). Let \( \{T(t, s), t \geq s\} \subset \mathcal{L}(X) \) be the family of evolution operators such that \( x(t) = T(t, s)x(s) \) when \( t \geq s \) and \( x(\cdot) \) is a mild solution of \( \dot{x} = Ax + B(\cdot)x \) in \([s, t]\). Then for \( t \geq s \) \( T(t+p, s+p) = T(t, s) \) and \( \sigma(T(s+p, s)) \setminus \{0\} \) is independent of \( s \) (Lemma 7.2.2 of [7]).

Suppose \( \sigma(T(s+p, s)) \cap S^1 = \emptyset \) for some (hence, every) \( s \in \mathbb{R} \), and define 
\[ I - P(t) = \frac{1}{2\pi i} \int_{|\mu| = 1} (\mu - T(t+p, t))^{-1} d\mu; \]
then \( P(t)^2 = P(t) = P(t+p) \) for all \( t \), \( T(t, s)P(s) = P(t)T(t, s) \) for \( t \geq s \). For any interval \( J \subset \mathbb{R} \), \( \{T(t, s), t \geq s \text{ in } J\} \) has an exponential dichotomy with projections \( \{P(t)\}_{t \in J} \); in this case, \( t \mapsto P(t) \) is strongly continuous. If \( r_{ess}(T(t+p, t)) < 1 \), the dichotomy has finite rank. (Most of the argument for this is in 7.2.3 of [7].)

(3) If \( \|T(t, s)\| \leq M e^{-\beta(t-s)} \) for \( t \geq s \) in \( J \), and some \( \beta > 0 \), we have a trivial dichotomy with projection zero.

The theory is much simpler with discrete time and we see, in Theorem 1.3, there is little loss in restricting attention to this case.

If \( \hat{J} \) is an “interval” in \( \mathbb{Z} \), \( \{T_n : n \in \hat{J}\} \subset \mathcal{L}(X) \), define
\[ T_{m,m} = I, \quad T_{n,m} = T_{n-1} \circ \cdots \circ T_{m+1} \circ T_m \quad \text{for } n > m \quad (2) \]
when \( m \) and \( n-1 \) are in \( \hat{J} \); then \( T_{n,m} T_{l,m} = T_{n,l} \) for \( m \geq m \geq l \) with \( l \) and \( n-1 \) in \( \hat{J} \). Let \( \hat{J}^+ = \hat{J} \) if \( \hat{J} \) is not bounded above, \( \hat{J}^+ = \hat{J} \cup \{1 + \max \hat{J}\} \) otherwise, so \( T_{n,m} \) is well defined for \( n \geq m \) in \( \hat{J}^+ \).

**Definition 1.2** If \( \hat{J} \) is an interval in \( \mathbb{Z} \), \( \{T_n : n \in \hat{J}\} \subset \mathcal{L}(X) \) and we define \( \{T_{n,m} : n \geq m \text{ in } \hat{J}^+\} \) as in (2) above, then \( \{T_n : n \in \hat{J}\} \) has a discrete dichotomy (with constants \( M \geq 1, \theta \in (0, 1) \) and projections \( P_n, n \in \hat{J}^+ \)) if:
(i) \( T_n P_n = P_{n+1} T_n \) for \( n \in \hat{J} \);

(ii) the restriction \( T_n | \mathcal{R}(P_n) \rightarrow \mathcal{R}(P_{n+1}) \) is an isomorphism for \( n \in \hat{J} \);

(iii) \( \|T_{n,m} (I - P_n)\| \leq M \theta^{n-m} \) for \( n \geq m \) in \( \hat{J}^+ \);

(iv) \( \|T_{n,m} P_m\| \leq M \theta^{m-n} \) for \( n \leq m \) in \( \hat{J}^+ \), where \( T_{n,m} P_m x = y \in \mathcal{R}(P_n) \) is defined by \( P_m x = T_{m,n} y \) (and well-defined, by (ii)).

Remarks: We often say merely "dichotomy" when discreteness is evident. We have \( T_{n,m} P_m = P_T T_{n,m} \) for all \( n \geq m \) in \( \hat{J}^+ \) and \( T_{n,m} | \mathcal{R}(P_m) \rightarrow \mathcal{R}(P_n) \) is an isomorphism.

If we define \( \hat{T}(t,s) = T_{n,m} \) when \( t \in [t_n,t_{n+1}) \), \( s \in [t_m,t_{m+1}) \), \( t \geq s \) [\( t_k = t_0 + kp \) for some fixed \( p > 0 \), \( t_0 \in \mathbb{R} \)] and if \( \hat{J} = \bigcup_{k \in \mathbb{Z}} [t_k,t_{k+1}) \), then \( \{ \hat{T}(t,s), t \geq s \) in \( \hat{J} \} \) is a family of evolution operators and it has a dichotomy (with exponent \( \beta \) and bound \( M \)) if and only if \( \{ T_n : n \in \hat{J} \} \) has a dichotomy with constants \( M \) and \( \theta = e^{-\beta p} \).

It is clear that, for any family of evolution operators \( \{ T(t,s) \mid t \geq s \) in \( J \} \), \( t_0 \in J \) and \( p > 0 \), if we have an exponential dichotomy with exponent \( \beta \), bound \( M \) and projections \( P(t) \), and if \( T_n = T(t_{n+1},t_n) \), \( t_n = t_0 + np \), \( \bigcup_{n \in \hat{J}} [t_n,t_{n+1}) \subset J \), then \( \{ T_n : n \in \hat{J} \} \) has a dichotomy with constants \( M \), \( \theta = e^{-\beta p} \) and projections \( P_n = P(t_n) \).

The converse also holds provided \( \sup \{ \|T(t,s)\| : 0 \leq t - s \leq 1 \} < \infty \). The following is a stronger version of Exercise 10, Section 7.6 of [7].

Theorem 1.3 Let \( \{ T(t,s) \mid t \geq s \) in \( J \} \subset \mathcal{L}(X) \) be a family of evolution operators with \( \sup \{ \|T(t,s)\| : 0 \leq t - s \leq 1 \} < \infty \), \( J \) a closed interval, \( p > 0 \), \( t_n = t_0 + np \) and \( \hat{J} \subset \mathbb{Z} \) an interval such that \( \bigcup_{n \in \hat{J}} [t_n,t_{n+1}) = J \) (or \( J \setminus \{ \max J \} \), if \( J \) is bounded above). Let \( T_n = T(t_{n+1},t_n) \) for \( n \in \hat{J} \).

If \( \{ T_n : n \in \hat{J} \} \) has a discrete dichotomy with constants \( M \geq 1 \), \( \theta = e^{-\beta p} \in (0,1) \) and projections \( \{ P_n, n \in \hat{J}^+ \} \), then \( \{ T(t,s), t \geq s \) in \( J \} \) has an exponential dichotomy with exponent \( \beta \), bound \( M' \), and projections \( \{ P(t), t \in J \} \) such that \( P(t_n) = P_n \) for \( n \in \hat{J}^+ \). Writing \( K_p = \sup \{ \|T(t,s)\| : 0 \leq t - s \leq p \} \), we have \( K_p \leq K_p^{p+1} \) and may use

\[
M' = \max(K_p^2 M \theta^{-2}, K_p^2 M^2 + K_p \theta^{-1}).
\]

The projections for the "interpolated" dichotomy are uniquely determined by the \( \{ P_n, n \in \hat{J}^+ \} \) and \( \{ T(t,s), t \geq s \) in \( J \} \), when we require \( P(t_n) = P_n \).

Proof: Let \( K = \sup \{ \|T(t,s)\| : 0 \leq t - s \leq p \) in \( J \} \). If \( t \in [t_n,t_{n+1}] \subset J \), define \( X(t) = T(t,t_n) \mathcal{R}(P_n) \), so \( X(t_n) = \mathcal{R}(P_n) \), \( X(t_{n+1}) = \mathcal{R}(P_{n+1}) \). Then

\[
T_n \Big|_{\mathcal{R}(P_n)} = T(t_{n+1},t) \Big|_{X(t) \to \mathcal{R}(P_{n+1})} \circ T(t,t_n) \Big|_{\mathcal{R}(P_n)} \to X(t). \quad (3)
\]
By definition, $T(t, t_n)|\mathcal{R}(P_n) \to X(t)$ is surjective, and it is injective by (3) and condition (ii) for a discrete dichotomy, so both factors on the right side of (3) are continuous bijections. Further, if $y \in \mathcal{R}(P_n)$, $y = T_{n, n+1}P_{n+1}x$ for some $x$ and $|y| \leq M\theta|P_{n+1}x|$ by condition (iv), $P_{n+1}x = T_ny$ and $|y| \leq M\theta|T_ny| \leq KM\theta|T(t, t_n)y|$ for all $y \in \mathcal{R}(P_n)$. Thus both factors on the right side of (3) are isomorphisms, $X(t)$ is a closed space and

$$||(T(t_{n+1}, t)|X(t) \to \mathcal{R}(P_{n+1}))^{-1}|| \leq KM\theta,$$

$$||(T(t, t_n)|\mathcal{R}(P_n) \to X(t))^{-1}|| \leq KM\theta.$$  

For $t \in [t_n, t_{n+1}]$, define

$$P(t) = (\text{inclusion } X(t) \subset X) \circ (T(t_{n+1}, t)|X(t) \to \mathcal{R}(P_{n+1}))^{-1} \circ P_{n+1} \circ T(t_{n+1}, t).$$

(4)

It is then easy to show that $\mathcal{R}(P(t)) = X(t)$, $P(t)^2 = P(t) \in \mathcal{L}(X)$, $P(t_{n+1}) = P_{n+1}$, $P(t_n) = P_n$ and $\|P(t)\| \leq K^2M^2\theta$. If $t \geq s$ are in $[t_n, t_{n+1}]$, $T(t, s)X(s) = X(t)$ by definition and

$$T(t_{n+1}, s)|_{X(s) \to \mathcal{R}(P_{n+1})} = T(t_{n+1}, t)|_{X(t) \to \mathcal{R}(P_{n+1})} \circ T(t, s)|_{X(s) \to X(t)}$$

so $T(t, s)X(s) \to X(t)$ is also an isomorphism. Further, by (4),

$$T(t_{n+1}, t)|_{X(t)} \circ T(t, s)P(s) = T(t_{n+1}, s)P(s) = P_{n+1}T(t_{n+1}, t) \circ T(t, s)$$

$$= T(t_{n+1}, t)|_{X(t)} \circ P(t)T(t, s)$$

so $T(t, s)P(s) = P(t)T(t, s)$. The equality holds for any $t \geq s$ in $J$, by an easy calculation, so (i) and (ii) of Definition 1.1 hold.

