Resenhas IME-USP 1994, Vol. 1, No. 4, 487 - 493.

A priori bounds for C^2 homeomorphisms of the circle

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Abstract: In this paper we establish C^2 a-priori bounds for the scaling ratios of critical circle mappings in a form that gives also a compactness property for the renormalization operator.

Key words: Scaling ratios, Poincaré metric, renormalization.

§1. Introduction

Understanding the geometric scalings of circle mappings can be viewed as a first step towards a smooth classification theory for such mappings. For diffeomorphisms, this problem was solved by Herman in [H]. Let us briefly state Herman's results in this context.

Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism, let $c \in S^1$ be a distinguished point and let

$$\rho(f) = [r_0, r_1, \cdots] = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{\cdots}}}$$

be the continued-fraction development of its rotation number. Let $\{q_n\}_{n\geq 0}$ be the sequence of return times of the forward orbit of c to itself, and for each $n \geq 0$ let J_n be the closed interval on the circle with endpoints $f^{q_n}(c)$ and $f^{q_{n+1}}(c)$ that contains c. Then c divides J_n into two intervals, I_n with endpoint $f^{q_n}(c)$ and I_{n+1} with endpoint $f^{q_{n+1}}(c)$. The ratio of lengths $s_n(f) = |I_{n+1}|/|I_n|$ is called the *n-th scaling ratio* of f. For example, if $f = R_\alpha$ is the rotation by $2\pi\alpha$ and $\alpha = \rho(f)$ is irrational, then writing $\rho_n = s_n(f)$ we have $\rho_0 = (1 - r_0\alpha)/\alpha$ and

$$\rho_{n-1} = \frac{1}{r_n + \rho_n}$$

for all $n \geq 1$. The main theorem of Herman states that if f is a sufficiently smooth diffeomorphism (at least C^3) with irrational rotation number α satisfying a Diophantine condition

$$|\alpha - \frac{p}{q}| \geq \frac{C}{q^{2+\beta}} \ ,$$

Partially supported by FAPESP's Projeto Temático "Transição de Fase Dinâmica e Sistemas Evolutivos".

Mathematics Subject Classification : Primary 58F03, 30C62, 26A16.

where C and β are positive constants, then f is smoothly conjugate to the rotation R_α. In particular, we have

$$\lim_{n\to\infty} \frac{s_n(f)}{\rho_n} = 1 , \tag{1}$$

in other words f and R_α have asymptotically the same scaling ratios.

If f is allowed to have critical points, then (1) is no longer true. For instance, if $\alpha = (\sqrt{5}-1)/2$ is the golden mean, then $\rho_n \to \alpha \simeq 0.625$, while taking f in the family

$$f_\theta \; : \; x \;\mapsto\; x + \theta - \frac{1}{2\pi}\sin 2\pi x \; (\mathrm{mod}\, 1) \tag{2}$$

so that its rotation number is the golden mean, computer-assisted work by Shenker shows that $s_n(f) \to 0.7760513\ldots$. One may wonder whether perhaps maps such as (2) can be used as models instead of rotations. Let us agree to call f a *critical circle mapping* if f is a local smooth diffeo at all points but one, the critical point, around which f is smoothly conjugate to a map of the form $x \mapsto x|x|^{s-1} + a$, where $s > 1$ is a constant called the type of the critical point; when $s = 3$ we say that the critical point is of cubic type. In [dF_2] (cf. also [dF_1]) we proved the following theorem.

Theorem 1. *If f and g are smooth critical circle mappings with critical points of cubic type and with the same irrational rotation number of bounded combinatorial type, then they have asymptotically the same scaling ratios, i.e.*

$$\lim_{n\to\infty} \frac{s_n(f)}{s_n(g)} = 1 . \qquad \square$$

The main ingredients in the proof of this theorem were of a holomorphic nature, but the starting point was the fact that the ratios of scaling ratios $s_n(f)/s_n(g)$ are bounded, and eventually universally so. This fact is known as the *real a-priori bounds for critical circle mappings*, and it was first proved in the C^3 case by Swiatek in [Sw_2] and by Herman & Yoccoz (unpublished). In this note we sketch the proof in the case of C^2 mappings. More precisely, we have the following theorem.

