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On Sum-free Sets of Natural Numbers

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Abstract. We give a brief survey on sum-free subsets of the natural numbers, highlighting recent results which may shed light on some old and new open problems in this area of combinatorial number theory.

§0. Introduction

Let A be a subset of a semigroup $(\mathcal{S}, \oplus)$. We say that A is *sum-free* if for every $x_1, x_2 \in A$ we have $x_1 \oplus x_2 \notin A$. More generally, we call A *k-sum-free* for some natural $k \geq 2$ if the equation $x_1 \oplus \cdots \oplus x_k = y$ has no solutions in A, and *strongly k-sum-free* if it is ℓ-sum-free for every $2 \leq \ell \leq k$. In this note we present some results and problems on the size, the structure, and the number of free-sets as well as several related concepts. We should mention also that we cover here only a small part of this interesting topic: for more information on sum-free sets the reader should refer to Cameron's excellent survey [10] and Calkin's dissertation [4] devoted to this subject.

§1. Infinite Subsets of the Natural Numbers

In the paper we deal mainly with *par excellence* the most natural case of sum-free sets: subsets of the natural numbers with the ordinary operation of addition. We shall start with a description of the structure of large infinite sum-free sets of natural numbers, where the size of a set $A \subseteq \mathbb{N}$ is measured in terms of its *upper density*

$$\bar{\mu}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} .$$

Since for every strongly k-sum-free set $A \subseteq \mathbb{N}$ and every $x \in A$ the sets A, $x+A, \ldots, (k-1)x+A$ must be disjoint, for each such subset A we have $\bar{\mu}(A) \leq 1/k$. A little more work is required when one would like to obtain the upper bound for the size of sets which are just k-sum-free. Note that this bound is no longer a decreasing function of k: the set of all odd numbers is k-sum-free for arbitrary large even k. Calkin and Erdős [8] conjectured that the maximum density of k-sum-free sets is determined by the above parity-type constraints and for each such set A we have $\bar{\mu}(A) \leq 1/\rho(k)$, where

$$(*) \qquad \rho(k) = \min\{\ell \colon \ell \not| \; k - 1\} .$$

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Łuczak and Schoen [17] showed that this is indeed the case.

Theorem 1. *If $A \subseteq \mathbb{N}$ is k-sum-free then $\bar{\mu}(A) \le 1/\rho(k)$, where $\rho(k)$ is given by $(*)$, whereas every strongly k-sum-free set has upper density at most $1/k$.*

The above upper bounds for the upper densities of both k-sum-free and strongly k-sum-free sets are achieved for sets of the type

$$\mathrm{Mod}(\ell; m) = \{n \colon n \equiv \ell \pmod m\}.$$

In particular, the set of odd numbers is the largest sum-free subset of the natural numbers. As a matter of fact, it is the maximum sum-free set in a very strong sense: if a sum-free set does not consist of odd numbers then its density drops down to $2/5$ independently of the smallest even number contained in it. An extension of this result for k-sum-free and strongly k-sum-free sets can be stated as follows.

Theorem 2.

(i) *For every sum-free set $A \subseteq \mathbb{N}$ which contains an even number we have $\bar{\mu}(A) \le 2/5$. Furthermore, all sets A with $\bar{\mu}(A) = 2/5$ containing even numbers are subsets of* $\mathrm{Mod}(1;5) \cup \mathrm{Mod}(4;5)$ *or* $\mathrm{Mod}(2;5) \cup \mathrm{Mod}(3;5)$.

(ii) *If $k \ge 3$ then every strongly k-sum-free set A with $\bar{\mu}(A) > 1/(k+1)$ is a subset of* $\mathrm{Mod}(\ell; k)$, *where $1 \le \ell \le k-1$ is relatively prime with k.*

(iii) *Each 3-sum-free set $A \subseteq \mathbb{N}$ with upper density larger than $2/7$ is a subset of one of the following four sets:* $\mathrm{Mod}(1;3)$, $\mathrm{Mod}(2;3)$, $\mathrm{Mod}(1;6) \cup \mathrm{Mod}(2;6)$, $\mathrm{Mod}(4;6) \cup \mathrm{Mod}(5;6)$.

