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Generating sets of certain automorphism groups

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Automorphisms of free groups. Let $F = F_n = < x_1, x_2, \ldots, x_n >$ be a noncyclic free group freely generated by a set $X = \{ x_1, x_2, \ldots, x_n \}$, $n \geq 2$. Then every endomorphism of F can be defined by prescribing the image set $W = \{ w_1, w_2, \ldots, w_n \}$ of its respective generators. We denote by End(F) the semigroup of all endomorphisms of F. An element $\eta \in$ End(F) may then be specified by:

$$\eta = \{ x_1 \to w_1, x_2 \to w_2, \ldots, x_n \to w_n \}.$$

If an endomorphism $\eta = \{ x_1 \to w_1, x_2 \to w_2, \ldots, x_n \to w_n \}$ is such that the image set $\{ w_1, w_2, \ldots, w_n \}$ is also a basis of F then η defines an automorphism of F. We denote by Aut(F) the group of all automorphisms of F. An element $\alpha \in$ Aut(F) may then be characterized as:

$$\alpha = \{ x_1 \to w_1, x_2 \to w_2, \ldots, x_n \to w_n \},$$

where $\{ w_1, w_2, \ldots, w_n \}$ is a basis for F.

Theorem (Nielsen, 1924). Let $F = F_n = < x_1, x_2, \ldots, x_n >$, $n \geq 2$, be a free group. Then Aut(F) can be generated by the following four automorphisms (three, if $n = 2$):

$$\alpha_1 = \{ x_1 \to x_2, x_2 \to x_3, \ldots, x_n \to x_1 \};$$

$$\alpha_2 = \{ x_1 \to x_2, x_2 \to x_1, x_i \to x_i, i \neq 1, 2 \};$$

$$\alpha_3 = \{ x_1 \to {x_1}^{-1}, x_i \to x_i, i \neq 1 \};$$

$$\alpha_4 = \{ x_1 \to x_1 x_2, x_i \to x_i, i \neq 1 \}.$$

[Remark. If $n \geq 4$, then Aut(F) can be generated by a set of two automorphisms (B. H. Neumann, 1932)].

Consider the natural homomorphism μ : Aut(F) $\to$ Aut(F/F'). The kernel of this homomorphism consists of all those automorphisms of F which induce identity automorphism modulo the commutator subgroup F' of F. These are the so-called IA-*automorphisms* of F. We denote by IA-Aut(F) the subgroup of all IA-automorphisms. Elements of IA-Aut(F) may be identified as

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$$\alpha = \{ x_1 \to x_1 d_1, x_2 \to x_2 d_2, \ldots, x_n \to x_n d_n \}, d_i \in F',$$

such that $\{ x_1 d_1, x_2 d_2, \ldots, x_n d_n \}$ is a basis of F. An inner automorphism of F is clearly an IA-automorphism. We denote by Inner-Aut(F) the subgroup of inner automorphisms of F. The centre of F is trivial, so Inner-Aut$(F) \cong F$. The following inclusions of normal subgroups of Aut(F) are now clear:

$$F \cong \text{Inner-Aut}(F) \leq \text{IA-Aut}(F) \leq \text{Aut}(F).$$

Since automorphisms of a free abelian group of rank n can be identified with $n \times n$ invertible matrices over the integers, it follos that

$$\text{Aut}(F_2)/\ \text{IA-Aut}(F_2) \cong \text{GL}(2, \mathbb{Z}).$$

When F is of rank 2, then IA-automorphisms and Inner-automorphisms coincide (Nielsen 1924) [for proof see, for instance, Lyndon and Schupp (1977)].

Remarks. The following additional comments are of general interest.

(i) (Bachmuth, Mochizuki and Formanek 1976). If F is free of rank 2 and R is a normal subgroup of F contained in F' such that the integral group ring $\mathbb{Z}(F/R)$ is a domain then Inner-Aut(F/R') = IA-Aut(F/R') . The case $R = F'$ was proved earlier by Bachmuth (1965) [see Gupta (1981) for an alternate proof].

(ii) (Meskin 1973). While $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ in GL(2,$\mathbb{Z}$) is of order 6, Aut(F_2) does not contain any element of order 6.

[Note that every finite subgroup of Aut(F_n) is embedded in GL($n, \mathbb{Z}$)].

(iii) (Magnus 1934). IA-Aut(F) is generated by a finite set of automorphisms:

$$\alpha_{ijk} = \{ x_i \to x_i[x_j, x_k], x_t \to x_t, t \neq 1 \},$$

for all i, j, k such that either $j = i$, or $j < k$ and $i \neq j, i \neq k$.