Theorem 2. *If $0 < \alpha < 1$ is an irrational number then there exist constants C_1 and C_2 such that*

- *(a) If f is a critical circle mapping with rotation number α, then for all sufficiently large n we have $|I_n| \geq C_1\,|I_{n+1}|$;*
- *(b) If f and g are critical circle mappings with rotation number α, then for all sufficiently large n we have $|\frac{s_n(f)}{s_n(g)} - 1| \leq C_2$.*

Bounds of the type given by the constants C_1 and C_2 in this theorem are called *beau bounds* by Sullivan in [Su]. We will prove here parts (a) and (b) for C^2-mappings with constants independent of n but possibly depending on f and g. One can then use this result to get *beau* bounds exactly as is done by Swiatek in [Sw_2].

§2. Cross-ratios, Poincaré length and Köbe's principle

We shall need certain abstract distortion tools that we proceed to explain. Following Sullivan [Su], we define the *Poincaré density* of an open interval $I = (a, b) \subseteq \mathbb{R}$ to be

$$\rho_I(x) = \frac{(b-a)}{(x-a)(b-x)}$$

Integrating $\rho_I(x)\,dx$ we get the *Poincaré metric* on I. Thus, the Poincaré length of $J = (c, d) \subseteq I$ is

$$\ell_I(J) = \int_J \rho_I(x)\,dx = -\log Cr[I, J] ,$$

where $Cr[I, J] = (a-c)(d-b)/(a-d)(c-b)$ is the cross-ratio of the four points a, b, c, d. If $f : I \to I^*$ is a diffeomorphism, then the derivative of f measured with respect to the Poincaré metrics in I and I^*,

$$D_I f(x) = f'(x)\,\frac{\rho_{I^*}(f(x))}{\rho_I(x)} ,$$

is called the *Poincaré distortion* of f. It is identically equal to one if f is Möbius, in which case f preserves cross-ratios. Now consider the symmetric function $\delta_f : I \times I \to \mathbb{R}$ given by

$$\delta_f(x, y) = \begin{cases} \log \dfrac{f(x) - f(y)}{x - y} & , x \neq y, \\ \\ \log f'(x) & , x = y \end{cases} .$$

Then an easy calculation shows that

$$\log D_I f(x) = \delta_f(x, x) - \delta_f(a, x) - \delta_f(x, b) + \delta_f(a, b) . \tag{3}$$

Note that when f is C^3 its Poincaré distortion is controlled by the second order mixed derivative of δ_f, since in that case

$$\log D_I f(t) = \int\int_Q \frac{\partial^2}{\partial x\,\partial y}\delta_f(x, y)\,dx\,dy ,$$

where Q is the square $(a, t) \times (t, b)$. Moreover, when $(x, y) \to (t, t)$ the integrand above becomes $-6\,Sf(x)$, where Sf is the Schwarzian derivative of f. This is consistent with the fact that maps with negative Schwarzian increase the Poincaré metric and consequently decrease cross-ratios. Now, for C^2 mappings we have the following infinitesimal version (originally due to Sullivan) of a result of de Melo & van Strien [MS].

Lemma 3. *Let $f : \overline{I} \to \mathbb{R}$ be a C^2-diffeomorphism onto its image. Then there exists a gauge function σ, depending only on the C^2 norm of f, such that $\nabla\delta_f$ is σ-Hölder, i.e.*

$$|\nabla\delta_f(z_1) - \nabla\delta_f(z_2)| \leq \sigma(|z_1 - z_2|) \tag{4}$$

for all z_1 and z_2 in $\overline{I} \times \overline{I}$. In particular, $\log D_I f(x) \leq |x-a|\sigma(|x-a|)$ *for all x in I.*

