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Billiards and Conditionally Invariant Probabilities[1]

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Abstract: We analyze the dynamics of a class of billiards (the open billiard on the plane) in terms of invariant and conditionally invariant probabilities. The dynamical system has a horse-shoe structure. The stable and unstable manifolds are analytically described. The natural probability μ is invariant and has support in a Cantor set. This probability is the conditional limit of a conditional probability μ_F that has a density with respect to the Lebesgue measure. A formula relating entropy, Liapunov exponent and Hausdorff dimension of a natural probability μ for the system is presented. The natural probability μ is a Gibbs state of a potential ψ (cohomologous to the potential associated to the positive Liapunov exponent (see (0.1))), and we show that for a dense set of such billiards the potential ψ is not lattice.

Key words: Open billiard, horse-shoe, invariant probability, conditionally invariant probability, entropy, Liapunov exponent, Hausdorff dimension, Gibbs state, non-lattice potential.

0. Introduction. We will present the analysis of the ergodic properties of a certain kind of billiard that we will call the open billiard. Definitons and theorems will be presented in the sequel. The proofs of the results that we consider will be presented in a forthcoming paper [12].

The main purpose of this paper is to give a partial answer to a question proposed by G. Pianigiani and J.Yorke [22] about probabilistic properties of trajectories of billiards:

> "There is a variety of phenomena in which trajectories appear chaotic for an extended period of time but then settle down. Consider a particularly difficult problem of this type. Picture an energy conserving billiard table with smooth obstacles so that all the trajectories are unstable with respect to the initial data. Now suppose a small hole is cut in the table so that the ball can fall through. We would like to investigate the statistical behavior of such phenomena. In particular, suppose a ball is started on the table in some random way according to some probability distribution. Let $p(t)$ be the probability that the ball stays on the table for at least time t and let $p_E(t)$ be the probability that the ball is in a measurable set E after time t. Does $\frac{p_E(t)}{p(t)}$ tend asymptotically to some constant $\mu(E)$ as t goes to infinity? And if it does, what are the properties of μ? Does it depend on the initial distribution? "

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quasi-stationary measures and its limit laws for billiard systems. For a Markov process analogous results were obtained firstly in [18] (see also [10]).

We will consider a class of billiards that we call open billiards and in this case we will present mathematical proofs of the results that answer the questions proposed above. For the open billiards there is no small hole where the ball can fall through, but the ball can get lost to infinity.

The first contribution in the direction of analyzing this type of problems in billiards was done by Pianigiani and Yorke in their mentioned paper, where they consider not billiards, but a related problem for one-dimensional C^2 expanding maps on the interval. They show the existence of a density F that plays an important role in the one-dimensional case. The measure $\mu_F = F(x)dx$ associated to this density is not invariant for the one-dimensional expanding map, but it is conditionally invariant. This result generalizes the Lasota-Yorke theorem to the case where the non-wandering set is a Cantor set. More recently P.Collet, S. Martinez and B. Schmitt [5] present another nice result related to the one-dimensional C^2 case. They showed that the measure μ_F obtained by Pianigiani-Yorke conditionally converges to a certain invariant measure ν.

We will apply these two results in the context of open billiards. In fact the C^2 case is not enough for our purposes, and we need a $C^{1+\epsilon}$ version that will be proved in a forthcoming paper. The dynamics of the billiard we consider is basically the one of a horse-shoe (if one considers a certain special metric). Stable and unstable manifolds can be precisely described and several results about a certain "natural" measure will be presented in the next sections.

We will be able to present a complete picture of all dynamic properties of the billiard we analyze.

The setting and our main results will be briefly presented in the next paragraphs.

The simplest example of the class of billiards we consider is the one given by three non-intersecting discs with equal radius and such that the centers of the disks are at the vertices of an equilateral triangle. This is a good example for the reader to have in mind (even if most of the results we obtain can be applied to more general open billiards).

This billiard is not what is usually called a Sinai billiard, since in our case most trajectories (in the Lebesgue sense) will go to infinity. The set of trajectories that remain on the table in the past and in the future defines a Cantor set. The main obstacles to extend the result presented here to a Sinai billiard (with a hole in the table where the ball could fall through) are the singularities that appear in the system due to the corners and the trajectories that are tangent to the boundaries of such billiards. Therefore we analyze open billiards where such pathologies do not occur.

What we call the "natural" measure μ (sometimes called the escape measure in the literature) was previously considered by C. Grebogi, E. Ott and J. Yorke (see for instance [19] section 5.6) and has the following description: suppose we are considering in the plane a certain expanding map whose non-wandering set is a Cantor set with Lebesgue measure zero. A natural generalization of the Bowen-

Ruelle-Sinai measure in this case is obtained in the following way. Given a set B contained in the Cantor set C, we are going to define the value $\mu(B)$. Consider a grid of squares with side ϵ . Denote by b_ϵ the number of squares that intersect B and c_ϵ the number of squares that intersect the Cantor set C. Now, when ϵ goes to zero, if there exists the limit

$$\lim_{\epsilon\to 0}\frac{b_\epsilon}{c_\epsilon}=\mu(B)$$

and if this limit is independent of the grid for any Borel set B, then we say that μ is a "natural" measure. This procedure is quite natural from the point of view of an experimental observer. Given what is left after n observations (this will produce a slightly distorted grid with a value ϵ inverse ly proportional to n), then one should consider the proportion of what is left of the set that one wants to measure over the full set that still remains. The role of the grid is to give a computable approximation of the Lebesgue measure. We would like to have a procedure allowing to obtain μ as a limit involving the Lebesgue measure (or a measure equivalent to Lebesgue measure).

We will present a precise definition of the probability μ as a Gibbs state [21] of the potential associated with the positive Liapunov exponent, but the reader should keep in mind the above procedure as the one that more precisely describes why the measure μ should be called "natural".