(iv) (Baumslag-Taylor 1968). IA-Aut(F_n)/ Inner-Aut(F_n) is torsion free for all $n \geq 3$.

(v) (Baumslag 1968). Aut(F_n) is residually finite.

(vi) (Grossman 1974). Aut(F_n)/ Inner-Aut(F_n) is residually finite.

(vii) (Formanek-Procesi 1990). Aut(F_n) is not linear for $n \geq 3$.

[Magnus and Tretkoff (1980) proved that Aut(F_2) is embedded in the quotient group Aut(F_n) / Inner-Aut(F_n), $n \geq 3$. No finite dimensional linear representation is known for this quotient. In fact, no linear representation is known for Aut(F_2)].

(viii) IA-automorphisms of an arbitrary 2-generator metabelian group G were studied in Gupta (1981) where it was shown, in particular, that IA-Aut(G) is itself metabelian.

Bachmuth, Baumslag, Dyer and Mochizuki (1987) addressed the presentation questions of these groups. They proved that IA-Aut(G) may not be finitely generated even as normal subgroup of Aut(G), and that Aut(G) may or may not be finitely presented when G is assumed to be finitely presented. Caranti and Scoppola (1991) have revealed some further facts about 2-generator metabelian groups $G = gp\{x, y\}$:

(a) Every map $\{ x \to xu, y \to yv \}$ extends to an endomorphism of G,

(b) Every map $\{ x \to xu, y \to yv \}$ extends to an automorphism of G if and only if G is nilpotent,

(c) Lower central series of IA-Aut(G) and Inner-Aut(G) coincide from second term onwards.

[In a subsequent paper they use module-theoretic constructions to study the lower central series of IA-Au(G)].

Primitivity in free groups. Let $F = F_n = < x_1, x_2, \ldots, x_n >$ be a free group. A word $w \in F$ is called *primitive* if it can be included in some basis of F. Since an automorphism maps a basis to a basis, w is primitive if and only if $\alpha(w) = x_1$ for some $\alpha \in \text{Aut}(F)$. For a given word w in F, testing to see if it is primitive is, in general, a very difficult problem. This problem was resolved by Whitehead (1936) through topological arguments using a very large but finite set of the so-called Whitehead (elementary) automorphisms. These are of two types:

Type I. $\alpha = \{ x_i \to x_{j\sigma}{}^{\pm 1}; \sigma,$ a permutation of $\{1, \ldots, n\} \}$.

[These form a finite subgroup of Aut(F_n) of order $2^n n!$].

Type II. Put $X = \{ x_1, x_2, \ldots, x_n\}$ and $X^{\hat{}} = \{ {x_1}^{-1}, {x_2}^{-1}, \ldots, {x_n}^{-1} \}$. For any choice of subset A of $X \cup X^{\hat{}}$ and for any choice of $a \in A, a \notin A$, a Type II automorphism $\alpha(A; a)$ is defined by the following rule:

$$\alpha(A; a) = \{ x_i \to a^{-1} x_i a, \text{ if } x_i \in A,\ {x_i}^{-1} \in A;\ x_i \to x_i a, \text{ if } x_i \in A,\ {x_i}^{-1} \notin A;$$
$$x_i \to a^{-1} x_i, \text{ if } x_i \notin A,\ {x_i}^{-1} \in A;\ x_i \to x_i, \text{ if } x_i \notin A,\ {x_i}^{-1} \notin A \}.$$

[For example, with $F = F_4,\ A = \{x_1, x_2, x_3, {x_1}^{-1}\},\ a = x_2$,

$$\alpha(A; a) = \{ x_1 \to {x_2}^{-1} x_1 x_2, x_2 \to x_2, x_3 \to x_3 x_2, x_4 \to x_4 \}$$

is a Whitehead automorphism of Type II.

Similary, with $F = F_4, A = \{x_1, x_2, {x_2}^{-1}, x_3, {x_4}^{-1}\}, a = x_1$,

$$\alpha(A; a) = \{\ x_1 \to x_1, x_2 \to {x_1}^{-1}x_2x_1, x_3 \to x_3x_1, x_4 \to {x_1}^{-1}x_4\ \}$$

is a Whitehead automorphism of Type II].

Theorem (Whitehead 1936). There is an algorithm to decide whether or not a given pair of words u, v in F are equivalent under an automorphism of F. More generally, there is an algorithm to decide if a given system $\mathbf{w} = \{\ w_1, w_2, \ldots, w_m\ \}$, $m \leq n$, of words in F is primitive.