Proof. We prove (4) with $\partial_x\delta_f$ replacing the gradient and some gauge function σ_x replacing σ. We have

$$\partial_x\delta_f(x,y) = \frac{f'(x)(x-y) - f(x) + f(y)}{(f(x)-f(y))\,(x-y)} . \tag{5}$$

Now let $\mu : \{(x,h) \in I \times \mathbb{R} : a \leq x+h \leq b\} \to \mathbb{R}$ be given by

$$\mu(x,h) = \begin{cases} \dfrac{f(x+h)-f(x)}{h} & , h \neq 0 \\ f'(x) & , h = 0 \end{cases} .$$

Then μ is C^1 provided f is C^2, and so we can write (5) as

$$\begin{aligned} \partial_x\delta_f(x,y) &= \frac{\mu(x,0)(x-y) - \mu(x,y-x)(x-y)}{\mu(x,y-x)\,(x-y)^2} \\ &= \frac{\partial_h\mu(x,\vartheta)}{\mu(x,y-x)} , \end{aligned} \tag{6}$$

for some $0 \leq \vartheta \leq |x-y|$. Let $m = \inf|\mu(x,h)| > 0$ and $M = \sup|\partial_h\mu(x,h)|$, both depending only on the C^1-norm of μ. Then if $z_i = (x_i, y_i) \in I \times I$ we have from (6)

$$|\partial_x\delta_f(z_1) - \partial_x\delta_f(z_2)| \leq \frac{M}{m^2}\,|\mu(x_1, y_1 - x_1) - \mu(x_2, y_2 - x_2)| .$$

Thus, writing $\varphi(z) = \mu(x, y-x)$, we can take

$$\sigma_x(t) = \sup_{|z_1 - z_2| \leq t} Mm^{-2}|\varphi(z_1) - \varphi(z_2)| .$$

We define σ_y for $\partial_y\delta_f$ in the same way. Then the sum $\sigma = \sigma_x + \sigma_y$ satisfies (4). Finally, from (3) and the mean-value theorem, we have

$$\log D_I f(x) \leq |x-a|\sigma_x(|x-a|) \leq |x-a|\sigma(|x-a|) , \tag{7}$$

and the Lemma is proved. ◻

We note that the above definitions and Lemma 3 still make sense when I is an interval in any Riemannian one-manifold. Therefore we have the following result. Recall that a circle homeo without periodic points is called *minimal* if the ω-limit set of any point is the whole circle.

Lemma 4. *Let f be a minimal C^2 circle homeomorphism, let N be a positive integer, and let $I = I(N) \subseteq S^1$ be an interval such that (a) I, fI, $f^2 I$, ..., $f^N I$ are k-quasidisjoint for some $k > 0$ independent of N; and (b) f restricted to each $f^i I$ is a diffeomorphism with C^2-norm uniformly bounded from below. Then the Poincaré distortion of f^N on I is bounded independently of N, and goes to zero as $N \to \infty$.*

Proof. The Poincaré distortion satisfies a chain rule. Therefore, if x is in I, we have by (7) and Lemma 3

$$
\begin{aligned}
|\log D_I f^N(x)| &= |\sum_{i=0}^{N-1} \log D_{f^i I} f(f^i(x))| \\
&\leq \sum_{i=0}^{N-1} \sigma(|f^i I|)\,|f^i I| \\
&\leq \sigma(\ell_N) \sum_{i=0}^{N-1} |f^i I| \,,
\end{aligned}
$$

where $\ell_N = \max_{0 \leq i < N} |f^i I|$. This last sum is bounded by k, while ℓ_N is also bounded independently of N. Since f is minimal, there are no wandering intervals and therefore ℓ_N goes to zero as N goes to ∞. □

This lemma tells us that, in the small, long compositions of uniformly C^2 diffeos defined over quasi-disjoint intervals are nearly projective. Now the so-callled *Köbe principle* says that if a diffeo is nearly projective over an interval I, then in a small subinterval J with definite *space* inside I the diffeo is in fact almost linear. The space $s(I,J)$ of J inside I is by definition the ratio between the length of the smallest of the two components of $I \setminus J$ and the length of J. Köbe's principle can be stated as follows.

