

# Flow Through Isotropic Granular (Non Consolidated) Porous Media

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**Abstract:** A generalization of the Navier-Stokes equations, proposed by Prieur du Plessis and Masliyah to model laminar flow through a rigid isotropic granular (non consolidated) porous medium of spatially varying permeability, is analyzed. Results concerning existence and regularity of solutions, similar to the ones holding the classical Navier-Stokes equations, are proved.

**Key words:** Granular Porous Media; Non Consolidated Porous Media; Faedo-Galerkin Method; Existence of Solutions.

## 1 Introduction

In this work we will prove results concerning existence and regularity of solutions of a system of partial differential equations corresponding to a generalization of the classical Navier-Stokes equations. The equations were derived by Prieur du Plessis and Masliyah [8] in order to model laminar incompressible flow through a rigid isotropic granular (non consolidated) porous medium of spatially varying permeability, and it has been considered a good model for many empirical situations occurring, for instance, in petroleum industry. The equations are the following:

$$\begin{cases} \rho u_t + \rho u \cdot \nabla \left( \frac{u}{n} \right) - \mu \Delta u + n \nabla p + \mu F(n)u = \rho n g & \text{in } Q_T, \\ \operatorname{div} u = 0 & \text{em } Q_T, \\ u(x, 0) = u_0(x) & \forall x \in \Omega, \\ u(x, t) = 0 & \forall t \in (0, T) \quad \forall x \in \partial\Omega. \end{cases} \quad (1.1)$$

Here,  $\Omega \subset \mathbb{R}^d$ , with  $d = 2$  or  $3$ , is an open, regular, bounded set corresponding to the granular region where the flow is occurring; the boundary of  $\Omega$  is denoted  $\partial\Omega$ .  $0 < T$  is the final time being considered in the problem.

The unknowns in the problem are  $u(x, t) \in \mathbb{R}^d$  and  $p(x, t) \in \mathbb{R}$ , which denote, respectively, the fluid velocity and the hydrostatic pressure at a point  $x \in \Omega$ , at time  $t \in [0, T]$ .

We assume that the fluid viscosity,  $\mu$ , is a positive constant. The density,  $\rho$ , without loss of generality, will be assumed to be normalized to be one.

The porosity  $n(x, t)$ , at a point  $x \in \Omega$ , at time  $t \in [0, T]$ , is defined, in rough terms, as the void volume divided by the total volume of small regions in the neighborhood of  $x$  at time  $t$ . Thus,  $n(x, t)$  assumes real values between zero and one. We observe that the porosity is one in cavities, where, therefore, the flow is free. At points  $(x, t)$  such that the porosity is zero, the material medium is

purely solid and can be excluded of the flow region. Throughout this work, we will assume that the porosity satisfies  $0 < n(x, t) \leq 1$ .

$F$  is a force term due to friction between the granular porous medium and the fluid. On physical grounds,  $F$  is a continuous function satisfying  $\lim_{z \rightarrow 1} F(z) = 0$  and  $\lim_{z \rightarrow 0} F(z) = \infty$  (see Prieur du Plessis and Masliyah in [8] for an expression for  $F$ .) We remark that our results will not depend on that particular expression for  $F$ .

A known external force field, like gravity, is denoted  $g(x, t)$  and may be acting on the flow.

In cartesian coordinates, we have

$$\Delta u = (\Delta u_1, \dots, \Delta u_d) \quad \text{and} \quad (u \cdot \nabla u)_i = \sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}$$

We observe that the above equations are different from those usually called “porous medium equations”. These last are associated to flows in consolidated porous media, in which the velocity and the pressure are related (generally by the Darcy’s Law, or its variants.)

From the technical point of view, equations (1.1) is related to the Navier-Stokes equations for nonhomogeneous fluids (see Antonzev and Kajikov [2] and Lions [6], [7].) as we will see in the paper, whose organization is the following:

The work is organized as follows:

In the next section (Section 2), we will fix the notation and recall certain results that will be used throughout the paper. In Section 3, by using the Faedo-Galerkin method in a Sobolev space context, we will obtain the existence of global weak solutions of (1.1). In Section 4 we consider the existence of strong solutions in the case that the medium porosity is a perturbation of a constant. As it is expected, since the same hold even for the case of the classical Navier-Stokes equations, we obtain the existence of strong solutions for small interval of time without requiring smallness of the data. Then, either in the case of spatial dimension two or spatial dimension three, but now with suitable smallness conditions on the data, we prove the existence of global in time strong solutions.