(iv) *If $k \ge 4$ then every k-sum-free set with $\bar{\mu}(A) > 1/(\rho(k)+1)$ is a subset of* $\mathrm{Mod}(\ell; \rho(k))$, *where $1 \le \ell \le \rho(k) - 1$ and $(\ell, \rho(k)) = 1$.*

The first part of Theorem 2 was proved by Łuczak [16], Schoen [19] and Deshouillers, Freiman, Sós and Tamkin [4] who showed it in a much stronger, finite version (see Theorem 6 below), (ii) and (iv) are due to Łuczak and Schoen [17], while the proof of (iii) was given by Schoen [19]. Note that bounds for densities in (ii) and (iii) are best possible and (iv) is sharp for every k for which $\rho(k)+1 \not| \, k-1$. On the other hand we do not know what is the maximum density of a k-sum-free set which contains a multiplicity of $\rho(k)$ when $\rho(k)+1$ divides $k-1$; one can expect that in this case the following conjecture holds.

Conjecture. *For every k-sum-free set $A \subseteq \mathbb{N}$ which contains a multiplicity of $\rho(k)$ we have $\bar{\mu}(A) \le 1/\rho_2(k)$, where*

$$\rho_2(k) = \min\{\ell \ge \rho(k) + 1 \colon \ell \not| \, k - 1\}\,.$$

Theorem 2i states that if a sum-free set contains an even number then its maximum possible density decreases from $1/2$ to $2/5$, and if such a set contains a multiplicity of ten its density must drop under $2/5$. Can we push down the density a little further if we assume that a sum-free set contains multiplicities of some other numbers? Moreover, each sum-free set A with density at least $2/5$

has a rather special "quasi-periodic" structure. Can we force such a structure in a sum-free set if we assume that its density is larger than, say, 7/20? Before we answer these questions let us make the notion of the "quasi-periodic structure" a little more precise. We say that a k-sum-free set A is *k-modular* with period s if $A = \bigcup_{i=1}^{t} \mathrm{Mod}(r_1; s)$ for some $s \geq 2$ and $1 \leq r_1 < \cdots < r_t \leq s-1$ (note that then the set $\{r_1, \ldots, r_t\}$ must be k-sum-free in the group $\mathbb{Z}_s$), and a *k-submodular* set of period s is a subset of k-modular set of this period.

Since our aim is to find critical densities above which k-sum-free and strongly k-sum-free sets are k-submodular, we first give a simple example of a large set which is not k-submodular. Let $\alpha \in (0,1)$ be an irrational number and a, b be real numbers such that $0 \leq a < b \leq 1$. We define $U(\alpha; a, b) \subseteq \mathbb{N}$ setting

$$U(\alpha; a, b) = \{n \in \mathbb{N} \colon \alpha n - \lceil \alpha n \rfloor \in (a, b)\} .$$

One can easily check that $U(\alpha; a, b)$ contains multiplicities of every natural number and so it is not k-submodular. Furthermore, $\bar{\mu}(U(\alpha; a, b)) = b - a$. Thus, $U(\alpha; 1/(k^2-1), k/(k^2-1))$ is a k-sum-free set with upper density $1/(k+1)$, and $U(\alpha; 1/(2k-1), 2/(2k-1))$ is a strongly k-sum-free set of density $1/(2k-1)$. The following result from [17] says that these two sets are, in a way, the maximum ones among the sets that are not k-submodular.

Theorem 3.