Verification of (iii) and (iv) is now straight-forward. For example, if $t \geq s$, $t \in [t_n, t_{n+1}]$, $s \in [t_{m+1}, t_m]$ with $n \geq m$, $y = T(s, t)P(t)x$, we have

$$y = (T(t_{n+1}, s)|_{X(s) \to \mathcal{R}(P_{n+1})})^{-1}T_{m+1, n+1}P_{n+1}T(t_{n+1}, t)x$$

so $|y| \leq M^2K^2\theta^{n-m+1}|x| \leq M^2K^2\theta^{(t-s)}|x|$, proving (iv).

Most of the following results treat only discrete dichotomies, and often with $J = \mathbb{Z}$ so the projections are uniquely determined.

**Theorem 1.4** Let $\{T_n\}_{n=\infty}^{\infty} \subset \mathcal{L}(X)$; then the following are equivalent.

(i) $\{T_n\}_{n=\infty}^{\infty}$ has discrete dichotomy.

(ii) For every bounded sequence $\{f_n\}_{n=\infty}^{\infty} \subset X$, there is a unique bounded sequence $\{x\}_{n=\infty}^{\infty} \subset X$ with $x_{n+1} = T_nx_n + f_n$ for all $n$. 
Proof: See [7], Theorem 7.6.5.

Remarks: The unique bounded solution is \( x_n = \sum_{k=0}^{\infty} G_{n,k+1} f_k \) where \( G_{n,m} = T_{n,m}(I - P_m) \) for \( n \geq m \), \( G_{n,m} = -T_{n,m} P_m \) for \( n < m \) so \( \|G_{n,m}\| \leq M \theta |n-m| \); the double sequence \( \{G_{n,m}\} \) is the Green function.

O. Perron [14], for ordinary differential equations on \( \mathbb{R}_+ \), and T. Li [9], for difference equations on \( \mathbb{Z}_+ \), obtained analogous conditions for finite dimensions, though the exponential bounds were not recognized until 1954 (Maizel). These results were greatly generalized by Massera and Schäffer [11]. Coffman and Schäffer [4] treated infinite-dimensional difference equations on \( \mathbb{Z}_+ \), with a more general notion of dichotomy. Slyusharchuk [16] gives a result like 1.4 (with partial proof) when \( T_n \in \mathcal{L}(X_n, X_{n+1}) \), the spaces depending on \( n \).

Simple examples (with \( X = C \)).

1. \( T_n = a, |a| \neq 1 \); the only bounded solution of \( x_{n+1} = ax_n + f_n \) [\( f \) bounded] is \( x_n = \sum_{k=0}^{\infty} a^{n-k-1} f_k \), if \( |a| > 1 \), or \( x_n = \sum_{k=0}^{n-1} a^{n-k-1} f_k \), if \( |a| < 1 \).

2. \( T_n = a, |a| = 1 \): \( x_n = a^n \) is a bounded non-trivial solution of \( x_{n+1} = ax_n \), so there is no dichotomy. If \( x_{n+1} - T_n x_n = a^n (\forall n) \) then \( x_n = x_0 a^n + na^{n-1} \) is unbounded for any \( x_0 \).

3. \( T_n = 2 \) (\( n \geq 0 \), \( T_n = \frac{1}{2} \) (\( n < 0 \)): \( \{T_n\}_{n \geq 0} \) and \( \{T_n\}_{n < 0} \) both have dichotomies, but there is no dichotomy on all \( \mathbb{Z} \) since \( x_{n+1} - T_n x_n = \delta_{n,0} \) has no bounded solution. In this example, \( x_{n+1} = T_n x_n (\forall n) \) with \( x_n \) bounded only when all \( x_n = 0 \). (Example of Slyusharchuk.)

Theorem 1.5 Suppose \( \{T_n\}_{n=0}^\infty \) has a discrete dichotomy with constants \( M \geq 1 \), \( \theta \in (0,1) \) and suppose \( M_1 > M, \theta < \theta_1 < 1 \) and

\[
0 < \varepsilon \leq \frac{\theta - \theta_1}{1 + \theta \theta_1} \left( \frac{1}{M} - \frac{1}{M_1} \right).
\]

Then any sequence \( \{S_n\}_{n=0}^\infty \subset \mathcal{L}(X) \) with \( \sup_n \|S_n - T_n\| \leq \varepsilon \) has a discrete dichotomy with constants \( M_1, \theta_1 \). If \( \{P_n^S\}, \{P_n^T\} \) are the corresponding projections, as \( \sup_n \|S_n - T_n\| \to 0 \),

\[
\sup_n \|P_n^S - P_n^T\| = O \left( \sup_n \|S_n - T_n\| \right).
\]

Proof: See [7], Theorem 7.6.7.

The argument for Theorem 1.5 uses the following lemma, stated as exercise 11 in Section 7.6 of [7].
Lemma 1.6 If \( a \geq 0, b \geq 0, 0 < r < r' \leq r_1, r_2 \leq 1 \) and
\[
b < (r' - r)/(1 + rr')
\]
and if \( \{g_n\}_{-\infty}^{\infty} \subset IR \) satisfies
\[
0 \leq g_n \leq ar_1^{|n|} + b \sum_{-\infty}^{\infty} r^{n-k-1} g_k \text{ for all } n \in \mathbb{Z},
\]
and \( g_n = 0(r_2^{-|n|}) \) as \( n \to \pm \infty \), then
\[
g_n \leq ar_1^{|n|}/(1 - b(1 + rr_1)/(r_1 - r)) \text{ for all } n \in \mathbb{Z}.
\]

Proof: As suggested in [7], we consider the map \( \Phi \) of real sequences
\[
\{f_n\} \xrightarrow{\Phi} \left\{ b \sum_{-\infty}^{\infty} r^{n-k-1} f_k \right\}
\]
and show it is a contraction in the norm \( \| \cdot \|_q, \| f \|_q = \sup_n |f_n|q^{n} \), when \( r' \leq q \leq 1/r' \). If \( S_n = \sum_{-\infty}^{\infty} r^{n-k-1} q^{n-|k|} \), then \( \| \Phi f \|_q/\| f \|_q \leq \sup_n S_n \). We have \( \sup_n S_n = q^{-2}S_2-n \) for \( n \leq 0 \), \( \sup_n S_n = S_0 = (1 + qr)/(q - r) \) if \( r < q \leq 1 \), \( \sup_n S_n = S_{+\infty} = q^2(1-r^2)/[(q-r)(1-qr)] \leq (q+r)/(1-rq) \) if \( 1 \leq q < r^{-1} \). Thus if \( \theta = b(1+rr')/(r'-r) \), \( \theta < 1 \) and \( \| \Phi f \|_q/\| f \|_q \leq \theta \) for \( r' \leq q \leq 1/r' \). If \( f_n = ar_1^{n} \), \( 0 \leq g \leq f + \Phi g \leq f + \Phi f + \cdots + \Phi^k f + \Phi^{k+1} g \), and \( \| \Phi^{k+1} g \|_2 \to 0 \) as \( k \to \infty \). For each \( n \), \( g_n \leq \sum_{k=0}^{\infty} (\Phi^n f)_n \) and \( \| f \|_{1/r_1} = a, \| \Phi^n f \|_{1/r_1} \leq b(1+rr_1)/\| f \|_{1/r_1} \), which gives the result.

Remark: Theorem 1.5, on the “roughness” of exponential dichotomies, may also be proved by continuity using Theorem 1.4 (as in [16]) or (at least for finite dimensions and invertible operators) by direct calculation as in Palmer [13] (or Coppel [5] for ODEs), where it is the beginning of the theory. “Roughness” theorems seem to start with Massera and Schäffer [11].

Sakamoto [15] gives a non-symmetric version of the lemma, for sequences \( O(\theta^2_n) \) in \( n \geq 0, O(\theta_{-n}^2) \) in \( n \leq 0 \).

Theorem 1.7 Suppose \( \{T_n\}_{n=\infty}^{\infty} \subset \mathcal{L}(X) \) is a bounded sequence and \( \{P_n\}, \{\tilde{P}_n\} \) are bounded sequences of projections in \( \mathcal{L}(X) \), and \( M \geq 1, \theta \in (0, 1) \) are constants such that, for all \( n \),
\[
\|P_n\| \leq M, \|\tilde{P}_n\| \leq M, \| I - \tilde{P}_n \| \leq M ;
\]
\[
T_n P_n = \tilde{P}_{n+1} T_n, \mathcal{R}(T_n P_n) = \mathcal{R}(\tilde{P}_{n+1}) ;
\]
\[
\| T_n x \| \leq \theta \| x \| \text{ if } P_n x = 0, \| T_n x \| \geq \theta^{-1} \| x \| \text{ if } P_n x = x .
\]

If \( \theta < \theta_1 < 1 \) and \( M_1 > M \), there exists \( \varepsilon > 0 \) depending on \( \theta, \theta_1, M, M_1 \) and \( \sup_k \| T_k \| \) such that: for any \( \{S_n\}_{n=\infty}^{\infty} \subset \mathcal{L}(X) \), if \( \| S_n - T_n \| \leq \varepsilon \) and \( \| \tilde{P}_n - P_n \| \leq \varepsilon \) for all \( n \), \( \{S_n\}_{n=\infty}^{\infty} \) has a discrete dichotomy with constants \( M_1, \theta_1 \).
Proof: See [7], Theorem 7.6.8. It suffices that \(4\varepsilon \leq \frac{\theta_1 - \theta}{1 + \theta_1 \theta} \left(\frac{1}{M} - \frac{1}{M_1}\right)/(1 + M \sup_k \|T_k\|).\)

Remark: Palmer proves a similar result for ODEs in [12] and for the case of finite-dimensional invertible operators \(T_n\) in [13].