In short, under suitable conditions, we were able to extend results holding in the case of the classical Navier-Stokes equations to equations (1.1).

## 2 Preliminaries

In this section we will fix the notation and, for the reader’s convenience, recall certain mathematical tools that have been used in the classical study of the Navier-Stokes equations.

For a Banach space  $X$ , in general, we will denote the norm of a vector  $v \in X$  by  $\|v\|_X$ . However, being  $\Omega$  an open, bounded and regular set on  $\mathbb{R}^d$  and  $1 \leq q \leq \infty$ , the norm of a function  $f \in L^q(\Omega)$  will be denoted  $\|f\|_q$ . Since the space

$L^2(\Omega)$  will be frequently used, the norm of  $f \in L^2(\Omega)$  will be denoted just by  $|f|$  to simplify the notation. The inner product of  $f, g \in L^2(\Omega)$  will be denoted  $(f, g) = \int_{\Omega} f(x)g(x)dx$  (in this paper, we will always work with real functional spaces.)

For  $k \in \mathbb{N}$ , we will denote by  $H^k(\Omega)$  the usual Sobolev space based on  $L^2(\Omega)$ . The closure of the class of  $C^\infty$ -functions with compact support in  $\Omega$  is  $H_0^k(\Omega)$ . The topological dual of  $H_0^k(\Omega)$  will be denoted by  $H^{-k}(\Omega)$ .

We will use several standard Sobolev imbedding results, which can be found in Adams[1] or several other references. In particular, the following inequalities can be found, for instance, in Temam [10], pp. 291 and 296: for any  $v \in H_0^1(\Omega)$ , there hold

$$\begin{aligned} (i) \quad & |v|_4 \leq 2^{\frac{1}{4}} |v|^{\frac{1}{2}} |\nabla v|^{\frac{1}{2}}, \quad \text{when } d = 2, \\ (ii) \quad & |v|_4 \leq 2^{\frac{1}{4}} |v|^{\frac{1}{4}} |\nabla v|^{\frac{3}{4}}, \quad \text{when } d = 3. \end{aligned} \quad (2.1)$$

Being  $(X, \|\cdot\|_X)$  a Banach space,  $-\infty \leq a < b \leq +\infty$  and  $1 \leq p < +\infty$ , we will denote by  $L^p(a, b; X)$  the class of functions defined on  $(a, b)$  and with values in  $X$  that are  $p$ -integrable in the sense of Bochner. With the norm  $\|f\|_{L^p(a, b, X)} = \left( \int_a^b \|f(t)\|_X^p dt \right)^{1/p}$ ,  $L^p(a, b, X)$  is also a Banach space. Similarly to the escalar case, we also define  $L^\infty(a, b, X)$  and  $\|f\|_{L^\infty(a, b, X)}$ .

The following functional spaces are used in the classical theory of the Navier-Stokes equations:

$$\begin{aligned} \mathcal{V} &= \{u \in (C_0^\infty(\Omega))^d : \operatorname{div} u \equiv 0\} \\ V &= \text{closure of } \mathcal{V} \text{ in } (H_0^1(\Omega))^d \\ H &= \text{closure of } \mathcal{V} \text{ in } (L^2(\Omega))^d \\ G &= \{u \in (L^2(\Omega))^d : u = \nabla p, \text{ for some } p \in H^1(\Omega)\} \end{aligned}$$

The following characterizations are known:

$$\begin{aligned} V &= \{u \in (H_0^1(\Omega))^d : \operatorname{div} u \equiv 0\}, \\ H &= \left\{ u \in (L^2(\Omega))^d : \operatorname{div} u \equiv 0 \text{ and } u \cdot \vec{N}|_{\partial\Omega} \equiv 0 \right\}. \end{aligned}$$

Here,  $\operatorname{div} u = 0$  and  $\nabla p$  are understood in the distributional sense;  $u \cdot \vec{N}|_{\partial\Omega}$  denotes the normal trace of  $u$  on  $\partial\Omega$  (see, for instance, Temam [10]).