We will also present a formula relating the entropy h_μ, the positive Liapunov exponent χ_μ and the Hausdorff dimension δ of the transverse measure (to be defined later):

$$h_\mu=\delta\ \chi_\mu.$$

For the general case of Axiom-A systems, a proof of this formula appears in [11]. Our result is analogous to the one obtained by Chernov-Markarian [3] for hyperbolic billiards, with a correction term δ due to the fractal structure of the Cantor set.

The Liapunov exponent of a point x will be expressed in terms of the time between bounces $t(x)$ and $k(x)$ (a continued fraction expression involving the time t between bounces of the trajectory by x, the curvature K of the boundaries of the billiard and the angles ϕ of the collisions with the boundary of the trajectory by x). More precisely for almost everywhere x, the Liapunov exponent χ_μ is equal to

$$\chi_\mu=\int\log|1+t(x)k(x)|d\mu(x).$$

The precise definitions will be presented in the next paragraphs.

By definition two functions (potentials) f and g are cohomologous (with respect to a map T from X to itself) if there exist a continuous function h such that $g-f=h\circ T-h$.

The probability μ can be defined as the Gibbs state associated with the potential

$$\psi(x)=-\log|1+t(x)k(x)|; \tag{0.1}$$

this potential is cohomologous to the potential given by minus log of the positive Liapunov exponent: $-\log f'_{|E^u}$ (where f is the billiard map to be defined on the next section). It is therefore natural to ask if the potential ψ is not lattice. We are able to show that for a dense set of billiards, this is so (see section 8). When one consider the statistics of the periodic orbits, it is important to know if the potential is lattice or not [21].

As the system is Axiom A, we are able to estimate the asymptotic growth rate of $n(r)$, the number of periodic trajectories with positive Liapunov exponent smaller than r. The value $n(r)$ grows like $\frac{r}{log r}$ (see[21] Theorem 6.9). In a related result, Morita[17] shows that $t(x)$ is not lattice for a general class of billiards.

The class of billiards we analyze here, apparently has some importance in the theory of quantum chaos (see [6],[7],[18],[19],[23]). The asymptotic growth rate of the number of periodic orbits is of indubitable relevance in this theory.

In [3], related results about quasi stationary measures for horse-shoe diffeomorphims were obtained.

1. The billiard map. Consider a finite number of closed curves δQ_i (where $Q_i, i = 1, 2, .., s, \quad s > 2$ are nonintersecting compact convex sets in the plane), that can be either $C^{r+1}, \quad r > 2$ with non-zero curvature or real analytic. We will call this system the open billiard.

We will say that the open ball billiard satisfies condition (M) if all curves are simple closed curves and the convex hull of $\delta Q_i \cup \delta Q_j$ does not intersect δQ_k for any triple of three distinct indices i, j, k. We will assume that all the billiards we consider here satisfy the condition (M).

We will denote by δQ the union of all $\delta Q_i, \quad i = 1, ..., s$ and by $n(q)$ the normal to the curve δQ at the point q. The normal will have norm one and point out to the outside of the curve.

Consider the dynamical system describing the free motion of a point mass in the plane, with elastic reflections on δQ (angle of incidence with the normal to the curve equal to the angle of reflection). The phase space of such a dynamical system is

$$M = \{(q, v); \quad q \in \delta Q, \quad |v| = 1, \quad < v, n(q) > \geq 0\}.$$

A coordinate system is defined on M by the arc length parameter r along δQ (therefore the state space in these coordinates has more than three connected components because $s > 2$) and the angle ϕ between $n(q)$ and v. Clearly $|\phi| \leq \pi/2$ and $< n(q), v >= \cos(\phi)$.

Consider the probability $d\lambda = c\cos(\phi)drd\phi$, where $c = 2|\delta Q|^{-1}$ is just a normalizing factor and $|\delta Q|$ stands for the total length of δQ.

Now we define the transformation map f in the following way:

$$f(x_0) = f(q_0, v_0) = (q_1, v_1)$$

with q_1 the point of δQ (if there exists such a point) where the oriented line through (q_0, v_0) first hits δQ and v_1 the angle with the normal $n(q_1)$ made by that line after reflection on the tangent line through $q_1 \in \delta Q$. Formally, $v_1 = v_0 - 2 < n(q_1), v_0 > n(q_1)$ (see fig 1). This transformation map f may not be defined for all $x_0 \in M$.

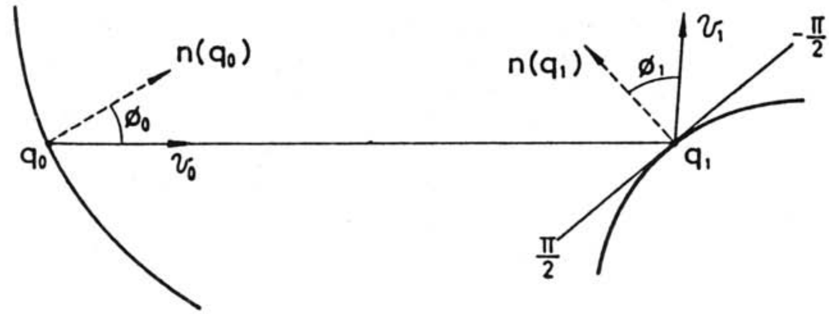

fig. 1

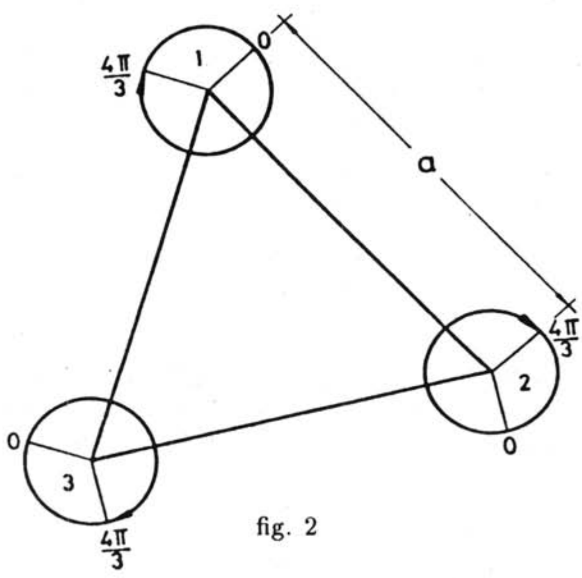

fig. 2

The measure λ is not globally invariant under f (any invariant measure is singular with respect to Lebesgue measure), but if x and $f(x)$ are in small open sets, then the image of the measure λ by f is preserved. f is a C^r diffeomorphism in these small neighbourhoods. The Euclidean length t (or time) between q_0 and q_1 is denoted by t_0. Hence, $q_1 = q_0 + t_0 v_0$ (a trajectory inside the billiard travels with constant velocity equal to one).