The following simple result allows us to apply Theorems 1.4, 1.5 to dichotomies defined only on \(\mathbb{Z}_+\) or \(\mathbb{Z}_-\). It is a simpler version of ex. 15, sec. 7.6 of [7].

**Theorem 1.8**

If \(\{T_n\}_{n \geq 0}\) has a discrete dichotomy with projections \(\{P_n\}_{n \geq 0}\) and constants \(M, \theta\), define \(\widetilde{T}_n = T_n\) for \(n \geq 0\), \(\widetilde{T}_n = \theta^{-1}P_0 + \theta(I - P_0)\) for \(n < 0\), \(\widetilde{P}_n = P_n\) for \(n \geq 0\), \(\widetilde{P}_n = P_0\) for \(n \leq 0\). Then \(\{\widetilde{T}_n\}_{-\infty}^\infty\) has a dichotomy with projections \(\{\widetilde{P}_n\}\) and constants \(M, \theta\).

If \(\{T_n\}_{n < 0}\) has a discrete dichotomy with projections \(\{P_n\}_{n \leq 0}\) and constants \(M, \theta\), define \(\widetilde{T}_n = T_n\) for \(n < 0\), \(\widetilde{T}_n = \theta^{-1}P_0 + \theta(I - P_0)\) for \(n \geq 0\), \(\widetilde{P}_n = P_n\) for \(n \leq 0\), \(\widetilde{P}_n = P_0\) for \(n \geq 0\). Then \(\{\widetilde{T}_n\}_{-\infty}^\infty\) has a dichotomy with projections \(\{\widetilde{P}_n\}\) and constants \(M, \theta\).

Proof: A straightforward calculation. We only note that the condition for a dichotomy in \(\mathbb{Z}_-\) uses \(P_n\) for \(n \leq 0\) but only \(T_n\) for \(n \leq -1\), so we may define \(\widetilde{T}_0\) conveniently in the second part. (This was overlooked in [7] 7.6, ex. 15, so our result is simpler.)

Remarks: A similar extension is possible for \(\{T_n\}\) defined only in a finite interval, \(a \leq n \leq b\). We may also treat continuous time. For example, suppose \(\{T(t, s), t \geq s \geq 0\}\) has a dichotomy with exponent \(\beta\), bound \(M\) and projections \(\{P(t), t \geq 0\}\). Define \(\widetilde{T}(t, s) = T(t, s)\) for \(t \geq s \geq 0\), \(\widetilde{T}(t, s) = e^{\beta(t-s)}P(0) + e^{\beta(s-t)}(I - P(0))\) for \(0 \geq t \geq s\), \(\widetilde{T}(t, s) = T(t, 0)\widetilde{T}(0, s)\) for \(t \geq 0 \geq s\). Then \(\widetilde{T}\) is a family of evolution operators which has a dichotomy with exponent \(\beta\), bound \(M\) and projections \(\{\widetilde{P}(t), t \in \mathbb{R}\}\), \(\widetilde{P}(t) = P(\max\{t, 0\})\).

In each of the following corollaries, we extend the sequence to \(\{T_n\}_{-\infty}^\infty\) as in Theorem 1.8 (for appropriate \(P_0\)), prove the extended sequence has a dichotomy (by Theorem 1.5, in the first case, or by Theorem 1.4), and then restrict to \(\mathbb{Z}_+\).

**Corollary 1.9** Assume \(\{T_n\}_{n \geq 0} \subset \mathcal{L}(X)\) [or \(\{T_n\}_{n < 0}\)] has a dichotomy with constants \(M, \theta\), and \(M_1 > M, \theta < \theta_1 < 1, 0 < \varepsilon \leq (1/M - 1/M_1)(\theta_1 - \theta)/(1 + \theta_1 \theta)\). If \(S_n \in \mathcal{L}(X)\) with \(\|S_n - T_n\| \leq \varepsilon\) for all \(n \geq 0\) [or \(n < 0\)], then \(\{S_n\}_{n \geq 0}\) [or \(\{S_n\}_{n < 0}\)] has a dichotomy with constants \(M_1, \theta_1\) and the corresponding projections satisfy \(\sup_n \|P^S_n - P^T_n\| = O(\sup_n \|S_n - T_n\|)\) as \(\sup_n \|S_n - T_n\| \to 0\).

**Corollary 1.10** Given \(\{T_n\}_{n \geq 0} \subset \mathcal{L}(X)\), the following are equivalent:

(i) \(\{T_n\}_{n \geq 0}\) has a dichotomy.
(ii) \( S_0 = \{ x_0 | \exists \text{ bounded } (x_n)_{n \geq 0} \subset X \text{ with } x_{n+1} = T_n x_n, \forall n \geq 0 \} \) splits in \( X \) (i.e., there is a closed subspace \( U_0 \) so that \( S_0 \oplus U_0 = X \)) and, for every bounded \( \{ f_n \}_{n \geq 0} \subset X \), there is a bounded \( \{ X_n \}_{n \geq 0} \subset X \) with \( x_{n+1} = T_n x_n + f_n, \forall n \geq 0 \).

**Corollary 1.11** Given \( \{ T_n \}_{n < 0} \subset \mathcal{L}(X) \), the following are equivalent:

(i) \( \{ T_n \}_{n < 0} \) has a dichotomy.

(ii) \( U_0 = \{ x_0 | \exists \text{ bounded } (x_n)_{n < 0} \subset X \text{ with } x_{n+1} = T_n x_n, \forall n < 0 \} \) splits in \( X \) and, for every bounded \( \{ f_n \}_{n < 0} \subset X \), there exists a bounded \( \{ x_n \}_{n \leq 0} \subset X \) with \( x_{n+1} = T_n x_n + f_n \) for all \( n < 0 \), and

\[
\{(x_n)_{n=0}^\infty \in \ell_\infty(\mathbb{Z}, X) \mid x_n = 0 \text{ for } n \geq 0, \ x_{n+1} = T_n x_n \text{ for all } n < 0\}
\]

consists only of the zero sequence.

**Remark:** The final hypothesis of (ii) in Corollary 1.11 is ugly but inevitable unless we change the definition of a dichotomy. With only the first two hypotheses, the map \( (x_n)_{n=0}^\infty \mapsto (x_{n+1} - T_n x_n)_{n=0}^\infty \) in \( \ell_\infty(\mathbb{Z}, X) \) (for the extended sequence) is surjective; its kernel is the set required to be zero by the final hypothesis. I can't find an example with the first two hypotheses true and the last false, nor prove the last unnecessary.

The more general notion of a dichotomy in Coffman and Schäffer [4] gives a result like 1.10 without assuming \( S_0 \) splits.

It is sometimes useful to know that the projections of a (continuous time) dichotomy are strongly continuous; this certainly holds if the evolution operators are strongly continuous.

**Theorem 1.12** Suppose \( \{ T(t,s), t \geq s \text{ in } J \} \subset \mathcal{L}(X) \) is a family of evolution operators which has an exponential dichotomy with projections \( \{ P(t), t \in J \} \) and assume, for some interval \( [a,b] \subset J \) and \( p \) with \( 0 < p < b - a \), that \( s \mapsto T(s+p,a) \) \( (a-p \leq s \leq a) \), \( s \mapsto T(s+p,s), (a \leq s \leq b-p) \), and \( s \mapsto T(b,s) \) \( (b-p \leq s \leq b) \) are strongly continuous. Then \( t \mapsto P(t) : [a,b] \rightarrow \mathcal{L}(X) \) is strongly continuous.

**Proof:** Suppose the dichotomy has exponent \( \beta \) and bound \( M \). We may extend \( T, P \) from \( [a,b] \) to all \( \mathbb{R} \), as in the remark following Theorem 1.8, so \( T \) has a dichotomy on all \( \mathbb{R} \) with exponent \( \beta \) bound \( M \) and projections \( \{ P(t), t \in \mathbb{R} \} \), and for the extension, \( s \mapsto T(s+p,s) \) is strongly continuous and \( \text{sup} \{ ||T(s+p,s)|| : s \in \mathbb{R} \} = K < \infty \). The new \( T, P \) agree with the original \( T, P \) in \( [a,b] \).

For each \( t \) and \( n \in \mathbb{Z} \), define \( T_n(t) = T(t + np, t) \), \( P_n(t) = P(t + np) \); each \( t \mapsto T_n(t) \) is strongly continuous, \( ||T_n(t)|| \leq K \), and \( \{ T_n(t) \}_{n=0}^\infty \) has a discrete dichotomy with constants \( M, \theta = e^{-\beta p} \) and projections \( \{ P_n(t) \}_{n=0}^\infty \). The Green functions satisfy

\[
G_{n,m}(t) - G_{n,m}(s) = \sum_{k=0}^\infty G_{n,k+1}(t) (T_k(t) - T_k(s)) G_{k,m}(s).
\]
With \( m = n = 0 \) and fixed \( x \in X, s \in \mathbb{R} \), and any \( N \)

\[
|P(t)x - P(s)x| \leq \sum_{|k| \leq N} M \theta^{k+1} |(T_k(t) - T_k(s))G_k,0(s)x| + 2K M^2(1 + \theta^2)(1 - \theta^2)^{-1}\theta^{2N+1}|x|.
\]

Given \( \varepsilon > 0 \), choose \( N \) large so the second term is \( < \varepsilon/2 \); for \( t \) near \( s \), the first is also \( < \varepsilon/2 \). Thus \( P(t)x \to P(s)x \) as \( t \to s \).