We remark that we identify  $H$  and its topological dual  $H'$ , obtaining the continuous and dense inclusions  $V \subset H \equiv H' \subset V$ . Thus, the inner product in  $H$  of  $f \in H$  and  $u \in V$  is the same that the duality product in  $V'$  e  $V$  of  $f$  and  $u$ :

$$(f, u)_{V', V} = (f, u) \quad \forall f \in H \quad \forall u \in V$$

Another very important result is that  $G = H^\perp$ , where  $H^\perp$  denotes the orthogonal space of  $H$  in the sense of  $(L^2(\Omega))^d$ . Thus, we have the following orthogonal decomposition:

$$(L^2(\Omega))^d = H \oplus G, \quad (2.2)$$

which defines the orthogonal projection of  $L^2(\Omega)$  on  $H$ . We denote this projection by  $P$  and then define the Stokes operator:

$$A = -P\Delta : \text{Dom}(A) = V \cap H^2(\Omega) \rightarrow H.$$

Important results holding for  $A$  (for their proof, see, for instance, Rautmann [9], pp. 427 and 428) are the following:

$A$  is a positive definite (symmetric) operator, with compact inverse  $A^{-1} : H \rightarrow H$ . Thus, the Stokes operator has a sequence of positive eigenvalues  $0 < \lambda_1 \leq \lambda_2 \leq \dots \lambda_i \rightarrow \dots + \infty$ , with corresponding eigenfunctions  $\{e_i\}_1^\infty$  forming an orthogonal basis both in  $H$  and  $V$ .

Another important result is the following:

**Lemma 2.1** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set with boundary of class  $C^3$ . Then there is a positive constant  $C$ , depending only on  $\Omega$ , such that, given  $g \in L^2(\Omega)$ , the unique solution  $v \in V$ ,  $q \in L^2(\Omega)/\mathbb{R}$  of the following Stokes Problem:*

$$\begin{cases} -\Delta v + \nabla \psi = g, \\ \text{div } v = 0, \\ u|_{\partial\Omega} = 0, \end{cases}$$

*is more regular and satisfies*

$$\|v\|_{H^2(\Omega)} + \|\psi\|_{H^1(\Omega)/\mathbb{R}} \leq C|g|$$

We remark that the above regularity of the boundary could be weakened. Since sharp regularity questions are not the main point in this paper, we will just assume hypotheses that are enough to prove the stated results.

We observe that (2.2) implies that for any  $v \in V \cap (H^2(\Omega))^d$  there is  $\psi \in H^1(\Omega)$  such that

$$-\Delta v = Av + \nabla \psi. \quad (2.3)$$

Writing this last equality as  $-\Delta v + \nabla(-\psi) = Av$  and recalling that since  $v \in V$  then  $\text{div } v = 0$  and  $v|_{\partial\Omega} = 0$ , Lemma 2.1 implies in particular that the decomposition (2.3) satisfies

$$\|v\|_{H^2(\Omega)} + \|\nabla \psi\| \leq C|g|. \quad (2.4)$$

For further considerations concerning the Stokes operators, see for instance, Heywood e Rannacher [5].

Concerning the nonlinearity corresponding to the convection term in the equations, the following results can be found in Temam [10], pp. 281 and 292, respectively.

**Lemma 2.2** *Let  $d \leq 4$  and  $u \in L^2(0, T, V)$ ; then the function  $Bu$  defined by*

$$\langle B(u)(t), v \rangle = b(u(t), u(t), v) = (u \cdot \nabla u, v), \forall v \in V, t \in [0, T],$$

*belongs to  $L^1(0, T, V')$ .*

**Lemma 2.3** For any  $u, v, w \in H_0^1(\Omega)$  we have

- (i)  $|(u \cdot \nabla v, w)| \leq 2^{\frac{1}{2}} |u|^{\frac{1}{2}} |\nabla u|^{\frac{1}{2}} |\nabla v| |w|^{\frac{1}{2}} |\nabla w|^{\frac{1}{2}}$ , when  $d = 2$ ,
- (ii)  $|(u \cdot \nabla v, w)| \leq 2 |u|^{\frac{1}{4}} |\nabla u|^{\frac{3}{4}} |\nabla v| |w|^{\frac{1}{4}} |\nabla w|^{\frac{3}{4}}$ , when  $d = 3$ .