[Rapaport (1958) gave an algebraic proof of Whitehead's theorem. See also McCool (1974) for a presentation of Aut(F) in terms of Whitehead automorphisms. Gersten (1984) gave another proof of the above theorem using graph theoretic methods and Whitehead automorphisms].

When $m = n$, algorithmic decidability of a system of n elements in the free group F of rank n reduces to decidability of a given endomorphism of F to be an automorphism. This, in turn, can be translated to a problem of invertibility of a given matrix over the free group ring $\mathbb{Z}F$. The following criterion is due to Joan Birman.

Theorem (Birman 1974). A given system $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ is primitive in F_n if and only if the $n \times n$ Jacobian matrix $J(\mathbf{w}) = (\partial w_i/\partial x_j)$ of Fox-derivates of the system is invertible over $\mathbb{Z}F$.

[If $w - 1 = \sum_i u_i(x_i - 1), u_i = \partial w/\partial x_i$ is the (left) Fox derivative of w with respect to x_i].

Remark. Umirbaev gives a similar criterion for a system of m elements, $m \leq n$, to be a part of a basis. Krasnikov (1978) has given a criterion for a system $\mathbf{w}= \{w_1, w_2, \ldots, w_n\}$ to generate F (modulo R') in terms of invertibility of the Jacobian matrix $J(\mathbf{w})$ over $\mathbb{Z}(F/R)$ (the case $R = F'$ is due to Bachmuth (1965) (see N. Gupta (1987) for proof).

Generating sets for Aut(F/V). Let V be a fully invariant subgroup of F contained in F'. Then Aut(F/V) is generated by T, the set of all tame automorphisms of F/V, together with the IA-automorphisms of F/V, where *tame automorphisms* are those induced by the automorphisms of the free group F. We abbreviate by writing $\mathrm{Aut}(F/V) = < T, \text{IA-Aut}(F/V) >$. As we know from the work of Nielsen, T is generated by at most 4 elementary automorphism. So, we need to concentrate on IA-Aut(F/V) in order to find a most economical set of generators of Aut(F/V). When $F = F_n, n \geq 4$, Bachmuth and Mochizuki have shown that $\mathrm{Aut}(F/F'') =< T >$, whereas for $n = 3$, Aut(F/F'') is infinitely generated. We study this question for the automorphisms of free nilpotent groups of class c on n generators. Let $F_{n,c} =< x_1, x_2, \ldots, x_n >, n \geq 2, c \geq 2$ denote the free nilpotent group of class c freely generated by the set $\{x_1, x_2, \ldots, x_n\}$. Then $F_{n,c} \cong F_n/\gamma_{c+1}(F_n)$, where $F_n = < f_1, f_2, \ldots, f_n; \emptyset >$ is the absolutely free group on $\{f_1, f_2, \ldots, f_n\}$. Since every automorphism of the free abelian group $F_{n,c}/\gamma_2(F_{n,c})$ is tame it follows that

$$\mathrm{Aut}(F_{n,c}) = < T, \text{IA-Aut}(F_{n,c}) > .$$

Consider an IA-automorphism of $F_n/\gamma_3(F_n)$ of the form

$$\alpha = \{x_1 \to x_1 d, d \in F_n{}', x_i \to x_i, i \neq 1\}.$$

Modulo $\gamma_3(F_n), d$ can be written as $d = \prod[x_1, x_i{}^{a_i}]\ d^\star = [x_1, w]\ d^\star$, where $d^\star$ does not involve x_1. Thus α assumes the form $\{x_1 \to x_1{}^w d^\star, x_i \to x_1, i \neq 1\}$, which is clearly a tame automorphism. Since every IA-automorphism of $F_n/\gamma_3(F_n)$ is a product of automorphisms of the form α above it follows that the automorphisms of $F_{n,c}/\gamma_3(F_{n,c})$ are all tame. Thus we have the modification,

$$\mathrm{Aut}(F_{n,c}) = < T, \mathrm{IA}^\star - \text{Aut-}(F_{n,c}) >,$$

where $\mathrm{IA}^\star - \mathrm{Aut}(F_{n,c})$ consist of IA-automorphisms of the form

$$\{\ x_i \to x_i d_i, d_i \in \gamma_3(F_{n,c}), i = 1, \ldots, n\ \}.$$

Problem: Together with T how many IA-Automorphisms of $F_{n,c}$ are required to generate $\mathrm{Aut}(F_{n,c}), c \geq 3$?