We return to discrete time.

More-or-less the following result has appeared in various places – we only mention Coppel [5] and ex. 22, sec. 7.6 of [7].

**Theorem 1.13** Suppose \( \{T_n\}_{n=0}^{\infty} \subseteq L(X) \). We have a discrete dichotomy on \( \mathbb{Z} \) if and only if the restrictions in both \( \mathbb{Z}_+ \) and \( \mathbb{Z}_- \) have dichotomies and also \( X = S_0 \oplus U_0 \) where

\[
U_0 = \{ x_0 | \exists \text{ bounded } \{ x_n \}_{n \leq 0} \subseteq X \text{ with } x_{n+1} = T_n x_n \text{ for } n < 0 \},
\]

\[
S_0 = \{ x_0 | \exists \text{ bounded } \{ x_n \}_{n \geq 0} \subseteq X \text{ with } x_{n+1} = T_n x_n \text{ for } n \geq 0 \}.
\]

In case the dichotomies in \( \mathbb{Z}_+, \mathbb{Z}_- \) have finite rank, \( X = S_0 \oplus U_0 \) means they have the same rank and also the only bounded solution of \( x_{n+1} = T_n x_n \) (all \( n \)) is the zero sequence.

**Proof:** If we have a dichotomy on \( \mathbb{Z} \) with projections \( \{ P_n \}_{n=0}^{\infty} \), it is clear the restrictions in \( \mathbb{Z}_+ \) and \( \mathbb{Z}_- \) have dichotomies. If \( P_0 x_0 = 0 \), \( x_n = T_n,0(I - P_0)x_0 \) is bounded as \( n \to +\infty \) so \( x_0 \in S_0 \). If \( x_0 \in S_0 \), \( x_n = T_n,0 x_0 \) is bounded and \( P_0 x_0 = T_{0,n} P_n x_n \to 0 \) as \( n \to \infty \) so \( x_0 \in \mathcal{N}(P_0) : S_0 = \mathcal{N}(P_0) \). Similarly \( U_0 = \mathcal{R}(P_0) \).

Now assume we have dichotomies in \( \mathbb{Z}_+ \), \( \mathbb{Z}_- \) with projections \( \{ P_n^+ \}_{n \geq 0} \), \( \{ P_n^- \}_{n \leq 0} \). As above, \( \mathcal{N}(P_0^+) = S_0 \) and \( \mathcal{R}(P_0^-) = U_0 \), and we assume \( U_0 \oplus S_0 = X \). Given bounded \( \{ f_n \}_{n=0}^{\infty} \subseteq X \), we show there is a unique bounded solution of \( x_{n+1} = T_n x_n + f_n \) (\( \forall n \)). In fact

\[
x_n = T_{n,0}(1 - P_0^+) x_0 + \sum_{0}^{\infty} G_{n,k+1}^+ f_k \quad \text{for } n \geq 0
\]

\[
x_n = T_{n,0} P_0^- x_0 + \sum_{-\infty}^{1} G_{n,k+1}^- f_k \quad \text{for } n \leq 0
\]

is the only candidate, and we only need to show these equations are consistent for a (unique) choice \( x_0 \), i.e.,

\[
(P_0^+ x_0, (1 - P_0^-) x_0) = \left( - \sum_{0}^{\infty} T_{0,k+1} P_k^+ f_k, \sum_{-\infty}^{1} T_{0,k+1} (1 - P_k^-) f_k \right)
\]
has a unique solution \( x_0 \). It suffices to show
\[
x_0 \mapsto (P_0^+ x_0, (I - P_0^-) x_0) : X \to \mathcal{R}(P_0^+) \times \mathcal{N}(P_0^-)
\]
is a bijection – which is equivalent to \( S_0 + U_0 = X \).

If \( P_0^+ x_0 = 0, (I - P_0^-) x_0 = 0 \) then \( x_0 \in \mathcal{N}(P_0^+) \cap \mathcal{R}(P_0^-) = S_0 \cap U_0 = \{0\} \).

If \( a = P_0^+ a, P_0^- b = 0, a - b \in X = S_0 + U_0 \) so \( a - b = s + u \) for some \( s \in S_0, u \in U_0 \), and then \( x_0 = a - s = b + u \) satisfies \( P_0^+ x_0 = P_0^+(a - s) = a, (I - P_0^-) x_0 = (I - P_0^-)(b + u) = b \).

The next result is due to X.-B. Lin [10] for continuous time.

**Theorem 1.14** Given \( \{T_n\}_{n<n_1} \subset \mathcal{L}(X) \) and \( n_0 < n_1 \), suppose \( \{T_n\}_{n<n_0} \) has a dichotomy with finite rank and projections \( \{P_n\}_{n<n_0} \) and assume \( T_{n_1,n_0} \mathcal{R}(P_{n_0}) \) is injective. Then \( \{T_n\}_{n<n_1} \) has a dichotomy with the same rank and projections \( \{\tilde{P}_n\}_{n<n_1} \) such that \( \|P_n - \tilde{P}_n\| \to 0 \) exponentially when \( n \to -\infty \).

Given \( \{T_n\}_{n>n_0} \subset \mathcal{L}(X) \) and \( n_0 < n_1 \), suppose \( \{T_n\}_{n>n_0} \) has a dichotomy with finite rank and projections \( \{P_n\}_{n>n_0} \) and assume the adjoint \( T_{n_1,n_0}^* \mathcal{R}(P_{n_1}^*) \) is injective. Then \( \{T_n\}_{n>n_0} \) has a dichotomy with the same rank and with projections \( \{\tilde{P}_n\}_{n>n_0} \) such that \( \|P_n - \tilde{P}_n\| \to 0 \) exponentially as \( n \to +\infty \).

If the constants of the original dichotomy are \( M, \theta \), we may use the same "\( \theta \)" for the extended dichotomy but a larger "\( M \)"; the exponential convergence is \( O(\theta^{2|n|}) \).

**Proof:** For the first case, define \( U_n = \mathcal{R}(P_n) \) for \( n \leq n_0 \), \( U_n = T_{n,n_0} \mathcal{R}(P_{n_0}) \) for \( n_0 \leq n \leq n_1 \). By hypothesis, \( \dim U_{n_1} = \dim U_{n_0} < \infty \) so \( \dim U_n \) is independent of \( n \) and each \( T_n|U_n - U_{n+1} \) is an isomorphism. Choose a closed space \( S_{n_1} \) so that \( S_{n_1} \oplus U_{n_1} = X \) and define \( S_n = T^{-1}_{n_1,n} S_n \) for \( n \leq n_1 \), a closed subspace of \( X \) with \( T_n S_n = S_{n+1} \cap \mathcal{R}(T_n) \subset S_{n+1} \) for \( n < n_1 \). If \( x \in S_n \cap U_n, T_{n_1,n} x \in S_{n_1} \cap U_{n_1} = \{0\} \) and \( T_{n_1,n} U_n \) is injective so \( x = 0 \). If \( x \in X, T_{n_1,n} x = u + s \) for some \( u \in U_{n_1}, s \in S_{n_1} \) and \( u = T_{n_1,n} u_n \) for some \( u_n \in U_n \) so \( T_{n_1,n}(x - u_n) = T_{n_1,n} T_{n_1,n} u_n \) or \( x \in S_n + U_n \). Thus \( X = U_n \oplus S_n \) and there is a projection \( \tilde{P}_n \) with \( \mathcal{R}(\tilde{P}_n) = U_n, \mathcal{N}(P_n) = S_n \).

We have
\[
\tilde{P}_{n+1} T_n = \tilde{P}_{n+1} T_n \tilde{P}_n + \tilde{P}_{n+1} T_n (1 - \tilde{P}_n) = \tilde{P}_{n+1} T_n \tilde{P}_n = T_n \tilde{P}_n
\]
and for \( n \leq n_0, \mathcal{R}(\tilde{P}_n) = \mathcal{R}(P_n) \) so \( \tilde{P}_n P_n = P_n, P_n \tilde{P}_n = \tilde{P}_n \).

If \( n \leq n_0 \)
\[
\tilde{P}_n = \tilde{P}_n P_n + \tilde{P}_n (1 - P_n) = P_n + T_{n,n_0} P_{n_0} \tilde{P}_{n_0} T_{n_0,n_0} (1 - P_n)
\]
so \( \|\tilde{P}_n - P_n\| \leq \|\tilde{P}_n - P_n\| M^2 \theta^{2(n_0 - n)} \to 0 \) as \( n \to -\infty \).

In particular \( K = \sup_{n \leq n_0} \|\tilde{P}_n\| < \infty \).

If \( n \leq m \leq n_0 \)
\[
\|T_{n,m} \tilde{P}_m\| = \|T_{n,m} P_m \tilde{P}_m\| \leq K M^m \theta^{m-n}
\]
and if \( m \leq n \leq n_0 \), similarly

\[
\|T_{n,m}(I - \tilde{T}_m)\| \leq (K + 1)M\theta^{n-m}
\]

There are finitely many other indices in \((n_0, n_1)\) and each \(T_n|\mathcal{R}(\tilde{T}_n) \to \mathcal{R}(\tilde{T}_{n+1})\)

is an isomorphism, so we get a dichotomy for \(\{T_n\}_{n<n_1}\).