(i) *For every $k \geq 2$ and $\epsilon > 0$ there exists $M = M(k, \epsilon)$ such that every k-sum-free set A with $\bar{\mu}(A) \geq 1/(k+1) + \epsilon$ is k-submodular with period M.*

(ii) *For every $k \geq 2$ and $\epsilon > 0$ there exists $M' = M'(k, \epsilon)$ such that every strongly k-sum-free set A with $\bar{\mu}(A) \geq 1/(2k-1) + \epsilon$ is k-submodular with period M'.*

We conclude with a few problems related to the structural characterization of dense sum-free sets given by Theorems 2 and 3 (although each of them can be generalized for k-sum-free sets here we concentrate on the simplest case when $k = 2$). For a sum-free set A let $\xi(A, n)$ be the number of natural numbers not larger than n which neither belong to A nor can be represented as a sum of two elements from A, i.e.,

$$\xi(A, n) = |\{1, 2, \ldots, n\} \setminus (A \cup (A + A))| .$$

Calkin and Erdős [8] proved that $\xi(U(\alpha; 1/3, 2/3), n) = O(\log n)$ and asked whether there exist sets A that are not 2-submodular and enjoy the property that $\xi(A, n) = o(\log n)$. In fact it is also not clear whether $U(\alpha; 1/3, 2/3)$ is, in a way, the maximum sum-free set which is not 2-submodular, i.e., we know no examples of sets A which are not 2-submodular, have density 1/3, and are such that for every irrational $\alpha \in (0, 1)$ the upper density of the symmetric difference of A and $U(\alpha; 1/3, 2/3)$ is positive.

Theorem 3 guarantees that each sum-free set of density larger than 1/3 is 2-submodular but we do not know much more about the structure of such sets. We mention just one interesting conjecture posed by Calkin. Let us call a sum-free set A *ultimately complete* if $\xi(A,n)$ is bounded from above by some constant, i.e., if there exists n_0 such that all natural numbers larger than n_0 which do not belong to A can be represented as a sum of two elements from A. Furthermore, we say that 2-modular sets A and B are *equivalent* if there exist natural numbers s, $t \le s$, $1 \le r_1 < \cdots < r_t \le s-1$ and u such that $1 \le u \le s-1$, $(u,s)=1$, $A = \bigcup_{i=1}^{t} \mathrm{Mod}(r_i; s)$ and $B = \bigcup_{i=1}^{t} \mathrm{Mod}(ur_i; s)$. Then Calkin's conjecture can be stated as follows.

Conjecture. *Every ultimately complete sum-free set A with $\bar{\mu}(A) > 1/3$ is a subset of a set equivalent to $\bigcup_{i=k+1}^{2k+1} \mathrm{Mod}(i; 3k+2)$ for some $k \ge 0$.*

§2. Difference Sets

For a set $A \subseteq \mathbb{N}$ the *difference set of A* is defined as

$$\mathrm{diff}(A) = \{x - y \colon x, y \in A, x > y\} .$$

In this section we consider certain properties of this special class of subsets of the natural numbers.

Let us start with the observation that $(x_{k+1}-x_k)+\cdots+(x_2-x_1) = (x_{k+1}-x_1)$, and so the difference set of any set with at least $k+1$ elements is not k-sum-free. Thus, we shall consider a property of difference sets which is a little stronger: we shall ask how large the difference set must be to guarantee that it contains a set $\{x_1, \ldots, x_k\}$ of k different elements together with the set

$$\mathrm{PS}(x_1, \ldots, x_k) = \{x_{i_1} + \cdots + x_{i_\ell} \colon 1 \le i_1 < \cdots < i_\ell \le k\}$$

of all its partial sums. It is not hard to prove that the difference sets of any set of natural numbers with a positive density has this property for every k. Bergelson, Erdős, Hindman and Łuczak [3] proved that, in fact, much more is true: the difference sets of such dense subsets has good additive and multiplicative properties at the same time.

Theorem 4. *Let $A \subseteq \mathbb{N}$ be such that $\bar{\mu}(A) > 0$ and let $f \colon \mathbb{N} \to \mathbb{N}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of* diff(A) *such that both sets*

$$\left\{ \sum_{n \in F} a_n x_n \colon F \subset \mathbb{N}, 0 < |F| < \infty \text{ and } a_n \in \{1, 2, \ldots, f(n)\} \right\}$$

and

$$\left\{ \prod_{n \in F} a_n x_n \colon F \subset \mathbb{N}, 0 < |F| < \infty \text{ and } a_n \in \{1, 2, \ldots, f(n)\} \right\}$$

are contained in diff(A).