Goryaga (1976) proved that if $n \geq 3.2^{c-2} + c$, then $\mathrm{Aut}(F_{n,c}) = < T, \theta_3, \ldots, \theta_c >$, where θ_k are defined by $\theta_k = \{x_1 \to x_1[x_1, x_2, \ldots, x_k], x_i \to x_i,\ i \neq 1\}$.

Andreadakis (1984) reduced the restriction on n in Goryaga's result significantly by proving that the same conclusion holds for $n \geq c$, i.e. for $n \geq c$, $\mathrm{Aut}(F_{n,c}) = < T, \theta_3, \ldots, \theta_c >$. Further improvement of Andreadakis'result is possible as is seen from the following result.

Theorem (C. K. Gupta & Bryant 1989). If $n \geq c$, then $\mathrm{Aut}(F_{n,c}) = < T, \theta_3 >$. In fact the following theorem is proved.

Theorem (Bryant & Gupta 1989). For $n \geq c - 1 \geq 2, \mathrm{Aut}(F_{n,c}) = < T, \delta_3 >$, where

$$\delta_3 = \{\ x_1 \to x_1[x_1, x_2, x_1],\ x_i \to x_i,\ i \neq 1\ \}.$$

[For $n \leq c - 2$, more IA-automorphisms seem to be required to generate $\mathrm{Aut}(F_{n,c})$. For instance, if $n \geq c - 2 \geq 2$ then $\mathrm{Aut}(F_{n,c}) = < T, \delta_3, \delta_4 >$, where

$$\delta_4 = \{x_1 \to x_1[x_1, x_2, x_1, x_1],\ x_i \to x_i,\ i \neq 1\}.$$

For $c + 1/2 \leq n \leq c - 1$, we give a specific generating set wich depends only on the difference $c - n$ (see Bryant and Gupta 1989). However, for $n \leq c/2$ we do not know any reasonably small generating set for $\mathrm{Aut}(F_{n,c})$].

Next we consider $M_{n,c}$, the free metabelian nilpotent of class c group, freely generated by $\{x_1, x_2, \ldots, x_n\}$. Then $M_{n,c} \cong F_n/\gamma_{c+1}(F_n)F_n''$, where, as before, F_n is the absolutely free group on the set $\{f_1, f_2, \ldots, f_n\}$. For $c \geq 3$, a complete description of IA-Aut$(M_{2,c})$ in terms of generators and defining relations has been given by Gupta (1981).

Theorem (Andreadakis and C. K. Gupta 1990). If $n \geq 2$ and $c \geq 3$ then for each $\alpha \in \mathrm{Aut}(M_{n,c})$ there exists a positive integer $a(\alpha)$ such that $\alpha^{a(\alpha)} \in < T, \delta_3, \ldots, \delta_c >$ where, in addition, the prime factorization of $a(\alpha)$ uses primes dividing $[c+1/2]!$

Automorphisms of free nilpotent Lie algebras. The corresponding problems for relatively free Lie algebras has been studied in Drensky and Gupta (1990). Let $F(n_c)$ resp. $M(n_c))$ denote the free nilpotent (resp. free metabelian and nilpotent) Lie algebra of class c on $\{x_1, \ldots, x_n\}$ over a field $\mathbf{K}$ of characteristic zero. Then

(i) If $n \geq c$, $\mathrm{Aut}(F(n_c)) = \mathrm{group}\{\mathrm{GL}(n, \mathbf{K}, \delta\}$, where $\delta(x_1) = x_1 + [x_1, x_2]$, $\delta(x_i) = x_i, i \neq 1$;

(ii) If $n \geq 2, c \geq 2$, $\mathrm{Aut}(M(n_c)) = \mathrm{group}\ \{GL(n, \mathbf{K}\ ), \delta\}$,

where δ is as before. We refer to our paper for details.

Automorphisms of free nilpotent of class 2 by abelian groups. Let $G = F/[F', F', F']$, $F = < x, y, u, v >$. Then we have the following result,

Theorem (Gupta and Levin 1989). $\mathrm{Aut}(G) =$ gp $\{T, \delta_0, \delta_1, \delta_2, \ldots\}$, where each $\delta_k = \{x \to x[[x, y]^{x^k}, [u, v]], y \to y, u \to u, v \to v\}$ is a non-tame automorphism of G. [The details of the fact that $\mathrm{Aut}(G)$ is generated by tame automorphisms and δ_k's are extremely technical and we refer to our paper. The non-tameness of δ_k is proved by showing that the Jacobian matrix $J(\mathbf{w})$ of the system $\mathbf{w} = (x[[x, y]^{x^k}, [u, v]], y, u, v)$ over the free group ring ZF is not invertible. This is achieved by building a homomorphism of the group $\mathrm{GL}(4, \mathbb{Z}F)$ into $\mathrm{GL}(2, \mathbb{Z}[t])$ which maps $J(\mathbf{w})$ to a non-invertible element of $\mathrm{GL}(2, \mathbb{Z}[t])$].