For the second case, let \(S_n = N(P_n)\) for \(n \geq n_1\), \(S_n = T_{n_1,n_0}^{-1}S_{n_0}\) for \(n_0 \leq n < n_1\); we show \(\text{codim}S_n = \text{codim}S_{n_0} < \infty\) for all \(n \geq n_0\), initially for \(n = n_0\). Suppose \(u_1, \ldots, u_m\) are independent relative to \(S_{n_0} : \sum_1^m c_ku_k \in S_{n_0} (c_k \in \mathbb{R})\) implies all \(c_k = 0\). Then \(\sum_1^m c_kT_{n_1,n_0}u_k \in S_{n_1}\) implies \(\sum_1^m c_kn_k \in S_{n_0}\) so the \(T_{n_1,n_0}u_k\) are independent relative to \(S_{n_1}\), \(\text{codim}S_{n_1} \geq \text{codim}S_{n_0}\) (and similarly \(\text{codim}S_{n_1} \geq \text{codim}S_{n_0}\) for \(n_0 \leq n \leq n_1\)). Let \(\xi_1, \ldots, \xi_m \in X^*\) be a basis for \(S_{n_1}^\perp = \mathcal{R}(P_{n_1}^*)\). If \(x \in S_{n_0} \cap T_{n_1,n_0}x \in S_{n_1} \perp \xi_k\) so \(T_{n_1,n_0}^*\xi_k \in S_{n_1}^\perp\). By hypothesis, the \(T_{n_1,n_0}^*\xi_k\) are independent so \(\text{codim}S_{n_0} \geq \text{codim}S_{n_1}\) and we have equality. If \(n_0 < n \leq n_1\), \(T_{n_1,n_0}^*|\mathcal{R}(P_{n_1}^*)\) is also injective so \(\text{codim}S_{n} = \text{codim}S_{n_1}\) for \(n_0 \leq n < n_1\), and equality is obvious for \(n \geq n_1\).

Choose \(U_{n_0}\) with \(U_{n_0} + S_{n_0} = X\) and define \(U_n = T_{n,n_0}U_{n_0}\) for \(n > n_0\). If \(x \in S_n \cap U_n\), \(x = T_{n,n_0}x_0\) for some \(x_0 \in U_{n_0}\) and also \(x_0 \in S_{n_0}\) so \(x_0 = 0\), \(x = 0\). Since \(N(T_{n,n_0}) \subset S_{n_0}\), \(T_{n,n_0}U_n \to U_n\) is a bijection and \(\dim U_n = \dim U_{n_0} = \text{codim}S_{n_0} = \text{codim}S_n\) for all \(n \geq n_0\), \(T_n|U_n \to U_{n+1}\) is an isomorphism and \(U_n \oplus S_n = X\). If \(\tilde{P}_n\) is the projection with \(\mathcal{R}(\tilde{P}_n) = U_n, N(\tilde{P}_n) = S_n\), we have \(\tilde{P}_{n+1} = T_n\tilde{P}_n\) for \(n \geq n_0\) and \(N(\tilde{P}_n) = S_n = N(P_n)\) for \(n \geq n_1\) so \(\tilde{P}_nP_n = \tilde{P}_n, P_n\tilde{P}_n = P_n\) for \(n \geq n_1\). Then for \(n \geq n_1\),

\[
\tilde{P}_n = P_n\tilde{P}_n + (I - P_n)\tilde{P}_n = P_n + T_{n,n_1}(I - P_n)\tilde{P}_n T_{n,n_1}P_n = P_n + O(\theta^{2(n-n_1)})
\]

and the proof is completed as in the first cases.

The last result of this section is due to Palmer [12] for ODEs. A continuous-time version for retarded FDEs is given by Lin [10]. I am unable to find a continuous-time version for PDEs which is not a disguised version of discrete time; Lemma 3.2 of [1] and Theorem 2.2 of [2] are certainly false as stated.

**Theorem 1.15** Let \(\{T_n\}_n \subset \mathcal{L}(X)\) and assume the restrictions in \(\mathbb{Z}_+\) and \(\mathbb{Z}_-\) both have dichotomies of finite rank. Define \(S_0, U_0\) as in Theorem 1.13 and define \(L : \mathcal{D}(L) \subset \ell_\infty(X) \to \ell_\infty(X)\) by

\[
x = (x_n)_n \in \ell_\infty(\mathbb{Z}, X) \text{ is in } \mathcal{D}(L) \text{ if sup } |x_{n+1} - T_nx_n| < \infty,
\]

and then \(Lx = (x_{n+1} - T_nx_n)_n\).

Then \(L\) is a closed operator, \(\dim N(L) = \dim(S_0 \cap U_0) < \infty, \mathcal{R}(L)\) is closed with \(\text{codim} \mathcal{R}(L) = \text{codim}(S_0 + U_0) < \infty,\) and \(L\) is Fredholm with index

\[
\text{ind } L = \dim N(L) - \text{codim } \mathcal{R}(L) = \dim U_0 - \text{codim } S_0 = (\text{rank } \mathbb{Z}_-) - (\text{rank } \mathbb{Z}_+).
\]
Finally, \( f \in \mathcal{R}(L) \) if and only if \( 0 = \sum_{k=-\infty}^{\infty} \langle \xi_{k+1}, f_k \rangle \) for all \( \xi \in \ell_\infty(\mathbb{Z}, X^*) \) with \( \xi_k = T_k^* \xi_{k+1} \) (\( \forall k \)). We remark that any such \( \xi \) has \( |\xi_k| \to 0 \) exponentially as \( k \to \pm \infty \), and there are only finitely many linearly independent \( \xi \).

**Proof:** Let \( (x_n)_{n=\infty}^\infty \in \mathcal{N}(L) \); \( (x_n) \) is a bounded solution of \( x_{n+1} = T_n x_n \) (\( \forall n \)) or \( x_n = T_{n,0}(1 - P_0^+) x_0 \) for \( n \geq 0 \), \( x_n = T_{n,0} P_0^- x_0 \) for \( n \leq 0 \) with \( x_0 \in \mathcal{R}(P_0^-) \cap \mathcal{N}(P_0^+) = \mathcal{U}_0 \cap S_0 \). The sequence is determined by \( x_0 \) so \( \dim \mathcal{N}(L) = \dim (\mathcal{U}_0 \cap S_0) \).

Let \( f \in \mathcal{R}(L) : f_n = x_{n+1} - T_n x_n \) (\( \forall n \)) for some bounded \( (x_n)_{-\infty}^\infty \), so

\[
x_n = \begin{cases} 
T_{n,0}(1 - P_0^+) x_0 + \sum_{k=0}^{\infty} G^{+}_{n,k+1} f_k & \text{for } n \geq 0 \\
T_{n,0} P_0^- x_0 + \sum_{k=-\infty}^{-1} G^{-}_{n,k+1} f_k & \text{for } n \leq 0
\end{cases}
\]

and \( x_0 \) satisfies

\[
(P_0^+ x_0, (I - P_0^-) x_0) = \left( -\sum_{k=0}^{\infty} T_{0,k+1} P^{+}_{k+1} f_k, \sum_{k=-\infty}^{-1} T_{0,k+1} (I - P^{-}_{k+1}) f_k \right).
\]

Conversely, any solution \( x_0 \) of the last equation determines a bounded \( (x_n)_{-\infty}^\infty = x \) with \( Lx = f \).

An argument similar to that in Theorem 1.13 shows \( (a, b) \in \mathcal{R}(P_0^+) \times \mathcal{N}(P_0^-) \) is in the range of \( x_0 \mapsto (P_0^+ x_0, (I - P_0^-) x_0) \) if and only if \( a - b \in \mathcal{N}(P_0^+) + \mathcal{R}(P_0^-) = S_0 + \mathcal{U}_0 \).

Thus \( f \in \mathcal{R}(L) \) if and only if, for all \( \xi \perp (S_0 + \mathcal{U}_0) \),

\[
0 = \sum_{k=-\infty}^{-1} \langle \xi, T_{0,k+1} (I - P_{k+1}^-) f_k \rangle + \sum_{k=0}^{\infty} \langle \xi, T_{0,k+1} P_{k+1}^+ f_k \rangle
\]
or \( 0 = \sum_{k=-\infty}^{\infty} \langle \xi_{k+1}, f_k \rangle \) where \( \xi_k = T_{0,k}^* (1 - P_{k}^-) \xi \) for \( k \leq 0 \), \( \xi_k = (T_{0,k} P_{k}^+)^* \xi \) for \( k > 0 \). Now \( \xi \perp (S_0 + \mathcal{U}_0) \) means \( \xi \in \mathcal{N}(P_0^+) \perp \mathcal{R}(P_0^-) \perp = \mathcal{R}(P_0^+) \cap \mathcal{N}(P_0^-) \) or \( \xi = P_0^{++} \xi = (I - P_0^-)^* \xi \), which is \( \xi_0 \), so \( \xi_{k-1} = T_{k-1}^* \xi_k \) for all \( k \). Conversely, a bounded solution of this equation has \( \xi_0 \in (S_0 + \mathcal{U}_0) \perp \). It only remains to note that

\[
\text{codim} \mathcal{R}(L) = \dim (S_0 + \mathcal{U}_0) = \text{codim} (S_0 + \mathcal{U}_0) = \text{codim} S_0 - \dim \mathcal{U}_0 + \dim (\mathcal{U}_0 \cap S_0).
\]

## 2 The shadowing lemma

Let \( X \) be a Banach space, \( V \) open in \( X \) and \( f : V \to X \) of class \( C^1 \). A set \( C \subset V \) is **invariant** if \( f(C) = C \).
Definition 2.1 A compact invariant $C \subseteq V$ is hyperbolic if, for every orbit $(y_n)_{n=\infty}^{\infty} \subseteq C \ [y_{n+1} = f(y_n)$ for all $n]$, $\{Df(y_n)\}_{n=\infty}^{\infty}$ has a discrete dichotomy with finite rank $m$ and constants $M, \theta$, where $m, M, \theta$ are independent of the orbit considered.