Nonetheless, the set diff(A) may have positive density even when A is very sparse. Thus, it is more natural to study properties of diff(A) under the weaker assumption that $\bar{\mu}(\text{diff}(A)) > 0$. However, no multiplicative properties similar to that from Theorem 4 can be shown in such a case: it turns out that for every $\epsilon > 0$ there exists a set A such that $\bar{\mu}(\text{diff}(A)) > 1 - \epsilon$ but for every $x, y \in \text{diff}(A)$ we have $xy \notin \text{diff}(A)$. On the other hand, additive properties of dense difference sets are partially characterized by the following result from [3].

Theorem 5. *If $A \subseteq \mathbb{N}$ and $\bar{\mu}(\text{diff}(A)) > 0$ then there exists $x_1, x_2, x_3 \in \text{diff}(A)$ such that $\{x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3\} \subset \text{diff}(A)$.*

Moreover, for every $\epsilon > 0$ there exists a set $A \in \mathbb{N}$ such that $\text{diff}(A) > 1/2 - \epsilon$ and for every $x_1, x_2, x_3, x_4, x_5 \in \text{diff}(A)$ at least one of the partial sums from $\text{PS}(x_1, x_2, x_3, x_4, x_5)$ does not belong to $\text{diff}(A)$.

It is not known whether from each difference set of positive density one can choose a set of four elements such that all of its partial sums are in the set. It is also not clear whether 1/2 which appears in the second part of Theorem 5 can be replaced by a smaller constant. More generally, the following question remains open.

Problem. *For every $k \geq 3$ find a smallest possible constant c_k for which the following holds: for every $\epsilon > 0$ and every set $A \subseteq \mathbb{N}$ with $\text{diff}(A) > c_k + \epsilon$ there exists $x_1, \ldots, x_k \in A$ such that $\text{PS}(x_1, \ldots, x_k) \subset \text{diff}(A)$.*

Thus, in particular, Theorem 5 states that $c_3 = 0$ and $c_5 \geq 1/2$.

§3. Sum-free Subsets of Finite Sets

One may hope that results analogous to Theorem 1 and 3 remain valid also for subsets of the set $[n] = \{1, 2, \ldots, n\}$. Nevertheless, when dealing with finite sum-free sets we must take into account "the boundary effect" which makes all structural characterization of such sets much more involved (if at all possible).

Let us consider the simplest case when $A = \{a_1, \ldots, a_k\}$ is a sum-free subset of $[n]$. Then for the set $B = \{a_k - a_i \colon 1 \leq i \leq k - 1\}$ we have $|B| = k - 1$ and, because A is sum-free, $A \cap B = \emptyset$. Thus $2k - 1 \leq n$ and consequently $|A| \leq \lfloor (n+1)/2 \rfloor$. Clearly, this upper bound is best possible since the set of all odd natural numbers not larger than n has $\lfloor (n+1)/2 \rfloor$ elements. Nonetheless, in the finite case another "extremal set" appears: the set

$$[\lceil (n+1)/2 \rceil, n] = \{i \colon \lceil (n+1)/2 \rceil \leq i \leq n\}$$

is sum-free and has $\lfloor (n+1)/2 \rfloor$ elements as well. Deshouillers, Freiman, Sós and Tamkin [4] proved that all sum-free subset of $[n]$ with more than $2n/5$ elements must be similar to above two extremal sets and characterized sum-free subsets of $[n]$ for which the critical density is attained.