A criterion for non-tameness and applications. The k-th left partial derivative ∂_k is defined linearly on the free group ring $\mathbb{Z}(F)$ $(= \mathbb{Z}(F_n))$ by: $\partial_k(x_k) = 1$; $\partial_k(x_i) = 0$, $i \neq k$; $\partial_k(uv) = \partial_k(u) + u\partial_k(v)$, $u, v \in \mathbb{Z}(F)$. In particular, for any $w \in \gamma_m(F)$, the partial derivative $\partial_k(w)$ lies in $\Delta^{m-1}(F)$, and hence, modulo $\Delta^m(F)$, it can be represented as a polynomial $f(X_1, X_2, \ldots, X_n\}$ in the non-commuting variables $X_i = x_i - 1$, $i = 1, \ldots, n$. For any $S_i, T_i \in \{X_1, \ldots, X_n\}$ we define an equivalence relation $\approx$ on monomials by : $S_1 \ldots S_k \approx T_1 \ldots T_k$ if one is a cyclic permutation of the other. Finally, a polynomial $f(X_1, \ldots, X_n\}$ is called *balanced* if $f(X_1, \ldots, X_n) \approx 0$, or equivalently, *the sum of the co-efficients of its cyclically equivalent terms is zero.* Then through a technical analysis of the invertibility of the Jacobian matrix associated with a basis of F we have the following useful test for an endomorphism of F to be an automorphism.

Criterion (Bryant, Gupta, Levin and Mochizuki 1990). Let $w = w(x_1, \ldots, x_n) \in \gamma_m(F_n)$ for some $m \geq 2$ and let α be an endomorphism of F_n defined by: $\alpha(x_1) \equiv x_1 w, \alpha(x_i) \equiv x_i(\bmod \gamma_{m+1}(F_n)),\ i = 2, \ldots, n$. Let $\partial_1(w) \equiv f(X_1, \ldots, X_n)$
$(\bmod \Delta^m(F))$. If α defines an automorphism of F_n then $f(X_1, \ldots, X_n)$ must be balanced.
[Using different methods, Shpilrain (1990) has also obtained a similar criterion].

Application 1. The following automorphism of free class-3 group $F_{n,3}$ of rank $n \geq 2$ is wild: $\alpha = \{x \to x[[x, y, x], y \to y, \ldots, z \to z\}$.
Proof (cf. Andreadakis 1968). We have,

$$\partial_x([x, y, x]) \equiv 2(y-1)(x-1) - (x-1)(y-1)(\Delta^3(F)).$$

Then $f = 2(y-1)(x-1) - (x-1)(y-1)$ is not balanced, and the proof follows.

Application 2. The following automorphism of free centre-by-metabelian group of rank $n \geq 4$ is wild :

$$\alpha = \{\ x \to x[[x, y], [u, v]], y \to y, \ldots, z \to z\ \}.$$

This answers a question of Stöhr (1987).

Proof Since, $[F'', F] \leq \gamma_5(F)$, it suffices to prove that α is not a tame automorphism of free class 4 group $F_{n,4}$. Indeed, $\partial_x([[x, y], [u, v]]) \equiv (y-1)([u, v] - 1)(\Delta^5(F))$ and $f = (y-1)(u-1)(v-1) - (y-1)(v-1)(u-1)$ is clearly not balanced.

Application 3. For each $k \geq 1$ the following automorphism of free class-2 by abelian group of rank 4 is wild:

$$\alpha_k = \{\ x \to x[[x, y, x^k], [u, v]], y \to y, u \to u, v \to v\ \}.$$

Proof (cf. Gupta-Levin (1989)). Since, $[F'', F'] \leq \gamma_6(F)$, it suffices to prove that α is not a tame automorphism of free class 5 group $F_{4,5}$. We have,

$$\begin{aligned}\partial_x([[x, y, x^k], [u, v]]) &\equiv (y-1)(x^k - 1)([u, v] - 1) - \\ &\quad (1 + x + \ldots + x^{k-1})([x, y] - 1)([u, v] - 1)(\Delta^5(F)) \\ &\equiv k(y-1)(x-1)([u, v] - 1) - k([x, y] - 1)(\Delta^5(F)).\end{aligned}$$

Then $f = k(y-1)(x-1)([u,v]-1) - k([x,y]-1)([u,v]-1)$ and the sum of the co-efficients of terms which are cyclically equivalent to $(x-1)(y-1)(u-1)(v-1)$ is minus k wich is non zero.