This is not the usual definition of "hyperbolic structure" but it is equivalent.

Lemma 2.2 Assume $f$ is $C^1$ on a neighborhood of the compact invariant set $C$ and $f|C$ is injective. Then $C$ is hyperbolic if and only if there exists continuous $P : C \to \mathcal{L}(X)$ with $P(y)^2 = P(y)$, $\text{rank } P(y) = \text{constant}$, $Df(y)P(y) = P(f(y))Df(y)$ and $Df(y)|\mathcal{R}(P(y)) \to \mathcal{R}(P(f(y)))$ an isomorphism for all $y \in C$, and for some constants $M \geq 1$, $0 < \theta < 1$, and any integer $n \geq 0$, $y \in C$,

$$|Df^n(y)(I - P(y))| \leq M\theta^n,$$
$$|Df^{-n}(y)P(y)| \leq M\theta^n$$

where, by definition,

$$Df^{-n}(y)P(y)z = w \in \mathcal{R}(P(f^{-n}y)) \text{ when } P(y)z = Df^n(f^{-n}y)w.$$  

Remark: $N(P) = \{(y, z)| y \in C, P(y)z = 0\}$ is the stable vector bundle, and $\mathcal{R}(P)$ the unstable bundle, of the usual definition.

Proof: It is clear that any such $P(\cdot)$ gives us a dichotomy on any orbit $\{f^n(y)\}_{n=\infty}^{\infty} \subseteq C$. Suppose $C$ is hyperbolic and $y \in C$: there is a dichotomy for $\{Df(f^n(y))\}_{n=\infty}^{\infty}$ with projections $\{P_n(y)\}_{n=\infty}^{\infty}$.

If $z = f(y)$, there is also a dichotomy for $\{Df(f^n(z))\}_{n=\infty}^{\infty}$ with projections $\{P_n(z)\}_{n=\infty}^{\infty}$, and $f^n(z) = f^{n+1}(y)$, so $P_n(z) = P_{n+1}(y)$ or $P_n(f(y)) = P_{n+1}(y)$, $P_n(y) = P_0(f^n(y))$ for all $n \in \mathbb{Z}$. Define $P(y) = P_0(y)$; then $P(f(y))Df(y) = Df(y)P(y)$ and $Df(y)|\mathcal{R}(P(y)) \to \mathcal{R}(P(f(y)))$ is an isomorphism for each $y \in C$.

If $n \geq 1$,

$$Df^n(y) = Df(f^{n-1}(y))\cdots Df(f(y))Df(y) = T_{n,0} \text{ when } T_k = Df(f^k(y)),$$

which gives the estimates claimed.

Proof of continuity of $P(\cdot)$ is more interesting. Since $C$ is compact and $f|C \to C$ is a continuous bijection, the inverse is also continuous. Suppose $\varepsilon > 0$, $N$ is a positive integer and $y \in C$; there is a neighborhood $V_y$ of $y$ so that, if $y' \in V_y \cap C$, $|f^n(y) - f^n(y')| \leq \varepsilon$ when $|n| \leq N$. Also there is a neighborhood $U$ of $C$, $K > 0$ and an increasing function $\omega_0(\cdot)$ with $\omega_0(t) \to 0$ as $t \to 0^+$ such that

$$|Df(x)| \leq K \text{ for } x \in U, \ |Df(x) - Df(y)| \leq \omega_0(|x - y|) \text{ for } x \in U, y \in C.$$  

We also assume $K$ exceeds the Lipschitz constant of $f|C$. Then $|Df(f^n(y)) - Df(f^n(y'))| \leq \omega_0(\varepsilon)$ for $|n| \leq N$, and it is bounded by $2K$ for all $n$. If $G_{n,m}(y)$,
\( G_{n,m}(y') \) are the Green's functions for the dichotomies, then for \( y' \in V_y \),

\[
|P(y) - P(y')| = \left| \sum_{k=-\infty}^{\infty} G_{0,k+1}(y)(Df(f^k(y)) - Df(f^k(y'))))G_{k,0}(y') \right|
\leq 2M^2\omega_0(\varepsilon)/(1 - \theta^2) + 4K M^2 \theta^{2N+1}/(1 - \theta^2),
\]

which is small if \( \varepsilon \) is small and \( N \) is large.

**Remark:** The notation \( U, K, \omega_0(\cdot) \) will be used below; they were defined with greater generality than is necessary here. Note that we avoid any hypothesis of uniform continuity of \( Df \) on an open set since \( f \) is merely \( C^1 \).

**Lemma 2.3** Suppose \( \{T_n\}_{n=0}^{\infty} \subset \mathcal{L}(X) \) has a discrete dichotomy with constants \( M, \theta \), and there exist \( g_n : X \to X \) with \( g_n(0) = 0 \), \( |g_n(x) - g_n(x')| \leq \gamma |x - x'| \) when \( |x| \leq r \), \( |x'| \leq r \), and \( \gamma M < (1 - \theta)/(1 + \theta) \). Finally suppose \( h_n \in X \) for \( n \in \mathbb{Z} \) with \( \sup_n |h_n| \leq r((1 - \theta)/(1 + \theta) - M\gamma)/M \). Then there is a unique sequence \( \{x_n\}_{n=0}^{\infty} \subset X \) such that

\[
|x_n| \leq r \text{ and } x_{n+1} = T_n x_n + g_n(x_n) + h_n \text{ for all } n \in \mathbb{Z}.
\]

**Proof:** Let \( S_r = \{ \text{sequences } \{x_n\}_{n=0}^{\infty} \subset X \mid \sup_n |x_n| \leq r \} \) and define

\[
(\Gamma(x))_n = \sum_{k=-\infty}^{\infty} G_{n,k+1}(h_k + g_k(x_k)) \text{ for } x \in S_r, \ n \in \mathbb{Z}.
\]

It is easily verified that \( \Gamma(S_r) \subset S_r \) and \( \Gamma \) is a contraction for the sup-norm in \( S_r \). The fixed point of \( \Gamma \) is the desired sequence.

**Theorem 2.4** (The shadowing lemma) Let \( X \) be a Banach space, \( V \) an open set in \( X \), \( f : V \to X \) a \( C^1 \) map. Assume there is a compact invariant set \( C \subset V \) which is hyperbolic and \( f|C \) is injective.

Then for any \( \varepsilon > 0 \), sufficiently small, there exists \( \delta > 0 \) such that, for each sequence \( \{y_n\}_{n=0}^{\infty} \subset C \) with \( |y_{n+1} - f(y_n)| \leq \delta, \forall n \) (a "\( \delta \)-pseudo-orbit") there is a unique orbit \( \{x_n\}_{n=0}^{\infty} \subset V, \ x_{n+1} = f(x_n) \) for all \( n \), such that \( |x_n - y_n| \leq \varepsilon, \forall n \).

**Proof:** The crucial point is to show, for \( 0 \leq \delta \leq \delta_0 \), given any \( \delta \)-pseudo-orbit \( \{y_n\}_{n=0}^{\infty} \subset C \), \( \{Df(y_n)\}_{n=0}^{\infty} \) has a dichotomy with constants \( M' \geq 1, \theta' \in (0,1) \), which are independent of the \( \delta \)-pseudo-orbit considered. Then the result will follow from the last lemma. In fact, \( x_n = y_n + z_n \) where we require \( |z_n| \leq \varepsilon \) and

\[
z_{n+1} - Df(y_n)z_n = g_n(z_n) + h_n
\]

with \( h_n = f(y_n) - y_{n+1}, \ g_n(z) = f(y_n + z) - f(y_n) - Df(y_n)z \). We have \( |h_n| \leq \delta, \ g_n(0) = 0 \), and for small \( \varepsilon > 0 \) so that \( B_{\varepsilon}(C) \subset U \)

\[
|Dg_n(z)| = |Df(y_n + z) - Df(y_n)| \leq \omega_0(\varepsilon) \text{ for } |z| \leq \varepsilon
\]
where \( \omega_0, U \) come from the proof of Lemma 2.2. Choose \( \varepsilon > 0 \) small so that 
\( B_{\varepsilon}(C) \subseteq U \) and also \( \omega_0(\varepsilon) < (1 - \theta')/(M'(1 + \theta')) \), and then \( \delta > 0 \) small so that 
\( \delta \leq \delta_0 \) and also \( \delta \leq \varepsilon((1 - \theta')/(M'(1 + \theta')) - \omega_0(\varepsilon)) \); then Lemma 2.3 applies.

To see we have a dichotomy along any \( \delta \)-pseudo-orbit \( \{y_n\}_{n=0}^\infty \subseteq C \), first choose the integer \( N \) so \( M\theta^N \leq \frac{1}{2} \); show \( \{DF^N(y_{Ni})\}_{n=0}^\infty \) has a dichotomy with constants 
\( 2M, 3/4 \) (by Theorem 1.7); then if \( V_{Ni+k} = DF(f^k(Y_{Ni})) \) for \( 0 \leq k < N, i \in \mathbb{Z} \), 
\( DF^N(y_{Ni}) = V_{Ni+N,Ni} = V_{Ni+N-1} \cdots o o V_{Ni+1} o V_{Ni} \) and we may "interpolate" a dichotomy for \( \{V_j\}_{j=-\infty}^\infty \), by Theorem 1.3, with constants \( M_1, \theta_1 = (3/4)^{1/N} \); and finally, since \( \sup_n |DF(y_n) - V_n| \to 0 \) as \( \delta \to 0 \), Theorem 1.5 gives the dichotomy with constants \( M' > M_1, \theta' \in (\theta_1, 1) \), for \( 0 < \delta \leq \delta_0 \), if \( \delta_0 \) is small.