Theorem 6. *For every positive number x there exist real numbers $n_0 = n_0(x)$ and $C = C(x)$ such that for every sum-free set $A \subseteq [n]$ with the largest element*

$n \geq n_0$ and cardinality $|A| \geq 2n/5 - x$ at least one of the following properties holds:

(i) *all elements of A are odd, i.e., $A \subseteq \mathrm{Mod}(1;2)$,*
(ii) $A \subseteq \mathrm{Mod}(1;5) \cup \mathrm{Mod}(4;5)$,
(iii) $A \subseteq \mathrm{Mod}(2;5) \cup \mathrm{Mod}(3;5)$,
(iv) *the smallest element of A is not smaller than $|A|$ and furthermore $|A \cap [n/2]| \leq (n - 2|A| + 3)/4$,*
(v) $A \subseteq [n/5 - C, 2n/5 + C] \cup [4n/5 - C, n]$.

Theorem 6 is a substantial strengthening of of Theorem 2i and its proof is much more involved. This might be the reason why we do not know a finite version of Theorem 3 for $k = 2$; as a matter of fact, it is not even clear how to state it properly.

Differences between finite and infinite cases become much more visible for $k \geq 3$. The size of the largest strongly k-sum-free subset of $[n]$ does not depend on k at all because $\big[\lceil (n-1)/2 \rceil, n\big]$ is strongly k-sum-free for every $k \geq 2$. The size of the largest k-sum-free subset $[n]$ in fact grows with k: it is easy to see that it cannot be larger than $\lfloor (k-1)(n+1)/k \rfloor$ and the example $\big[\lceil (n+1)/k \rceil, n\big]$ shows that this estimate is sharp. However, no results analogous to Theorem 7 valid for $k \geq 3$ has been shown so far. In particular, the question about the maximum size of a k-sum-free (or strongly k-sum-free) subset of $[n]$ which contains a small natural number, say, smaller than $n/100k$, remains open.

Finally, let us mention a problem on finite sum-free subsets of sets different from $[n]$. For a natural number $k \geq 2$ and a finite set of natural numbers A, we denote the sizes of the largest subsets of A which are k-sum-free and strongly k-sum-free by, respectively, $\nu_k(A)$ and $\nu_{\leq k}(A)$, and set

$$a_k = \min\{\nu_k(A)/|A| \colon A \text{ is a finite subset of the natural numbers}\}$$

and

$$a_{\leq k} = \min\{\nu_{\leq k}(A)/|A| \colon A \text{ is a finite subset of the natural numbers}\}\ .$$

Füredi observed that the set $\{1,2,3,4,5,6,8,9,10,18\}$ shows that $a_2 \leq 2/5$, and it was proved by Erdős [15], and independently by Alon and Kleitman [2], that $a_2 \geq 1/3$. One can easily mimic their argument to show that for every $k \geq 2$ we have $a_k \geq 1/(k+1)$ and $a_{\leq k} \geq 1/(2k-1)$, but all known upper bounds for a_k tend to 1 as $k \to \infty$. Thus, in particular, at this moment we cannot even decide whether the value of a_k tends to 0 or to 1 as $k \to \infty$.

§4. The Number of Sum-free Sets

The number of all sum-free subsets of $[n]$ is clearly bounded from above by the number of all subsets of $[n]$ and bounded from below by the number of subsets of the largest sum-free subset of $[n]$, which, let us recall, has $\lfloor (n+1)/2 \rfloor$ elements.

Cameron conjectured that the latter bound is closer to the truth, i.e., that there are only $2^{(1+o(1))n/2}$ sum-free subsets of $[n]$. His conjecture was settled in the affirmative by Calkin [5], and, independently by Alon [1]. Recently, Calkin and Taylor [9] generalized this result for k-sum-free sets proving the following theorem.

Theorem 7. *For every $k \geq 2$ there exists a constant c such that for every natural number n the set $[n]$ contains at most $c2^{(k-1)n/k}$ k-sum-free subsets.*