Non-tame automorphisms of free polynilpotent groups. The tame range TR $\{\mathrm{Aut}F_n(\mathbf{V}),\ n \geq 1\}$ of the automorphismgroups of the free groups of a variety $\mathbf{V}$ of groups has been defined by Bachmuth and Mochizuki (1987/89) as the least integer $d \geq 1$ such that all automorphisms of a free $\mathbf{V}$-group $F_k(\mathbf{V})$ of rank $k \geq d$ are tame. If no such d exists then TR $\{\mathrm{Aut}F_n(\mathbf{V}), n \geq 1\}$ is defined to be infinite.

For the variety $\mathbf{M}$ of metabelian groups, Bachmuth and Mochizuki (1985) proved that TR $\{\mathrm{Aut}F_n(\mathbf{M}), n \geq 1\} = 4$. They raised questions about the possible values of the tame range of automorphism groups of certain relatively free soluble groups defined by outer-commutator words. We give a complete answer to their question by proving the following theorem.

Theorem (Gupta-Levin 1991). With the three know exceptions: $\mathbf{M}$, the variety of metabelian groups, $\mathbf{A}$, the variety of abelian groups and $\mathbf{N}_2$, the variety of nilpotent groups of class at most 2, the range TR$\{\mathrm{Aut}F_n(\mathbf{V}), n \geq 1\}$ is infinite for any variety $\mathbf{V}$ defined by an outer-commutator word.

Outline An outer commutator u has one of the following three types:

(i) $u = [a_1, \ldots, a_m]$;
(ii) $u = [[a_1, \ldots, a_r], [b_1, \ldots, b_s]], r \geq s \geq 2, r + s = m$;
(iii) $u = [[a_1, \ldots, a_r], v, \ldots, w], 2 \leq r < m$, where $v = v(b_1, \ldots, b_s), \ldots, w = w(c_1, \ldots c_t)$ are outer commutators of weights $s, \ldots, t$, respectively, with $s \geq 2, t \geq 1, r + s + \ldots + t = m$.

Let $F = F_n$ be free of rank $n \geq m$. With $(x_1, \ldots, x_m) = (a_1, \ldots, a_m)$ if u is of type (i); $(x_1, \ldots, x_m) = (a_1, \ldots, a_r, b_1, \ldots, b_s)$ if u is of type (ii); and $(x_1, \ldots, x_m) = (a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_t)$ if u is of type (iii), let $U, V, \ldots, W$ be the fully invariant closures in F of $u, v, \ldots, w$ respectively. Then, clearly

$$U = \gamma_m(F), U = [\gamma_r(F), \gamma_s(F)] \text{ or } U = [\gamma_r(F), V, \ldots, W]$$

according as u is of type (i), (ii) or (iii). Define $u^\star = [x_1, \ldots, x_{m-1}]$ if u is of type (i), $u^\star = [[x_1, \ldots, x_{r-1}], [x_{r+1}, \ldots, x_{r+s}]]$ if u is of type (ii), $u^\star = [[x_1, \ldots, x_{r-1}], v, \ldots, w]$ if u is of type (iii). Define $\mu = \{x_1 \to x_1 u^\star, x_i \to x_i,\ i \neq 1\} \in \mathrm{End}(F)$. Then μ induces an automorphism of F/U. The proof of the theorem consists in showing that if $U \neq \gamma_2(F),\ \gamma_3(F), F''$ then μ induces a non-tame automorphism of F/U. We may assume $m \geq 4$, anf if $m = 4$ then $U = \gamma_4(F)$. Since $u^\star \in \gamma_{m-1}(F)$ and $U \leq \gamma_m(F)$, it suffices to prove that μ induces a non-tame automorphism of $F/\gamma_m(F)$. This is achieved by a direct application of the criterion.

Lifting primitivity of relatively free groups F/U Recall that a system $\mathbf{w}$=$\{w_1, \ldots, w_m\}, m \leq n$, of words in a free group $F = < x_1, \ldots, x_n >$ is said to be *primitive* if it can be included in some basis of F. Let U be a fully invariant subgrooup of F. We say that a system $\mathbf{w} = \{w_1, \ldots, w_m\}, m \leq n$, of words in F is *primitive mod* U if the system $\{w_1 U, \ldots, w_m U\}$ of cosets can be extended to some basis for F/U. Given a system $\mathbf{w} = \{w_1, \ldots, w_m\}, m \leq n$, which is primitive modulo U we wish to study the possibility of lifting this system to a primitive system of F. For, instance, if F is free of rank $n \geq 4$ then every automorphism is tame (Bachmuth & Mochizuki) and consequently, every primitive system mod F'' lifts to a primitive system of F. Whereas, for $n = 3$ the system $\{x[x, y, x], y, z\}$ is primitive mod F'' but can not be lifted to a primitive system of F (Chein). We consider the problem of primitivity lifting of certain relatively free nilpotent groups.