The map f is called the billiard map. We are interested in analyzing trajectories with infinite bounces. The trajectories that do not have this property are the ones that in some finite (positive or negative) time escape to infinity.

We will denote by $x_i = (q_i, v_i) \in M$, $i \in \mathbf{N}$ the successive hits of a trajectory beginning at time 0, $x_0 = (q_0, v_0)$, with the boundary δQ, that is, $f(q_i, v_i) = (q_{i+1}, v_{i+1})$. We are interested among other things in properties for trajectories with $x_0 = (q_0, v_0)$ in a set of full μ-measure (μ stands for the natural measure).

Given a trajectory beginning at $x_0 = (q_0, v_0) \in \delta Q$, we will denote by $K_i = K(x_i)$, $i \in \mathbf{N}$, the curvature of δQ at q_i. For instance, if one considers the model where all Q_i, $i = 1, .., s$, are disks, then the K_i are all constants. The angle between $n(q_i)$ and v_i will be denoted by ϕ_i and finally, t_i denotes the Euclidean distance between the bounces q_i and q_{i+1}, $i \in \mathbf{N}$ (see fig 1). The backward orbit $x_i = (q_i, v_i)$, $i \in \mathbf{Z}$, is analogously defined. In any case, the main property is $f(x_i) = x_{i+1}$, $i \in \mathbf{Z}$.

In the case we are considering, if f is defined for $x_0 = (q_0, v_0) \in M$, then it is also defined in an open neighbourhood of x_0 unless the trajectory through x_0 hits the image $f(x_0) = f(q_0, v_0) = (q_1, v_1) = x_1$ in a position tangent to δQ, that is, $v_1 = \pi/2$ or $v_1 = -\pi/2$. In this case f is defined in an left or right open neighbourhood. When we speak about neighbourhoods we are considering any one of the possible cases described above. The set of points $x_0 = (q_0, v_0) \in M$ whose forward or backward trajectory is tangent to δQ for some x_i, $i \in \mathbf{Z}$ has λ-measure zero.

If $\tilde{x}_1 = (\tilde{q}_1, \tilde{v}_1) = f(\tilde{x}_0)$ is defined for $\tilde{x}_0 = (\tilde{q}_0, \tilde{v}_0)$, then for all $x_0 = (q_0, v_0)$ in a neighbourhood of $\tilde{x}_0$ the derivative matrix is given by (see [3],[14])

$$f'(x_0) = \begin{pmatrix} \frac{t_0 K_0 + \cos\phi_0}{-\cos\phi_1} & \frac{t_0}{\cos\phi_1} \\ K_1 \frac{t_0 K_0 + \cos\phi_0}{\cos\phi_1} + K_0 & -\frac{K_1 t_0}{\cos\phi_1} - 1 \end{pmatrix} \tag{1.1}$$

Note that when the image of (q_0, v_0) by f is tangent to δQ (that is, $q_1 = \pi/2$ or $q_1 = -\pi/2$), then the entries of the above matrix become infinity.

2. The open billiard with three circumferences. We will consider now a particular example where the hypotheses of all results presented in this paper are satisfied. Consider three circular disks of radius one (fig 2) whose centers are located in the vertices of an equilateral triangle of side $a > 2$.

The more natural system of coordinates to consider in this problem is to denote by r the angle of the q coordinate in each circle.

In this case the phase space is given by three rectangles M_1, M_2 and M_3, where each one is a copy of a rectangle with base $0 \leq r \leq 4\pi/3$ and height $-\pi/2 \leq \phi \leq \pi/2$. (see fig. 2 and 3).

We will denote by δQ_i the circle corresponding to the set M_i, $i = 1, 2, 3$.

As an example notice that the point $(\pi/2, 0) \in M_1$ is a periodic point with period 2, because $f(\pi/2, 0) = (5\pi/6, 0) \in M_2$ and $f(5\pi/6, 0) = (\pi/2, 0) \in M_1$.

There exist several trajectories that are not periodic but have infinitely many bounces. The map f is not defined everywhere (see fig 3); for example, it is not defined at the point $(4\pi/3, 0)$.

In fact the map f and its inverse f^{-1} are not defined outside the dashed region in fig 3. The horse-shoe structure of the map f will be more carefully explained later.

Notice that if $a \leq 2$ then the billiard is an example of a classical Sinai billiard, because different components of the boundary intersect with non zero angle. The statistical properties of this kind of billiards have been extensively studied.

If $2 < a \leq 4/\sqrt{3}$, then it is easy to see that for such open billiards the condition (M) defined above is not satisfied. The case $a = 2$ is extremely interesting but it will not be analyzed here.

The three circles open billiard subject to the condition $a > 4/\sqrt{3}$ satisfies the condition (M) and it is under the assumptions of the theorems that we will prove later. It is the simplest example of such a class of open billiards. Apparently, the results we present in the next sections can be also extented to the case $2 < a < 4/\sqrt{3}$. We will indicate why we believe this is true (see the end of section 3).