Let us consider now the infinite case. At first sight there is nothing non-trivial to ask about: clearly there is an uncountable number of sum-free sets (it is enough to take all sets consisting of odd numbers) and Calkin [6] noticed that, in fact, there is an uncountable number of sum-free subsets of the natural numbers of positive density, which are maximal and are not 2-submodular. However, Cameron [10] proposed a more sophisticated way of looking at infinite sum-free subsets of the natural numbers. Let us define a *distance* $d(A, B)$ between two infinite sets $A, B \subseteq \mathbb{N}$ setting $d(A, B) = 0$ if $A = B$, and $d(A, B) = 2^{-n}$ if $A \cap [n] = B \cap [n]$ but $A \cap [n+1] \neq B \cap [n+1]$. It is easy to see that the above function is a metric on the family of all subsets of the natural numbers. Thus, we may ask about the Hausdorff dimension d_k of the family of all k-sum-free sets in such a metric space. One can easily observed that if by $f_k(n)$ we denote the number of all k-sum-free subsets of $[n]$ then

$$(**) \qquad d_k \leq \log_2 \left(\liminf_{n\to\infty} (f_k(n))^{1/n} \right).$$

Hence, since the family of all sets which contains only odd numbers has Hausdorff dimension $1/2$, from Theorem 7 it follows that $d_2 = 1/2$. As a matter of fact, having in mind Theorem 2, one may expect that a little more is true and "most" sum-free subsets of the natural numbers consist only of odd numbers, i.e., that the following conjecture holds.

Conjecture. *The Hausdorff dimension of the family of all sum-free subsets of the natural numbers containing at least one even number is strictly smaller than $1/2$.*

For $k \geq 3$ Theorem 7 and $(**)$ give $k - 1/k$ as the upper bound for d_k but, because the size of the maximum k-sum-free subset of $[n]$ is strongly influenced by the boundary effect, probably this upper bound does not approximate the dimension d_k well enough. On the other hand, the Hausdorff dimension of the set of all subsets of $\mathrm{Mod}(k; 1)$ is $1/k$ and thus we have a lower bound of $1/k$ for d_k. It is tempting to conjecture that it gives the correct value of d_k, i.e., that $d_k = 1/k$.

§5. Generating Sum-free Sets

How to generate a sum-free subset of the natural numbers? The most straightforward procedure uses a natural bijection θ between all infinite zero-one sequences and all sum-free subsets of the natural numbers. It is based on the simple observation that a sum-free subset of the natural numbers can be generated from a zero-one "decision sequence". More precisely, for a given infinite zero-one sequence

$\sigma = \sigma_1\sigma_2\dots$ we recursively construct two non-decreasing sequences of subsets of the natural numbers $A_1 \subseteq A_2 \subseteq \dots$ and $U_1 \subseteq U_2 \subseteq \dots$ in the following way. If $\sigma_1 = 0$ we set $A_1 = \{1\}$ and $U_1 = \emptyset$, while for $\sigma_1 = 1$ we put $A_1 = \emptyset$ and $U_1 = \{1\}$. Now let us suppose that we have found A_n and U_n and let x_n denote the smallest natural number which belongs to neither A_n nor U_n and is not a sum of two elements from A_n. Then, we put $A_{n+1} = A_n$ and $U_{n+1} = U_n \cup \{x_n\}$ when $\sigma_n = 0$, whereas for $\sigma_n = 1$ we set $A_{n+1} = A_n \cup \{x_n\}$ and $U_{n+1} = U_n$. Finally, we set $\theta(\sigma) = \bigcup_{n=1}^{\infty} A_n$. Thus, for example, $\theta(00000\dots) = \emptyset$, $\theta(11111\dots) = \mathrm{Mod}(1;2)$ while $\theta(10101\dots) = \mathrm{Mod}(1;3)$.

We call a sequence $\sigma = \sigma_1\sigma_2\dots$ *ultimately periodic* with period s if there exist n_0 and s such that $\sigma_n = \sigma_{n+s}$ for every $n \geq n_0$. Similarly, if a set of natural numbers A is such that for some n_0 and s and all $n \geq n_0$ we have $n \in A$ if and only if $n + s \in A$ we say that A *ultimately periodic* with period s. Cameron [10] noticed that if $\theta(\sigma)$ is ultimately periodic then σ is also ultimately periodic and asked whether the reverse implication holds as well. Calkin [6] found some partial evidence that probably this is not the case but the question still remains open with the sequence $\sigma = (01001)$ of period five as the simplest candidate for a periodic sequence for which the sum-free set $\theta(\sigma)$ is not ultimately periodic. We know also that even if the answer to Cameron's question is negative the period of $\theta(\sigma)$ grows quickly with the period of σ. For instance, for $\sigma = (0110011)$ the period of $\theta(\sigma)$ is 10710 (see [6] for further examples and a more elaborate treatment of this problem).