Lifting primitivity of free metabelian nilpotent groups Let $\mathbf{w} = \{w_1, \ldots, w_m\}$, $m \leq n$, be primitive mod $U = \gamma_{c+1}(F)F''$. We wish to lift this system to a primitive system of F. This is not always possible. For example if $F =< x, y, z >$, the system $\{x[y, z, x, x], y\}$ is primitive mod $\gamma_4(F) = (\gamma_4(F)F'')$ but the extended system $\mathbf{w} = \{x[x, y, x]u, yv, zw\}$ is not primitive in F for any choice of u, v in $\gamma_4(F)$ and w in F'. This can be seen using Bachmuth - Birman's criteria by verifying that the Jacobian matrix $J(\mathbf{w})$ of the system is not invertible. For $n \geq 4$, we can take advantage of Bachmuth and Mochizuki's result which reduces the problem of lifting primitivity mod U to that of mod F''. Thus we can restrict to free metabelian nilpotent-of-class-c groups $M_{n,c}$ and need only study the lifting of primitivity mod $\gamma_{c+1}(M_n)$ to the free metabelian group $M = M_n =< x_1, \ldots, x_n >$. An outline of the procedure for lifting a single element w (which is primitive mod $\gamma_{c+1}(M)$) to a primitive element of M as follows:

Step 1. Since w is primitive mod $\gamma_{c+1}(M)$ there is an automorphism of M which maps w to an element of the form $x_1 v, v \in M'$. Thus we assume $w = x_1 v, v \in M'$. Further, since every automorphism of free class-2 group lifts, we may assume $c \geq 3$.

Step 2. Working by induction on c, we may assume $v = \prod[x_1, x_i]^{p_{1i}} \prod[x_j, x_k]^{q_{jk}}$, where $p, q \in \Delta^{c-2}(M'), 2 \leq j < k \leq n, q_{jk}$ independent of x_1.
(Notation : $[x_i, x_j]^{g+h} = [x_i, x_j]^g [x_i, x_j]^h$).

Step 3. There is an automorphism μ of M which maps x_1 to $x_1 \prod[x_j, x_k]^{-q_{jk}}$. Applying μ to w, if necessary, we may assume $w = x_1 \prod[x_1, x_i]^{p_{1i}}$.

Step 4. For each $p \in \Delta^{c-2}(M), c \geq 3$, consider the system g=$\{g_1, \ldots, g_n\}$ with $g_1 = x_1[x_1, x_2]^p [x_2, x_3]^{(x_2-1)p}$, $g_3 = x_3[x_1, x_2]^{-p^2}[x_2, x_3]^{p-(x_2-1)p^2}, g_i = x_i, i \neq 1, 3$. Let $J(\mathrm{g})$ be the Jacobian matrix of the system g over $\mathbb{Z}(M/M')$. It is easily seen that with $\pi = \theta p$ (under $\theta : \mathbb{Z}M \to \mathbb{Z}(M/M')$, the matrix $J(\mathrm{g})$ has the form,

$$
\begin{matrix}
1+(a_2-1)\pi & * & -(a_2-1)^2\pi & 0 & \dots & 0 \\
0 & 1 & 0 & 0 & \dots & 0 \\
-(a_2-1)\pi^2 & * & 1-(a_2-1)\pi+(a_2-1)^2\pi^2 & 0 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 1 & \dots & 0 \\
0 & 0 & 0 & 0 & \dots & 1
\end{matrix}
$$

The determinant of $J(\mathbf{g})$ is easily seen to be 1, so $J(\mathbf{g})$ is invertible. Since $p \in \Delta^{c-2}(M)$, it follows that $g_1 = x_1[x_1, x_2]^p$ is primitive in M for all $p \in \Delta^{c-2}(M)$. Consequently, $w = x_1 \prod [x_1, x_i]^{p_{1i}}$ is primitive in M.
We thus have proved,

Theorem (Gupta, Gupta and Roman'kov 1992). If F is free of rank ≥ 4 then every primitive element mod $\gamma_{c+1}(F)$ can be lifted to a primitive element of F.