The dynamics of f in the case $a > 4/\sqrt{3}$ is the same as of a shift of finite type. This can be seen as follows. Denote by π : domain of $f \to \{1, 2, 3\}$ the map that assigns to each $x = (q, v) \in M$ the value i such that $q \in \delta Q_i$. Given a certain sequence $\theta_i \in \{1, 2, 3\}$, $i \in \mathbf{Z}$, such that for any i, $\theta_i \neq \theta_{i+1}$, $i \in \mathbf{Z}$, there exists a unique $x_0 = (q_0, v_0)$ such that

$$\pi(f^n(x)) = \pi(q_n, v_n) = \theta_n .$$

It is also true that $\pi \circ f(x) = \sigma \circ \pi(x)$, where σ is a shift of finite type on three symbols $\{1, 2, 3\}$. In other words, π is a conjugacy of f with the shift σ. Therefore, the dynamics of f is the one of a shift of finite type (remember that $\theta_i \neq \theta_{i+1}$, but this is the only restriction). This result was shown by Morita[17]. We will need to analyze metrical questions and therefore we will need more delicate properties and estimates about the dynamics; the fact that f is conjugated to the shift σ is not enough. Among other problems, we will need to take special attention when the entries of the matrix (1.1) become infinity due to tangencies of the orbit, etc...

Morita[17] also shows that the ceiling function $t(x)$ (the time between bounces) is Hölder continuous and non-lattice. We will consider here another potential ψ (different from t) that is natural in the setting we are working in. We will also show that for a dense set of values $a > 4/\sqrt{3}$, the potential ψ is not lattice. This allows one to estimate the growth number of periodic trajectories subject to weights, as in [21].

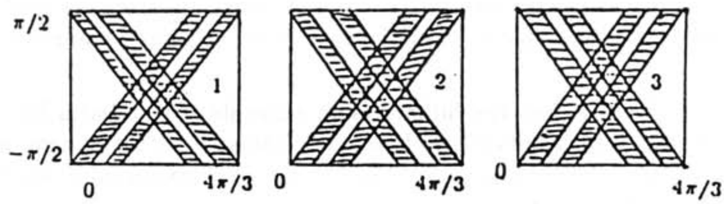

fig. 3

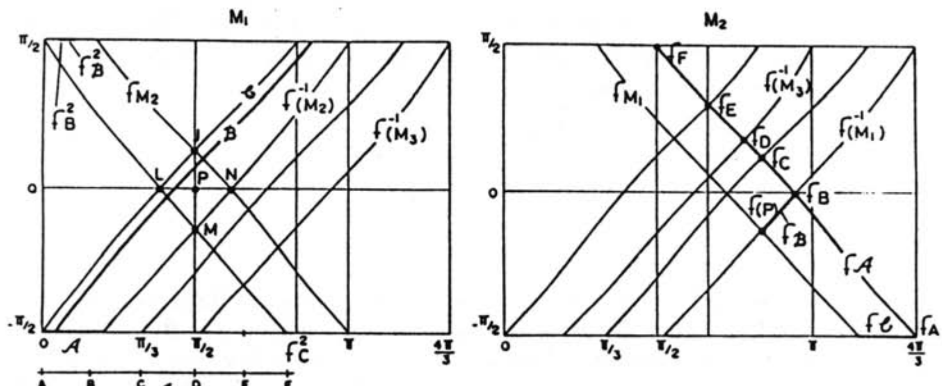

fig. 4

3. Trajectories with infinitely many bounces. Our first goal is to analyze geometrical and dynamical properties of the set of points that have infinitely many bounces in the past and in the future. This subset of M will have the structure of the product of two Cantor sets. We will begin considering the trajectories such that there exist infinitely many bounces in the future. We need therefore to analyze the set

$$\cap_{0\leq j} f^{-j}(M_{i_j}),\quad i_j \neq i_{j+1},\quad \forall j \in \mathbf{N}.$$

We will carefully analyze the case $a > 4/\sqrt{3}$, even if at the end of our reasoning, we will be able to indicate why we believe it is also true for $a > 2$.

From the symmetry of the problem, it follows that we have to analyze the structure of the set M_1 intersected with $\cap_{0\leq j} f^{-j}(M_{i_j})$, where $i_0 = 1$, because for the other connected components M_2 and M_3 the structure is basicall y the same (we have of course to assume respectively that $i_0 = 2$ or $i_0 = 3$).

In fig 4, we represent some of the backward iterates.

Note that the line $\mathcal{A} = \{(r, -\pi/2),\ \ \pi/3 \leq r \leq \pi/2\} \subset M_1$ iterated by f goes on the curve $f(\mathcal{A}) \subset M_2$ shown in fig 4. The curve $f(\mathcal{A})$ can be also parametrized by r, given by the projection $(\phi, r) \rightarrow r$, over $\pi/2 \leq r \leq 4\pi/3$ (see fig 4). We draw two strips in M_1 corresponding to the pre-images $f^{-1}(M_2)$ and $f^{-1}(M_3)$ in fig 4. There are also two other important strips, the ones corresponding to the images $f(M_2)$ and $f(M_3)$ in M_1 (see the first square in fig. 3). We only draw in fig. 4 the set $f(M_2)$ in order to make more clear the other curves and sets that we will describe in the sequel. The intersections of these four strips are four non-linear rectangles in M_1 that correspond to the cylinders (with coordinates θ in the shift) $\{2,1,2\}, \{3,1,2\}, \{2,1,3\}$ and $\{3,1,3\}$.

Similar pictures can be drawn in M_2 and M_3. From this picture the reader can realize the horse-shoe structure of the dynamics of f (see also fig. 3). It is important to point out that the distortion could be very bad close to the boundaries and this requires a more delicate analysis. In other words we need extra care with the almost tangent trajectories because in this case the expanding properties are not s o good. This question will appear in the next sections.