Now, we state an abstract compactness criterion, due to Antonzev and Kajikov [2], whose proof can be found, for instance, in Lions [7].

**Lemma 2.4** Let  $0 < T < \infty$  and  $\Upsilon$  be a class of functions for which there are positive constants  $c_1$  and  $c_2$  such that for any  $v \in \Upsilon$ , there hold  $\|v\|_{L^2(0,T;V)} +$

$$\|v\|_{L^\infty(0,T;H)} \leq c_1 \text{ and } \int_0^{T-\delta} |v(t+\delta) - v(t)|^2 dt \leq c_2 \delta^{1/2} \text{ as } \delta \rightarrow 0+$$

Then,  $\Upsilon$  is relatively compact subset of  $L^p(0, T; L^q(\Omega))$ , for any  $p \in [2, \infty)$  and  $q \in [2, 6)$  satisfying  $\frac{1}{p} + \frac{3}{2q} > \frac{3}{4}$ .

We now recall the well known Gronwall's lemma (Hale [3]), as well as one of its generalizations.

**Lemma 2.5** Let  $f(t)$  be a nonnegative absolutely continuous function on  $[0, T]$ , satisfying for almost every  $t$  the following differential inequality

$$f'(t) \leq \phi(t)f(t) + \psi(t),$$

where  $\phi(t)$  e  $\psi(t)$  are nonnegative integrable functions on  $[0, T]$ . Then,

$$f(t) \leq \exp\left(\int_0^t \phi(s)ds\right)[f(0) + \int_0^t \psi(s)ds] \text{ for } 0 \leq t \leq T.$$

The following generalization of the above lemma can be found in Heywood [4], p. 656:

**Lemma 2.6** Let  $\Phi(t)$ ,  $\Psi(t)$ ,  $f(t)$  and  $h(t)$  be given nonnegative functions on  $[0, T]$ . Let  $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a locally Lipschitz continuous function and  $\Phi(0) = \Phi_0 \geq 0$ . Suppose that for almost every  $t$  it holds

$$\begin{aligned} \Phi'(t) + \Psi(t) &\leq g(\Phi(t)) + f(t), \\ \Phi(0) &= \Phi_0 \geq 0. \end{aligned}$$

Then,  $\Phi(t) \leq F(t, \Phi_0)$ , for  $t \in [0, T(\Phi_0)]$ , where  $F(\cdot, \Phi_0)$  is the solution of the initial value problem

$$\begin{aligned} F'(t) &= g(F(t)) + f(t), \\ F(0) &= \Phi_0, \end{aligned}$$

and  $[0, T(\Phi_0)]$  is the maximal interval in which  $F$  can be defined.

When  $g$  is a nondecreasing function, we also have

$$\int_0^t \psi(s)ds \leq \tilde{F}(t, \Phi_0),$$

where  $\tilde{F}(t, \Phi_0) = \Phi_0 + \int_0^t [g(F(s, \Phi_0)) + f(s)]ds$ .

Finally as it is usual in works involving estimates for partial differential equations, in this paper  $C$  will denote a positive generic constant depending only on the data of the problem. Only when it will be necessary to distinguish certain constants to clarify the argument, we will use other symbols to denote constants.

### 3 Weak Solutions

With the objective of obtaining a suitable (“conservative”) weak form of (1.1), we proceed as follows:

First we introduce the functional space

$$\Phi = \left\{ \phi : \phi \in L^2(0, T; V), \frac{\partial \phi}{\partial t} \in L^\infty(0, T; V'), \phi(T) = 0, \frac{\partial \phi_i}{\partial x_j} \in L^\infty(0, T; L^{3/2}(\Omega)) \right\}. \quad (3.1)$$