By induction, \( |y_{n+m} - f^m(y_n)| \leq \delta(1 + K + \cdots + K^{m-1}) \) for \( m \geq 1, n \in \mathbb{Z} \). 
\( |y_{n+m} - f^m(y_n)| \leq |y_{n+m} - f(y_{n+m-1})| + |f(y_{n+m-1}) - f(f^{m-1}(y_n))| \leq \delta + K|y_{n+m-1} - f^{m-1}(y_n)| \). Let \( K^* = 1 + K + \cdots + K^{N-1} \); then \( |y_{Ni+N} - f^N(y_{Ni})| \leq K^* \delta \) for all \( i \in \mathbb{Z} \). Now \( P : C \to \mathcal{L}(X) \) is uniformly continuous and has modulus of continuity \( \omega_p(\cdot) \) so 
\[
|P(f^N(y_{Ni})) - P(y_{Ni+N})| \leq \omega_p(K^* \delta) \tag{1}
\]

We apply Theorem 1.7 with \( T_i = DF^N(y_{Ni}), P_i = P(y_{Ni}), \tilde{T}_{i+1} = P(f^N(y_{Ni})) \), so \( T_i P_i = \tilde{T}_{i+1} T_i \). We have \( |\tilde{T}_{i+1} - P_{i+1}| \leq \omega_p(K^* \delta) \), which is uniformly small for small \( \delta \), so we have a dichotomy for \( \{T_i\} = \{DF^N(y_{Ni})\}_{n=0}^\infty \) with constants \( 2M \) and 3/4. (It suffices that \( 32M(1 + MK^N)\omega_p(K^* \delta_0) \leq 1 \).)

Define \( V_j \) (\( j \in \mathbb{Z} \)) as above, \( V_{Ni} = DF^N(y_{Ni}) \), and let \( \tilde{T}(t, s) = V_{n,m} \) when \( t \geq s, t \in [n, n+1], s \in [m, m+1] \). By Theorem 1.3, \( \{\tilde{T}(t, s), t \geq s\} \) has a dichotomy so also \( \{V_i\}_{i=-\infty}^\infty \) has a dichotomy with constants \( M_1, \theta_1 = (3/4)^{1/N} \). (We may use \( M_1 = \max(2MK^2N(\frac{1}{2})^2, \frac{4}{3} K^N + 4M^2 K^{2N}) \).)

Finally, if \( n = Ni + k (0 \leq k < N, i \in \mathbb{Z}) \)
\[
|DF(y_n) - V_n| = |DF(y_{Ni+k}) - DF(f^k(y_{Ni}))| \leq \omega_0(K^* \delta) \tag{2}
\]

Given \( M' > M_1, 1 > \theta' > \theta_1 \), for \( 0 < \delta \leq \delta_0 \) (and small \( \delta_0 \)) \( \{DF(y_n)\}_{n=0}^\infty \) has a dichotomy with constants \( M', \theta' \), for any \( \delta \)-pseudo-orbit \( \{y_n\}_{n=0}^\infty \subseteq C \). (It suffices that \( \omega_0(K^* \delta_0) \leq \frac{\theta' - \theta}{1 + \theta' \theta_1} \frac{1}{M_1 - \frac{1}{M'}} \).)

3 A transverse homoclinic orbit

Theorem 3.1 Assume \( V \subseteq X \) is open, \( f : V \to X \) is \( C^1 \) and also:

(i) \( x_0 \in V \) is a hyperbolic fixed point of \( f \) \( (f(x_0) = x_0, \sigma(DF(x_0)) \cap S^1 = \emptyset) \) with finite rank, i.e. \( r_{ess}(DF(x_0)) < 1 \) or \( \sigma(DF(x_0)) \cap \{\lambda \in \mathbb{C} : |\lambda| \geq 1\} \) consists only of isolated eigenvalues of finite multiplicity;

(ii) there exists a homoclinic orbit \( \{y_n\}_{n=-\infty}^\infty \subseteq V \setminus \{x_0\}, y_{n+1} = f(y_n) \) for all \( n \), and \( y_n \to x_0 \) as \( n \to \pm \infty \).
(iii) the homoclinic orbit is transverse, i.e.

$$\eta_{n+1} = Df(y_n)\eta_n \text{ for all } n, \sup_{n} |\eta_n| < \infty \text{ imply all } \eta_n = 0.$$ 

Then $C = \{x_0, y_n \ (n \in \mathbb{Z})\}$ is a compact invariant hyperbolic set, $f|C$ is injective, and for $\varepsilon > 0$ sufficiently small, there is $N_\varepsilon$ such that, for any positive integer $N \geq N_\varepsilon$ and for each $\sigma \in S = \{0, 1\}^\mathbb{Z}$, there is a unique orbit $x^\sigma [x^\sigma_{n+1} = f(x^\sigma_n), \forall n]$ such that $|x^\sigma_n - \eta^\sigma_n| \leq \varepsilon, \forall n$, where

$$\eta^\sigma_n = \begin{cases} x_0 & \text{if } \sigma(i) = 0 \\ y_j & \text{if } \sigma(i) = 1 \end{cases}, \quad n = (2N + 1)i + j, -N \leq j \leq N,$$

and $\sigma \mapsto x^\sigma$ is injective from $S$ to $\ell_\infty(\mathbb{Z}, X)$.

Define $d(\sigma, \sigma') = \sum_{t=1}^{\infty} 2^{-n} \Delta (\max_{|t| \leq n} |\sigma(k) - \sigma'(k)|)$ with $\Delta(t) = t/(1 + t)$ and $\sigma, \sigma' \in S$. Then $(S, d)$ is a compact metric space, homeomorphic to the "middle thirds" Cantor set, and $x^\sigma : S \to X$ is continuous, for each $n \in \mathbb{Z}$.

The zero sequence $\sigma = 0$ gives $x^\sigma_n = x_0$ for all $n$.

If $\sigma \in S \setminus \{0\}$ has finite support ($\sigma(i) = 0$ for all large $|i|$) then $x^\sigma$ is a homoclinic orbit, $x^\sigma_n \to x_0$ as $n \to \pm \infty$.

If $\sigma$ is periodic with period $p$, $n \mapsto x^\sigma_n$ is periodic with period $(2N + 1)p$. If $\sigma \neq 0$, $\sigma$ has least period $p$ if and only if $x^\sigma$ has least period $(2N + 1)p$.

(iv) Assume also that $f|V$ is injective.

Then $\sigma \mapsto x^\sigma_0$ is injective with compact image $K$; $K$ is a topological Cantor set and the restriction $\varphi$ of $\sigma \mapsto x^\sigma_0 : S \to K$ is a homeomorphism; $f^{2N+1}(K) = K$, and $f^{2N+1}|K \to K$ is $\varphi \circ \beta \circ \varphi^{-1}$ where $\beta : S \to S$ is the Bernoulli shift, $\beta(\sigma)(n) = \sigma(n + 1)$ for $n \in \mathbb{Z}, \sigma \in S$.

Suppose, finally, that (i), (ii) and (iv) hold but perhaps not (iii) and also:

(v) $Df(x) \in \mathcal{L}(X)$ is injective with dense range [$Df(x)^* \text{ injective}$] for each $x \in V$.

Then the global stable and unstable manifolds $W^s(x_0), W^u(x_0)$ are $C^1$ immersed submanifolds of $V$, $\dim W^u(x_0) = \text{codim} W^s(x_0) < \infty, y_n \in W^s(x_0) \cap W^u(x_0)$, and (iii) is equivalent to saying $W^s(x_0) \cap \cap_{j=0}^{\infty} W^u(x_0)$ for some (or every) $n \in \mathbb{Z}$.

Proof: It is clear that $C$ is compact and invariant. For any $n, f(y_n) = y_{n+1} \neq x_0 = f(x_0)$, and if $f(y_m) = f(y_p)$ for some $p > m, n \mapsto y_n$ is periodic for $n \geq m$ and does not tend to $x_0$ as $n \to +\infty$; thus $f|C$ is injective.

Since $x_0$ is hyperbolic with finite rank, $\{Df(x_0)\}_{n=\infty}^{\infty}$ has a dichotomy with finite rank and constant projection $P_{\infty}$. By Corollary 1.9, since $Df(y_n) \to Df(x_0)$ as $n \to \pm \infty$, for some positive integer $N$, both $\{Df(y_n)\}_{n=\infty}^{N}$ and $\{Df(y_n)\}_{n=1}^{\infty}$
have dichotomies, both with the same rank (since the projections are close to $P_\infty$). Let $\eta_{-N} \in U(-N)$; there is a sequence $\{\eta_n\}_n = Df(y_n)$ with $\eta_{n+1} = Df(y_n)\eta_n$ ($\forall n$) and $\eta_n$ is bounded as $n \to -\infty$. If $T_{N,-N}\eta_{-N} = 0 = \eta_N$ [$T_k = Df(y_k)$], then $\eta_n = 0$ for all $n \geq N$ so, by hypothesis (iii), all $\eta_n = 0$, $\eta_{-N} = 0$. Thus $T_{N,-N}[U(-N)$ is injective and, by Theorem 1.14, we may extend the dichotomy to $\{Df(y_n)\}_{n \leq N}$, still with the same rank. Hypothesis (iii) and Theorem 1.13 say we have a dichotomy for $\{Df(y_n)\}_{n = \infty}$, so $C$ is hyperbolic.