We draw the curve $\mathcal{A}$ in the left square of fig. 4 and its image $f(\mathcal{A})$ in the right square of fig. 4. To be more explicit about the dynamics of f we denote by A, B, C, D, E, F points in the curve $\mathcal{A}$. Note the position of the images of these points in the set $f(\mathcal{A})$ in the right square in fig. 4. Note also the curve $\mathcal{B}$ and its image $f(\mathcal{B})$ (see fig. 4). The curve $\mathcal{C}$ represents positions (r_0, ϕ_0) whose image $f(r_0, \phi_0) = (r_1, \phi_1)$ will hit the circle 2 in a tangent positon $(\phi_1 = -\pi/2)$ (see fig. 4)

The strip that appears in M_1 between the two strips $\{1,2\}$ and $\{1,3\}$ corresponds to the trajectories of M_1 that are lost in the middle of the two circles M_2 and M_3. The two other components in M_1, external to $\{1,2\}$ and $\{1,3\}$, correspond to the trajectories that are lost between M_1 and M_2 or between M_1 and M_3. The cylinders $\{1,2,1\}$, $\{1,2,3\}$, $\{1,3,1\}$ and $\{1,3,2\}$ correspond in M_1 to four strips contained in the two strips $\{1,2\}$ and $\{1,3\}$ (see fig 5). These four strips are strictly inside the two previous ones.

Inductively, the cylinders $\{1,i_1,i_2,..,i_n\}$, $i_j \neq i_{j+1}$, $j \in \{1,2,..,n-1\}$, correspond to $\cap_{j=1,..,n} f^{-j}(M_{i_j})$ and are 2^n thin increasing strips going from the bottom to the top of M_1.

Note the important geometrical property presented in fig 4, showing how the set $\mathcal{A}$ goes by f into the curve $f(\mathcal{A})$. The boundary of $M_1 \cap f^{-1}M_2$ goes by f into the upper and lower boundary of M_2. The correct understanding of the geometrical position of all these boundaries and its images by f is essential for the next sections.

The intersection of an infinite sequence of nested sets is given generically by

$$\cap_{j=1}^{\infty} f^{-j}(M_{i_j}), \quad i_j \neq i_{j+1}, \quad j \in \mathbf{N},$$

and it is a curve coming from the bottom to the top of M_1 (in order to prove this property, which follows from expansiveness, we need to use an analytical expression that will be shown in section 4 and 5). We will show finally that the union of all such possible nested sequences of sets can be parametrized as the product of a Cantor set by such curves.

4. Analytical expressions. We will now obtain the analytical expression of the differential equations satisfied by the invariant curves that generate the Cantor set, which were mentioned in the last paragraph.

To illustrate our reasoning, we will first obtain the equation of the curve $\mathcal{B} \subset M_1$ through x such that $f(\mathcal{B}) \subset M_2$ and $f^2(\mathcal{B}) \subset M_1$, with $\phi_2 = \pi/2$, $\frac{d\phi_2}{dr} = 0$ (we are using the notation $f^i(x) = x_i = (r_i, \phi_i)$). This curve $\mathcal{B}$ contains the 2-periodic point p_{121}. It follows from [3], [14] that

$$\frac{d\phi_1}{dr_1}(x) = K_1(x) + \frac{\cos\phi_1(x)}{t_1(x)},$$

$$\frac{d\phi_0}{dr_0}(x) = K_0(x) + \cos\phi_0(x)\frac{1}{t_0(x) + \frac{1}{\frac{2K_1(x)}{\cos\phi_1(x)} + \frac{1}{t_1(x)}}}.$$

The last equation describes the parametrization ϕ_0 of $\mathcal{B}$.

Now, by induction, it follows that the boundary of the strips that successively appear when we remove the trajectories that go to infinity at time n, is given by

$$\frac{d\phi_0}{dr_0} = K_0 + \cos\phi_0\frac{1}{t_0 + \frac{1}{\frac{2K_1}{\cos\phi_1} + \frac{1}{t_1 + \frac{1}{\frac{2K_2}{\cos\phi_2} + \frac{1}{\dots + \frac{1}{\frac{2K_n}{\cos\phi_n} + \frac{1}{t_n}}}}}}}.$$

We omitted the reference to the point x in the above formula.

When n goes to infinity the above equation will converge to the equation of the parametrization of the curve of points $y \in M_1$ with the same future specification of bounces θ_i, $i \in \mathbf{N}$ as x .

The continued fraction that appears multiplying $\cos\phi_0$ is given by

$$k^s(x) = \frac{1}{b_1(x) + \frac{1}{b_2(x) + \frac{1}{b_3(x) + \dots}}}, \tag{4.1}$$

with

$$b_{2k}(x) = \frac{2K(f^k(x))}{\cos\phi(f^k(x))} = \frac{2}{\cos\phi(f^k(x))}, \qquad b_{2k+1}(x) = t(f^k(x)). \tag{4.2}$$

This continued fraction converges if $K(f^k(x)) > 0$ and $\sum_{k=0}^{\infty} t(f^k(x)) = \infty$ (see [4],[17]). For the open billiard we consider here, this is the case because $K(f^k(x)) = 1$ and $t(f^k(x)) > a - 2$ for all k. Therefore, the curves that in the future have infinitely many bounces are defined by the differential equation

$$\frac{d\phi}{dr}(x) = K(x) + k^s(x)\cos\phi(x).$$

We point out that this is also true for the billiards considered by Morita, when the obstacles are convex and the condition (M) is true.

We will use the notation

$$\frac{d\phi^s}{dr}(x) = K(x) + k^s(x)\cos\phi^s(x) \tag{4.3}$$

to enhance that this differential equation determines the parametrization $\phi^s(r)$ in the variable r, of the stable manifold $(r, \phi^s(r))$ through $x_0 = (r_0, \phi_0)$. Note that the differential equation is non-autonomous because we take derivatives in r, but k depends on (r, ϕ).