Lemma 5. *Let $f : I \to \mathbb{R}$ be a C^2-diffeo onto its image, and let $J \subseteq I$ be such that $s(I,J) > 0$. Then there exists a constant C depending only on $s(I,J)$ and the Poincaré distortion of f such that*

$$
\left|\log \frac{f'(x)}{f'(y)}\right| \leq C|x - y|
$$

for all x and y in J. Moreover, for fixed space, C goes to zero with the Poincaré distortion.

Proof. See [MS, Ch. IV]. □

§3. The a-priori bounds

Now we use these ideas to give a brief sketch of the proof of Theorem 2. The first return map to J_n consists of f^{q_n} restricted to I_{n+1} and $f^{q_{n+1}}$ restricted to I_n.

This pair is called the *n-th renormalization* of f. Our problem is to bound the C^1-norms of these renormalizations (after we rescale both maps by a linear map taking I_n onto an interval of unit size) by a constant depending on f but not on n. The key point is to get uniform space around the two intervals containing the critical value of f, namely $f(I_n)$ and $f(I_{n+1})$. Once this is accomplished, Lemmas 4 and 5 give C^1 control of the renormalizations of f independently of n. For this purpose consider the collections

$$\mathcal{A}_n = \{I_n, fI_n, f^2 I_n, \ldots, f^{q_{n+1}-1} I_n\}$$

and

$$\mathcal{B}_n = \{I_{n+1}, fI_{n+1}, f^2 I_{n+1}, \ldots, f^{q_n - 1} I_{n+1}\} .$$

We have the following combinatorial facts.

Lemma 6. *For each $n \geq 0$, the union $\mathcal{A}_n \cup \mathcal{B}_n$ is a partition of S^1.* □

Lemma 7. *For each i in the range $1 \leq i \leq q_{n+1} - 1$, the inverse composition $f^{-i+1} : f^i(I_n) \to f(I_n)$ extends as a diffeomorphism to an interval $J_{i,n}$ containing $f^i(I_n)$ and its two nearest neighbors in $\mathcal{A}_n \cup \mathcal{B}_n$.* □

The fundamental observation of Swiatek in [Sw_1] is that the smallest interval in $\mathcal{A}_n \cup \mathcal{B}_n$ already has universal space around itself. More precisely, we may assume without loss of generality that $f^i I_n \in \mathcal{A}_n$ is the smallest interval in $\mathcal{A}_n \cup \mathcal{B}_n$. Then by definition of space we have

$$s(J_{i,n}, f^i(I_n)) \geq 1 ,$$

where $J_{i,n}$ is given by Lemma 7. Using Lemma 4, we transfer this space to space around $f(I_n)$ as follows. We view the composition f^{-i+1} as made-up of factors of two types. There are *bounded factors*, namely those whose domains are far away from the critical point and which have uniformly bounded C^2-norms, and there are *singular factors*, namely those whose domains fall inside a fixed neighborhood of the critical point. The singular factors have positive Schwarz derivative, and therefore can only *increase* space. The sub-compositions between two singular factors are made-up of bounded factors and therefore distort space by an additive, uniformly bounded amount, by Lemma 4. Hence the whole composition has uniformly bounded distortion of space, which gives us space for $f(I_n)$ inside $f^{-i+1}(J_{i,n})$. This fact plus a similar argument produces space around $f(I_{n+1})$ also. Finally, we can use Lemma 5 to get that the C^1 norm of $f^{q_{n+1}}$ restricted to I_n is uniformly bounded, thereby proving (a) and (b). Lemma 5 does not apply immediately to the whole composition, for not all factors in it are bounded. It is necessary to segregate the singular factors from the bounded factors, shuffling them appart by conjugating the bounded ones by the singular ones. This can be safely done because such conjugations are bounded operators in the space of C^2 diffeos with the C^2 topology.

Acknowledgement. The author wishes to thank Dennis Sullivan for several useful conversations on these and related matters.

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