In [10] Cameron investigated properties of random sum-free sets $\theta(\sigma^{\mathrm{rand}})$, where σ^{rand} is a sequence of independent random bits. He proved that for any 2-modular set A the probability that $\theta(\sigma^{\mathrm{rand}}) \subseteq A$ is strictly positive and asked whether $\theta(\sigma^{\mathrm{rand}})$ is 2-submodular with probability one. It turns out that this is not the case: Calkin and Cameron [7] proved that the event that the only even element of $\theta(\sigma^{\mathrm{rand}})$ is two holds with positive probability, which easily implies the negative answer to the above question. Another problem posed in [10] concerns the upper density of $\theta(\sigma^{\mathrm{rand}})$. Cameron proved that with probability one it is less than $1/3$ but conjectured that in fact it is bounded from above by $1/4$. Theorem 2 suggests that maybe even a somewhat stronger conjecture holds: it is possible that with probability one either $\theta(\sigma^{\mathrm{rand}}) \subseteq \mathrm{Mod}(1;2)$ or $\bar{\mu}(\theta(\sigma^{\mathrm{rand}})) \leq 1/5$.

§6. Related Results and Concepts

We conclude with few brief remarks about possible extensions of some of theorems presented in the previous sections. Clearly, there are basically two ways of generalizing these results: either we may study a modified notion of sum-free sets for sets of natural numbers, or try to obtain analogous theorems for semigroups other than the natural numbers.

An example of a result of the first type is a construction of Choi [13] who showed that there exists a sequence of subsets of the natural numbers A_n such that for the size of $u(n)$ of the maximal subset of A_n which is k-sum-free *for every* $k \geq 2$ we have $u(n) = O(|A_n|/\log\log|A_n|)$. Another strengthening of the notion of sum-free sets came from Erdős [15] who considered the function $g(n)$ defined as the maximum number such that every set $A \subseteq \mathbb{N}$ of n elements contains a subset $B \subseteq A$ of size $g(n)$ such that none of the sums of any two distinct elements of B belongs to A (not just to B!). An elementary application of a greedy procedure shows that $g(n)$ grows at least linearly with $\log n$. On the other hand, Choi [12] proved that for every $\epsilon > 0$ we have $g(n) = o(n^{2/5+\epsilon})$.

Numerous extensions of results from §3 for sum-free subsets of different semigroups have been proved by Erdős [15] and Alon and Kleitman [2]. In particular they noticed that every finite subset A of real numbers different from zero contains a sum-free subset with more than $|A|/3$ elements. Alon and Kleitman proved also that for general Abelian groups 1/3 must be replaced by 2/7, and, as was observed by Rhemtulla and Street [18], the constant 2/7 is best possible.

Theorem 8. *For any finite Abelian group G and any set A of non-zero elements of G there exists $B \subseteq A$ such that B is sum-free and $|B|/|A| > 2/7$. Furthermore, 2/7 cannot be replaced by a smaller constant.*

We conclude by one more result from [2] on measurable sets of the n-dimensional torus.

Theorem 9. *Let T be an n-dimensional torus treated as a semigroup with the natural operation of addition and let m denote the usual Lebesgue measure on T. Then for every $\epsilon > 0$ and every measurable sum-free set $A \subseteq T$ there exists a measurable set $B \subseteq A$ such that $m(B) \geq (1/3 - \epsilon)m(A)$. Furthermore, the above statement is not true when 1/3 is replaced by a smaller constant.*

Let us remark that no results similar to Theorems 8 or 9 are known for k-sum-free subsets when $k \geq 3$.

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