In an analogous way one can show that the curve through x_0, given by the set of points (r, ϕ) with infinitely many bounces in the past (the unstable manifold passing through x) is parametrized by $(r, \phi^u(r))$, with $\phi^u(r)$ given by:

$$\frac{d\phi^u}{dr}(x) = K(x) - k^u(x)\cos\phi^u(x), \tag{4.4}$$

where

$$k^u(x) = a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \frac{1}{a_4(x) + \cdots}}}, \tag{4.5}$$

and

$$a_{2k+1}(x) = \frac{2}{\cos\phi(f^{-k}(x))}, \quad a_{2k}(x) = t(f^{-k}(x)), \quad k \in \mathbf{N}.$$

5. The hyperbolic structure: stable and unstable manifolds. Consider in the descending strip of type $\{1,2,1\}$, the unstable manifold of the 2-periodic point $p = p_{121} = (\pi/2, 0) = f^2(\pi/2, 0) \in M_1$. The stable manifold is given by

$$\gamma^s(p) = \{z; \pi(f^{2n}(z)) = 1, \quad \pi(f^{(2n+1)}(z)) = 2, \quad \forall n \in \mathbf{N}\}$$

and the unstable manifold through p is given by

$$\gamma^u(p) = \{z; \pi(f^{-2n}(z)) = 1, \quad \pi(f^{-(2n+1)}(z)) = 2, \quad \forall n \in \mathbf{N}\}.$$

More generally, consider the 2-periodic points p_{iji} in M_i, $i \neq j$, $i, j \in \{1,2,3\}$; there is a total of 6 such periodic points of period 2.

Unstable manifolds are defined by graphs of decreasing functions and stable manifolds are described by graphs of increasing functions. This follows from the inclination of the parametrizations ϕ given by the analytical expressions (4.3) and (4.4) of the differential equations described in section 4.

Let $\gamma^u(p_{iji})$ be the unstable manifold through p_{iji} intersected with the set M_i and

$$\gamma^u_{ij} = \gamma^u(p_{iji}) \cap f^{-1}(M_j). \tag{5.1}$$

Note that the curve $\gamma^u(p_{iji})$ goes from the bottom to the top of M_i, but for $\gamma^u_{ij} = \gamma^u_{ij}$ this is not true.

Denote by $M' = \cup_{i,k\neq j} M_{ijk}$ the union of the twelve quadrilaterals, where $M_{ijk} = f(M_i) \cap M_j \cap f^{-1}(M_k)$. The dynamics of the trajectories that do not go to infinity can be studied in M'. These quadrilaterals are far away from $\phi = \pm\pi/2$ and hence, for $x \in M'$, $\cos\phi(x) > c_1 > 0$.

Now we define the p-length of a general curve $\gamma \subset M$ by

$$p(\gamma) = \int_\gamma \cos\phi dr. \tag{5.2}$$

More precisely, if γ is defined by $(r, \phi(r))$, $r_0 \leq r \leq r_1$, then

$$p(\gamma) = \int_{r_0}^{r_1} \cos\phi(r) dr.$$

If γ is any decreasing curve ($\phi'(r) < 0$), and f is continuous in γ, then

$$p(f(\gamma)) = \int_{r_0}^{r_1} \left(\frac{t(r)(K(r) - \phi'(r))}{\cos\phi} + 1 \right) \cos\phi dr, \tag{5.3}$$

where $t(r) = t(x)$ is the distance to the next bounce beginning at $x = (r, \phi(r))$. Since $p(\gamma)$ is of order $\cos\phi_0 dr$ (for small γ), for small γ passing through $x_0 = (r_0, \phi_0)$, $\frac{p(f(\gamma))}{p(\gamma)}$ is approximately equal to $1 + \frac{t(r_0)(K(r_0) - \phi'(r_0))}{\cos\phi(r_0)}$ with $x_0 \in \gamma$.

This property will lead us to define a kind of partial derivative $\delta f^p_\gamma(x_0)$ using the p-length defined above.

Definition 1. Given a curve γ through x_0, we define *the p-derivative of γ at* x_0 as the limit

$$\delta f^p_\gamma(x_0) = \lim_{p(\gamma)\to 0} \frac{p(f(\gamma))}{p(\gamma)}.$$

The dynamics of f is the one of a horseshoe diffeomorphism (see [12]) and this result is obtained using the p metric defined above. Therefore all considerations in chapter 2 of [20] can be applied and we conclude that there exist a $C^{1+\epsilon}$ foliation of stable and unstable foliations around the non-wandering set.

It easily follows (see [20], chapter 2) that the projection along stable (and unstable) leaves is $C^{1+\epsilon}$. This property explains why we will need in the future a

$C^{1+\epsilon}$ version of the results of class C^2 that were previously obtained by other authors [5] [22].

6. Expanding transformations and invariant measures. We will state in this section the $C^{1+\epsilon}$ results that we will need in section 7. These results will be proved in the appendix.

A piecewise continuous map T is transitive on components if for every two maximal sets B, C where T is continuous, there exists $n = n(B, C) \in \mathbf{N}$ such that $T^n B \cap C \neq \emptyset$.

We will say that a probability measure μ, defined on the elements of a σ-álgebra $\mathcal{A}$ of A, is conditionally invariant with respect to $T : A \to TA$ if $\mu(T^{-1}C) = \alpha\mu(C)$ for every element $C \in \mathcal{A}$, for some positive constant α.

It results $\alpha = \mu(T^{-1}A)$. Hence μ is conditionally invariant if and only if

$$\mu(T^{-1}C|T^{-1}A) = \mu(C).$$

This implies that $\alpha^n = \mu(T^{-n}A)$ for every $n \geq 0$.

We will represent by μ_F, the probability measure $d\mu_F = F d\nu$ where ν is another fixed probability measure on A, and $\int_A F d\nu = 1$.

Hypothesis A: Assume $T : \bar{A} \to \mathbf{R}$, $B = A \cap T^{-1}A$, is such that

a) $A = \bigcup_{i=1}^{k} A_i$ where A_i are disjoint open intervals;

b) $A \subset T(A)$ (strictly);

c) $A \cap T(\partial A) = \emptyset$;

d) $\bar{A}$ is endowed with some metric d, such that the derivative T_d of T with respect to this metric, is well defined on B; i. e.: there exists

$$(6.1) \qquad T_d(x) = \lim_{y \to x} \frac{d(Ty, Tx)}{d(y, x)}$$

for every $x \in B$;

e) T_d is γ-Hölder continuous on B; i. e.: there exist $k > 0$ and $0 < \gamma < 1$, such that $|T_d(x) - T_d(y) \leq k d^\gamma(x, y)$ for every x, $y \in B$;

f) there exist $M > \beta > 1$ such that $\inf\{T_d(x) : x \in B\} \geq \beta$ and $\sup\{T_d(x) : x \in B\} \leq M$.

g) $T_{|\bar{A}_i}$ is an homeomorphism for every $i = 1, \ldots, k$.