Now, formally we divide the first equation in (1.1) by  $n$  and observe that

$$\frac{u'}{n} = \frac{d}{dt} \left( \frac{u}{n} \right) + \frac{un'}{n^2}.$$

Then, we take the inner product in  $L^2(Q_T)$  of the resulting equation with a function  $v \in \Phi$ . After some integrations by parts, we have the following weak formulation for (1.1): given  $u_0 \in H$  and  $g \in L^2(0, T; V')$ , find  $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$  such that, for any  $\phi \in \Phi$ , it holds

$$\left\{ \begin{aligned} & - \int_0^T \left( \frac{u}{n}, \frac{\partial \phi}{\partial t} \right) dt + \int_0^T \left( \frac{n'}{n^2} u, \phi \right) dt \\ & - \sum_{i,j=1}^d \int_{Q_T} \frac{u_j}{n} \frac{u_i}{n} \frac{\partial \phi_i}{\partial x_j} dx dt + \sum_{i,j=1}^d \int_{Q_T} \frac{\partial}{\partial x_j} \left( \frac{1}{n} \right) u_j \frac{u_i}{n} \phi_i dx dt \\ & + \mu \int_0^T (\nabla u, \nabla \left( \frac{1}{n} \right) \phi) dt + \mu \int_0^T (\nabla u, \frac{1}{n} \nabla \phi) dt \\ & + \mu \int_0^T \left( \frac{F(n)}{n} u, \phi \right) dt = \int_0^T \langle g, v \rangle dt + \left( \frac{u_0}{n(\cdot, 0)}, \phi(0) \right). \end{aligned} \right. \quad (3.2)$$

We remark that we will assume conditions on  $n$  that will guarantee that  $n(\cdot, 0)$  makes sense. The condition  $\frac{\partial \phi_i}{\partial x_j} \in L^\infty(0, T; L^{3/2}(\Omega))$  in the definition of the functional space  $\Phi$  (see (3.1)) is introduced in order that the third term in (3.2) make sense (this could be relaxed a little.)

A solution of the above problem will be called a *weak solution* of (1.1).

Concerning (3.2), we have the following result:

**Theorem 3.1** Let  $\Omega \subset \mathbb{R}^d$ , with  $d = 2, 3$  be an open bounded set, with Lipschitz boundary and  $0 < T \leq \infty$ . Also, let be given  $u_0 \in H$ ,  $g \in L^2(0, T; V')$  and a continuous function  $F : (0, 1] \rightarrow \mathbb{R}^+$ . Suppose that the porosity  $n : Q_T = \Omega \times (0, T) \rightarrow (0, 1]$  satisfies

$$\begin{aligned} n &\in L^\infty(Q_T), \text{ with } 0 < n_0 \leq n(x, t) \leq 1, \quad \forall (x, t) \in Q_T \\ n' &\in L^2(0, T; L^{\frac{3}{2}}(\Omega)) \cap L^1(0, T; L^\infty(\Omega)) \\ \nabla n &\in L^2(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; L^3(\Omega)) \end{aligned}$$

Then, there exists a solution  $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$  of (3.2).

**Remark 3.1** The hydrostatic pressure can be recovered by proceeding exactly as in Temam [10]; we find  $p \in L^2(0, T; L^2(\Omega))$  such that  $(u, p)$  satisfy, in the sense of  $L^2(0, T; H^{-1}(\Omega))$ , the following equations

$$\begin{cases} u_t + u \cdot \nabla \left( \frac{u}{n} \right) - \mu \Delta u + n \nabla p + \mu F(n) u = n g, \\ \operatorname{div} u = 0. \end{cases}$$

The corresponding initial condition makes sense in a form that is a little weaker than that expressed in (1.1): first we observe that, with the help of (2.2) it is easy to show that

$$\frac{d}{dt} \left( \frac{u}{n} \right) \in L^1(0, T; V').$$

Thus, we have  $\frac{u}{n} \in C(0, T; V')$ , and so we can talk about  $\frac{u}{n}(0)$ . Then, in a standard way it is possible to prove that

$$\frac{u}{n}(0) = \frac{u_0}{n(\cdot, 0)}.$$

With a little more of regularity, for either  $u$  or  $n$ , which is the case of the strong solutions to be constructed in the next section, for instance, this last equality would imply that  $u(0) = u_0$ .

**Remark 3.2** Obviously, variants of the above conditions on the porosity are possible.

For instance, the condition  $n' \in L^2(0, T; L^{3/2}(\Omega))$  is enough to guarantee that the second term in (3.2) makes sense in spatial dimension  $d = 3$  (and also when  $d = 20$ . However, when  $d = 2$ , this condition could be relaxed.)