As noted in the proof of the shadowing lemma, for small $\delta_0 > 0$ if $\{\eta_n\} = Df(y_n) \subset C$ is any $\delta$-pseudo-orbit with $0 < \delta \leq \delta_0$, $\{Df(\eta_n)\}_{n = \infty}$ has a dichotomy with constants (say $M, \theta$) independent of the pseudo-orbit considered. Choose $\varepsilon > 0$ sufficiently small so the shadowing lemma (2.4) applies, $B(\varepsilon) = U, \omega(\varepsilon) M < (1 - \theta)/4 [\omega_0, U$ from Lemma 2.2] and $|x_0 - y_0| > 2\varepsilon$ and $|y_n - y_0| > 2\varepsilon$ for all $n \not= 0$. Choose $\delta$ in $0 < \delta < \delta_0$ so we may apply the shadowing lemma and $N_\varepsilon > 0$ so that $|y_{\pm N} - y_0| < \delta/2$ for $N \geq N_\varepsilon$; choose any integer $N \geq N_\varepsilon$. Then for every $\sigma \in S$, $\{\eta_n^\sigma\}_{n = \infty}$ is a $\delta$-pseudo-orbit so there is a unique orbit $\{x_n^\sigma\}_{n = \infty}$ with $|x_0^\sigma - y_0^\sigma| \leq \varepsilon$ for all $n$. If $\sigma \not= s^\sigma$ in $S$, there exists $i$ with $\sigma(i) \not= s(i)$ so, if $n = (2N + 1)i, |x_n^\sigma - x_n^s| \geq |y_0 - y_0| - 2\varepsilon > 0$, and $\sigma \not= x^\sigma$ is injective. It is clear that $\eta_n^{\sigma+2N+1} = \eta_n^{\sigma}(0)$ and $|x_n^{\sigma} - x_n^{\sigma} - \eta_n^{\sigma})| \leq \varepsilon, |J^{\sigma+2N+1}(x_n^{\sigma}) - \eta_n^{\sigma+2N+1}| \leq \varepsilon$ for all $n$, so by uniqueness $f^{2N+1}(x_n^{\sigma}) = x_n^{\sigma}$ for all $n$.

Again by uniqueness, if $\sigma$ is periodic with period $p$, $n \mapsto x_n^\sigma$ is periodic with period $p(2N + 1)$, while $\sigma \equiv 0$ gives $x_0^\sigma = x_0$ for all $n$. If $\sigma \not= 0$ and $n \mapsto x_n^\sigma$ is periodic with period $M$, we show $M$ is a multiple of $2N+1$. Suppose $M = (2N + 1)k + m$ for some integers $k \geq 0$ and $m$ with $|m| \leq N$. If $m = 0$, it follows as above that $\beta^k \sigma = \sigma$; if $m \not= 0$, we find a contradiction. Now $|\eta_n^\sigma - \eta_n^{\sigma+M}| \leq 2\varepsilon$ so $|\eta_n^\sigma - \eta_n^{\sigma+M}| \leq 2\varepsilon$ for all $n$. There exists $i$ with $\sigma(i) = 1$ and we choose $n = (2N + 1)i$ so $\eta_n^\sigma = y_0$. If $\beta^k \sigma(i) \not= 1, |x_0 - y_0| \leq 2\varepsilon$, which is false; if $\beta^k \sigma(i) = 1, |y_0 - y_m| \leq 2\varepsilon$, which is also false unless $m = 0$.

Regarding the symbol space $S$: if $\psi(0) = 0, \psi'(t) \geq 0, \psi''(t) \leq 0$ and $\psi(t) \not= 0$ for $t > 0$, then $\psi$ is strictly increasing and $\psi(a + b) \leq \psi(a) + \psi(b)$ for $a, b \geq 0$. Since $t \mapsto t/(1 + t)$ has these properties, $d$ is a distance (metric) for $(S)$, and $(S, d)$ is clearly complete. For any integer $N$ let $S_N = \{\sigma \in S \mid \sigma(i) = 0 \text{ for } |i| > N\}$, a finite set; given any $\sigma \in S$ there exists $\sigma_N \in S_N$ with $\sigma(i) = \sigma_N(i)$ for $|i| \leq N$ so $d(\sigma, \sigma_N) \leq 2^{-N}$, so $S$ is totally bounded, hence compact. Define $\theta : S \to \mathbb{R}$ by

$$\theta(\sigma) = 2\sigma(0) + \sum_{n=1}^{\infty} \frac{2\sigma(n)}{3^{2n-1}} + \frac{2\sigma(-n)}{3^{2n}} ;$$

$\theta$ is a continuous map whose image is the set of numbers in $[0,3]$ whose ternary expansion contains only 0 and 2, i.e. the "middle thirds" Cantor set $C$. The restriction $\theta|S \to C$ is a continuous bijection, hence a homeomorphism.

Now we show $\sigma \mapsto x_n^\sigma : S \to X$ is continuous for each $n$. Since $M_0(\varepsilon) < (1 - \theta)/4$, we may choose $\theta_1$ in $0 < \theta_1 < 1$ so that $M_0(\varepsilon)(\theta_1^{-1} + \theta^{-1}) \leq 1/2$. Given $\sigma \in S$ and a positive integer $m$, suppose $\overline{\sigma} \in S$ with $d(\overline{\sigma}, x^\sigma) < 2^{-m}$; then
\( \sigma(i) = \sigma(i) \) for \( |i| \leq m \). Let \( m' = m(2N + 1) \) so \( \eta_k^\sigma = \eta_k^\sigma \) for \( |k| \leq m' \). In notation like that of the shadowing lemma

\[
x_n^\sigma = \eta_n^\sigma + \sum_{i=-\infty}^{\infty} C_{n, k+1}(g(\eta_k^\sigma, x_k^\sigma) + h_k^\sigma)
\]

and similarly for \( \eta^\sigma, x^\sigma \). If \( |n| \leq m' \),

\[
x_n^\sigma - x_n^\sigma = \sum_{i=-\infty}^{\infty} (C_{n, k+1}(g(\eta_k^\sigma, x_k^\sigma) + h_k^\sigma) - C_{n, k+1}(g(\eta_k^\sigma, x_k^\sigma) + h_k^\sigma))
\]

Now

\[
|G_{ij}^\sigma - C_{ij}^\sigma| = \left| \sum_{|k| > m'} C_{i, k+1}(Df(\eta_k^\sigma) - Df(\eta_k^\sigma))C_{k, j}^\sigma \right|
\]

\[
\leq 2M^2K \sum_{|k| > m'} \theta^{|i-k-1|} |\theta| |j-k|.
\]

\[
\leq 2M^2K \times \left\{ C'\theta_1^{|i-j|} \quad \text{for all } i, j \right\}
\]

where \( C' = \sup_{i,j} \sum_{\infty} \theta^{|i-k-1|} |\theta|^{i-k-1} |\theta|^{j-k} \) \( \leq \infty \), by the calculation in Lemma 1.6. Then for \( |n| \leq m' \)

\[
|x_n^\sigma - x_n^\sigma| \leq \sum_{|k| \leq m'} \left. \frac{2M^2K\theta}{1 - \theta^2} \theta^{2m' - |n+k+1|} (\delta + \varepsilon \omega_0(\varepsilon)) \right| + \sum_{|k| > m'} \left. 2M^2K \theta_1^{k+1-|n|} (\delta + \varepsilon \omega_0(\varepsilon)) \right| + \sum_{|k| \leq m'} \left. M\theta^{n-k-1} 2(\delta + \varepsilon \omega_0(\varepsilon)) + \sum_{|k| \leq m'} M\theta^{n-k-1} \omega_0(\varepsilon)|x_k^\sigma - x_k^\sigma| \right|.
\]

This implies, for a constant \( C'' = O(\delta + \varepsilon \omega_0(\varepsilon)) \),

\[
\max_{|k| \leq m'} |x_k^\sigma - x_k^\sigma| / (\theta_1^{m'-k} + \theta_0^{m'+k}) \leq C'' ;
\]

with fixed \( n \) and \( m \to \infty \) (or \( \sigma \to \sigma \)), we see \( x_n^\sigma \to x_n^\sigma \).
If \( \sigma \) has finite support, \( d(\beta^i(\sigma),0) \to 0 \) as \( i \to \pm \infty \), so
\[
x_n^\sigma = x_j^\beta(\sigma) \to x_j^0 = x_0 \quad [n = (2N + 1)i + j, \ |j| \leq N] \quad \text{as} \quad n \to \pm \infty.
\]

(iv) Assume also that \( f \) is injective. Then the orbit \([x_n^\sigma]_{-\infty}^{\infty}\) is determined by any of its points so \( \sigma \mapsto x_0^\sigma : S \to X \) is a continuous injection with compact image \( K \subset V \), and the restriction \( \varphi \) of \( \sigma \mapsto x_0^\sigma : S \to K \) is a homeomorphism. Also \( f^{2N+1}(x_0^\sigma) = x_0^{\beta(\sigma)} \) so \( f^{2N+1}(K) \to K \) is \( \varphi \circ \beta \circ \varphi^{-1} \).

Finally assume (i), (ii), (iv) and (v) but not (iii). Theorem 6.1.9 of [7] shows the global manifolds \( W^s(x_0), W^u(x_0) \) are immersed \( C^1 \) submanifolds of \( V \) with complementary dimensions, \( \dim W^u(x_0) = \text{codim} W^s(x_0) < \infty \), and \( y_n \in W^u(x_0) \cap W^s(x_0) \) for all \( n \). We have
\[
T_{y_n} W^s(x_0) = \{ \eta_n \in X \mid \exists \{ \eta_k \}_{k \geq n} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k \geq n \}
\]
\[
T_{y_n} W^u(x_0) = \{ \eta_n \in X \mid \exists \{ \eta_k \}_{k \leq n} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k < n \}
\]
(These are merely interpretations of the difference equations defining the derivatives: see Theorem 5.2.2 of [7] for the case of continuous time).

Then \( W^s(x_0) \cap W^u(x_0) \) is equivalent to \( T_{y_n} W^s(x_0) \cap T_{y_n} W^u(x_0) = \{ 0 \} \), i.e.
\[
(\eta_k)_{-\infty}^{\infty} \text{ bounded with } \eta_{k+1} = Df(y_k)\eta_k \text{ for all } k \text{ implies } \eta_n = 0 \text{ (so all } \eta_k = 0 \)
\]
which is (iii).

References


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