Let be ν the probability measure induced by the metric d on Borel sets of A.

The proof of the next two theorems for the case $C^{1+\epsilon}$ will be presented in [12].

Theorem 1. (see Pianigiani-Yorke[22] for the case C^2). Let $T : \bar{A} \to \mathbf{R}$ satisfies *Hypothesis A* a)-g). Then

i) There exists a γ-Hölder continuous function $F : A \to \mathbf{R}$,

$$F \in \mathcal{K} = \{G \in C^0(A), \inf_{x\in A} G > 0, \sup_{x\in A} G < \infty, \int G d\nu = 1\}$$

such that μ_F is absolutely continuous with respect to the measure ν, and conditionally invariant with respect to T.

ii) If T is also transitive on components, then there exists a unique $F \in \mathcal{K}$ such that μ_F is conditionally invariant with respect to T.

iii) If, furthermore T^n is transitive on components for all $n \in \mathbf{N}$ then, for every $g \in \mathcal{K}$,

$$\lim \frac{P_1^n g}{\|P_1^n g\|_1} \to F.$$

Here $\| \|_1$ means the L^1 norm on $L^1(A,\nu)$ and
$P_1 : L^1(A,\nu) \to L^1(TA,\nu)$ is the Perron-Frobenius operator defined by

$$(6.2) \qquad P_1 g(x) = \sum_{y:T_y=x} (T_d(y))^{-1} g(y) = \frac{d(\mu_g \circ T^{-1})}{d\nu}.$$

We consider $P_1 : L^1(A,\nu) \to L^1(A,\nu)$ by taking the restriction of $P_1 f$ to A. Then $\int_A f.(g\circ T)d\nu = \int_{TA}(P_1 f) g d\nu$ for $f \in L^1(A,\nu)$, $g \in L^1(TA,\nu)$, and

$$\int_{T^{-n}(A)} f.(g \circ T^n) d\nu = \int_A (P_1^n f) g d\nu,$$

for every $f, g \in L^1(A,\nu)$.

See proof in [12].

We define now the operator $Q : L^1(A,\nu) \to L^1(A,\nu)$ by

$$(6.3) \qquad Qg(x) = [\alpha F(x)]^{-1} P_1(gF)(x)$$

where $\alpha = \mu_F(T^{-1}A) = SF d\nu$. Since $P_1 F = \alpha F$, we have that $Q1 = 1$.

The reader familiar with Thermodynamic Formalism (see [21]) will recognize the operator Q as the Ruelle-Perron-Frobenius operator obtained from the potential

$$\log \frac{|T_d|^{-1}(x) F(x)}{\alpha F(T(x))}.$$

This potential is cohomologous to the potential $-\log |T_d(x)|$. The procedure of defining Q by (6.3) above is usual in Thermodynamic Formalism when one knows the eigenfunction F and the eigenvalue α. This procedure is sometimes called normalization of the operator.

We refer the reader to [21] where the theory of Thermodynamic Formalism developped initialy by Bowen, Ruelle and Sinai is carefully described.

In terms of the variational problem of the pressure the two cohomologous potentials will determine the same Gibbs state.

The reader should take care with the different domains where the two operators are defined: the Perron-Frobenius operator of Lasota-Pianigiani-Yorke is defined over L_1 functions and the Ruelle-Perron-Frobenius operator of Thermodynamic Formalism is defined over Holder continuous functions. The most surprising property of the Pianigiani-Yorke result is the existence of the derivative of F in a full neighbourhood of the Cantor set under the C^2 hypothesis. Under the $C^{1+\epsilon}$ hypothesis, we will show in [12] that F will be Holder continuous.

Now we will need another result related to Theorem 1.

Theorem 2. (see Collet, Martinez, Schmitt[5] for the C^2 case). Let $T : \bar{A} \to \mathbf{R}$ satisfies *Hypothesis A* a)-g) and suppose that T^n is transitive on components for all $n \in \mathbf{N}$. Then

i) $Q^n g(x) \to \mu(g)$ for every γ-Hölder continuous function g on A. $\mu(g)$ defines a probability measure μ, with support on $K = \bigcap_{n \geq 0} T^{-n} A$;

ii) The conditional probability measure of staying in A, when the evolution occurs with probability μ_F, $\mu_F(C|T^{-n}A) \to \mu(C)$ when $n \to +\infty$, for every Borel set $C \subset A$;

iii) μ is Gibbsian with potential $-\log T_d(x)$; i. e.

$$c^{-1}[(T^n)_d(z)]^{-1}\alpha^{-n} \leq \mu\left[\bigcap_{l=0}^{n-1} \bar{T}^{-l}(\bar{A}_{i_l})\right] \leq c[(T^n)_d(z)]^{-1}\alpha^{-n}$$

for every $i_0, ..., i_{n-1} \in \{1, 2, ..., k\}$, every $n \in \mathbf{N}$ and some $z \in \bigcap_{l=0}^{n} T_i^{-l} \bar{A}_l$.

So, $(A, T, \mathcal{A}, \mu)$ is a Kolmogorov system, satisfies the property of exponential decay of correlations, and

$$\log\alpha = h_\mu(T) - \int_A \log T_d(x) d\mu(x) = \sup\{h_\eta(T) - \int \log T_d(x) d\eta(x) :$$

η is an invariant probability measure $\}$,

where $h_\eta(T)$ is the entropy of T with respect to η.