Likewise, for $n \geq 4$ and $m \leq n-2$, it can be proved that every primitive system $\mathbf{g}=\{g_1, \dots, g_m\}$ mod $\gamma_{c+1}(M_n)$ can be lifted to a primitive system of M_n, yielding the following theorem. We refer to Gupta-Gupta-Roman'kov (1991) for details.

Theorem For $n \geq 4$ and $m \leq n-2$, every primitive system $\mathbf{g}=\{g_1, \dots, g_m\}$ mod $\gamma_{c+1}(F_n)F''$ can be lifted (via $\gamma_{c+1}(F_n)F''$) to a primitive system of F_n.

Remarks.

(1) The restriction $m \leq n-2$ in the above theorem can not be improved. To see this choose $g_1 = x_1[x_1, x_3, x_3], g_i = x_i, i \neq 1, 3$. Then for any choice of $g_3 = x_3 u, u \in M_n{}'$, and any choice of elements $w_i \in \gamma_4(M_n), i = 1, \dots, n$, the Jacobian matrix $J(\mathbf{g})$ of the system $\mathbf{g}=\{g_1 w_1, \dots, g_n w_n\}$ can be seen to be non-invertible.

(2) When rank of F is 3 the metabelian approach does not apply as the metabelian group $M = M_3 = < x, y, z >$ admits wild automorphisms (Chein 1968), The proof that every primitive element of $M_{3,c}, c \geq 3$, can be lifted (via $\gamma_{c+1}(M)F''$) to a primitive element of F_3 is quite technical and we refer to our paper for details.

(3) Since, every IA-automorphism of M_2 is inner (Bachmuth), $g = x_1 u$ can be lifted to a primitive element of M_2 if and only if u is of the form $[x_1, v]$. Thus, for $c \geq 3$, not every primitive element of $M_{2,c}$ can be lifted to a basis of M_2.

(4) The existence of non-tame automorphisms of M_3 was first shown by Chein (1968). Specifically, the automorphism $\{x \to x[y, z, x, x], y \to y, z \to z\}$ of M_3 can not be lifted to an automorphism of the free group F_3. It is easily seen that every endomorphism in M_3 of the form $\{x \to x[y, z]^{p(x,y,z)},\ y \to y,\ z \to z\}$ is an automorphism of M_3. So, for each $p(x, y, z) \in \mathbb{Z}M_3$ the element $x[y, z]^{p(x,y,z)}$ is primitive in M_3 can be lifted to a primitive element of F_3 (Gupta, Gupta and Romankov). Two natural questions are:

(i) Can every primitive element of M_3 be lifted to a primitive element of F_3 ? [Roman'kov (1993) gave a negative answer to this question.]

(ii) Is primitivity in $M_n, n \geq 2$, algorithmically decidable? [The answer is yes (see Gupta et al (1994)). The corresponding solution for $M_n, n \geq 4$, is due to Timoshenko (1989).]

Lifting primitivity of free nilpotent groups.

Let $\mathbf{w}=\{w_1, \ldots, w_m\}, m \leq n$, be primitive mod $U = \gamma_{c+1}(F_n)$. We wish to lift this system to a primitive system of F_n. Here we do not have tha facility of working modulo F'' so certain further restrictions on m may be necessary. We have the following result.

Theorem (Gupta and Gupta 1992). For $m \leq n+1-c$, every primitive system $\mathbf{w}=\{w_1, \ldots, w_m\}, m \leq n, \text{mod}\gamma_{c+1}(F_n)$ can be lifted to a primitive system $\mathbf{w}^\star$ of F_n.

Remark. For $c \geq 4, n = c-1$, it would be of interest to know whether every primitive element mod $\gamma_{c+1}(F)$ can be lifted to a primitive element of F. The simplest case of the problem is to decide whether or not, for $n = 3, c = 4$, the element $x_1[x_1, x_2, x_2, x_3]$ can be lifted to a basis of F.

We conclude with the following more general question.

Question. Is Aut(F/U) always tame for relatively free countable infinite rank groups defined by outer-commutator words? In particular, are all automorphisms of free polynilpotent groups of countable infinite rank tame?
[Affirmative answers are now known in the following cases: free metabelian groups (Brayant and Groves 1992), for free nilpotent groups (Bryant and Macedonska 1989) and for free (nilpotent of class c) - by - abelian groups and certain central extensions (Bryant and Gupta (1993)].

NOTE. The manuscript in prepared by freely using the material from a series of Lectures I gave in Parma, Italy in 1991.

The reader is also refered to a recent survey article on lifting automorphisms (Gupta-Shpilrain 1995).

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