See proof in [12]

Now we will make some comments about different properties claimed by the above Theorem.

The precise meaning of the limit in property i) will be explained later in the appendix. The conditions above allows one to apply the Riesz Theorem, defining in this way a probability μ such that $\mu(g) = \int g(x) d\mu(x)$. The measure μ is invariant for T and therefore the support of μ is the non-wandering set of T (having in this case a Cantor set structure on the line). The property ii) is the more important one. It claims that if we calculate $\mu_F(V|T^{-n}(A))$, the part of V

in $T^{-n}(A)$ (the subset of A that still remains in A after n iterations), then when n goes to infinity, the system will determine in the limit a certain measure $\mu(V)$. The analogy of the natural measure we mention before and the measure μ we just defined (and satisfying property ii)) is transparent.

Property iii) is also very important because a Gibbsian measure has several nice properties: the system is Kolmogorov (therefore ergodic), there exists exponential decay of correlation, etc.... (see[24]).

Both Theorems can be formulated for $T : \bar{A} \to \mathbf{R}^n, A = \cup A_i$, where $A_i \subset \mathbf{R}^n$ are disjoint connected uniformly arcwise-bounded sets. This means that there exists a number b such that any two points in each A_i can be joined by a polygonal line of lengt at most b (see[24]).

7. Measures for open billiards: invariant and conditionally invariant. Now we will return to the considerations of section 5, and show how the results of section 6 can be applied to the open billiard.

Note for example that for the point $(\pi/2, 0) \in M_1$, $f(\pi/2, 0) = (5\pi/6, 0) \in M_2$ and $f(5\pi/6, 0) = (\pi/2, 0) \in M_1$. Therefore the 2-periodic orbit $(\pi/2, 0) \in M_1$ and $(5\pi/6, 0) \in M_2$ has an unstable manifold with two components. Since there exist three pairs of 2-periodic orbits, we will consider six small pieces of unstable curves around these six periodic points.

Formally, let p_{iji} the 2-periodic point such that $p(f^{2n}(p_{iji})) = i$ and $p(f^{2n+1}(p_{iji})) = j$, where $n \in \mathbf{Z}$. The local unstable manifold of p_{iji} is defined by

$$\gamma^u(p_{iji}) = \{z \in M_i; p(f^{-2n}(z)) = i, \quad p(f^{-(2n+1)}(z)) = j, \quad \forall n \in \mathbf{N}\}.$$

Let us write $\gamma_{ij} = \gamma^u(p_{iji}) \cap f^{-1}(M_j)$, therefore, as we have seen in (5.1),

$$f(\gamma_{ij}) = \gamma^u(p_{jij}).$$

Note that the image of each one of the γ_{ij} six small pieces of unstable manifold is the full unstable manifold (from bottom to the top) through another 2-periodic point.

Denote also by $\Pi^s_{jk} : M_j \cap f^{-1}(M_k) \to \gamma_{jk}$ the projection along stable fibers; as we mentioned before, this projection is $C^{1+\epsilon}$ (this is the reason for the need of $C^{1+\epsilon}$ theorems in the present paper).

Theorem 3: - *Consider the system described in section 2. Let A be the set $\cup_{i\neq j}\gamma_{ij}$. We define T on $\bar{A}$ as a continuous extension of its values on the twelve connected pieces of curves $\gamma_{ij} \cap f^{-2}(M_l)$, $l \neq j$. On these curves, T is defined by $T(x) = f(x)$, if $f(x) \in \gamma_{ji}$, and $\Pi^s_{jk} f(x)$, if $f(x) \in f^{-1}(M_k)$, $k \neq i$.*

Then $T : \bar{A} \to T(\bar{A})$ satisfies the Hypothesis A and also the hypotheses of Theorems 1 and 2.

See proof in [12].

Remark 1: Note that now A is a union of pieces of curves in $\mathbf{R}^2$ and not a union of intervals in $\mathbf{R}$ as in Theorems 1 and 2, but the proof of the analogous result is the same.

Remark 2: To be more precise, we will need to consider f^N, a high iterate of f as having the hypotheses of Theorems 1 and 2 satisfied, but this is no problem for our purposes, as will be explained later.

Now we will construct the natural two dimensional measure for the open billiard problem.

Remember that $M' = \cup_{i,k\neq j} M_{ijk}$.

Theorem 4: *Consider the system $f : M' \to M'$ described in section 2 and 5. Then there exists a conditionally invariant posite measure μ^+ and a measure $\mu_1^+(V)$ such that*

$$\lim_{n\to\infty} \mu^+(V|f^{-n}(M')) = \mu_1^+(V),$$

for every Borel set $V \subset M'$. The measure μ_1^+ is invariant under f, supported on K_1 K_2 and (M', f, μ_1^+) is a K-system.

See proof in [12]

8. The non-lattice property of the potential ψ. We say that a potential B is lattice if there exist an integer valued function G, γ a real positive constant and a continuous function g such that $B = g \circ T - g + G\gamma$.

When one wants to prove asymptotic growth rate properties of the periodic orbits (see[21]), the proofs are different for the lattice and non-lattice potential. It is possible to obtain such properties by means of Tauberian Theorems combined with Fourier Series arguments (in the case of lattice potentials) or Fourier Transforms arguments (in the case of non-lattice potentials). The lattice potentials appear only in very special situations. One should expect that in general the potentials that ocurr in mathematical problems are non-lattice. This is the Claim of the main Theorem of the present section.

In this section we will show that for the billiard given by three circles of radius one centered at the corners of an equilateral triangle with side a, the Liapunov exponent potential is not lattice for a dense set of possible values a. This claim is equivalent to showing that the potential ψ defined before is not lattice, because these potentials (up to a minus sig) are cohomologous as shown in (7.1). From this result, we obtain the asymptotic growth rate property of Liapunov exponents of periodic orbits that was mentioned in section 0 (see [21]). The next theorem shows the existence of a dense set of interesting examples.

Theorem 5 - *The potential ψ is non-lattice*

See proof in [12].

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