The result of Theorem 3.1 is also true, with slight modifications in the proof, when the conditions on  $n$  are instead

$$\begin{aligned} n &\in L^\infty(Q_T), \text{ with } 0 < n_0 \leq n(x, t) \leq 1, \quad \forall (x, t) \in Q_T \\ n' &\in L^\infty(0, T; L^{\frac{3}{2}}(\Omega)), \\ \nabla n &\in L^\infty(0, T; L^3(\Omega)), \end{aligned}$$

with the last two with small enough norms.

In the special case of dimension two ( $d = 2$ ), the conditions on  $n$  can also be replaced by

$$\begin{aligned} n' &\in L^1(0, T; L^\infty(\Omega)) \cap L^2(0, T; L^p(\Omega)) \quad \text{for some } p > 1, \\ \nabla n &\in L^2(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; L^q(\Omega)) \quad \text{for some } q > 2; \end{aligned}$$

or even

$$\begin{aligned} n' &\in L^\infty(0, T, L^p(\Omega)) \quad \text{for some } p > 1, \\ \nabla n &\in L^\infty(0, T, L^q(\Omega)), \quad \text{for some } q > 2, \end{aligned}$$

both with small enough norms.

We also remark that the above smallness conditions depend only on  $n_0$  and  $\mu$ .

**Proof of Theorem 3.1:** By a standard argument, it is enough to prove the result when  $0 < T < \infty$ . Thus, in the rest of the proof, we will assume this condition.

We will use the Faedo-Galerkin method. For this, let  $(v_i)_{i \in \mathbb{N}}$  be a basis of  $V$  and for each  $m \in \mathbb{N}$ , define the  $m$ -dimensional subspace  $V_m = \text{span}[v_1, v_2, \dots, v_m]$ . Now, restricting (3.2) to  $V_m$ , by the regularity of the elements of  $V_m$ , we can do integration by parts again and then obtain the following approximate problem:

for each  $m \in \mathbb{N}$ , we have to find  $u_m(x, t) = \sum_{i=1}^m c_{i,m}(t)v_i(x)$  such that for all  $v \in V_m$ , it holds

$$\begin{cases} \left( \frac{u'_m}{n}, v \right) + \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), v \right) + \mu (\nabla u_m, \nabla \left( \frac{1}{n} \right) v) \\ + \mu (\nabla u_m, \frac{1}{n} \nabla v) + \mu \left( \frac{F(n)}{n} u_m, v \right) = \langle g, v \rangle, \\ u_m(0) = u_{0m}, \end{cases} \quad (3.3)$$

where  $u_{0m}$  is a given sequence converging to  $u_0$  in  $H$ , and thus there exists a constant  $C$  such that  $|u_{0m}| \leq C|u_0|$ .

As it is usual with the Faedo-Galerkin method, for each  $m \in \mathbb{N}$ , the above corresponde to a initial value problem for a system of ordinary differential equations for the coefficients  $c_{i,m}(t)$ ,  $i = 1, \dots, m$ . Caratheodory's local existence and uniqueness theorem (Hale [3]) then guarantees the existence of  $\tau_m > 0$  and  $u_m : [0, \tau_m) \rightarrow V_m$ , solution of (3.3).

Now, we proceed with the derivation of suitable estimates for  $u_m$ .

By doing the time derivative in the first term and taking  $v = u_m$ , after some



computations, we obtain

$$\begin{aligned}
& \frac{d}{dt} \left| \frac{u_m}{\sqrt{n}} \right|^2 + 2\mu \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + 2\mu \left| \sqrt{\frac{F(n)}{n}} u_m \right|^2 \\
& \leq 2 \langle g, u_m \rangle - \left( \frac{n'}{n^2} u_m, u_m \right) + 2\mu \left( \frac{\nabla n}{n^2} \nabla u_m, u_m \right) \\
& \leq 2\epsilon^{-1} \|g\|_{V'}^2 + \epsilon \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + n_0^{-2} |n'(t)|_\infty \left| \frac{u_m}{\sqrt{n}} \right|^2 \\
& \quad + 2\epsilon^{-1} n_0^{-2} |\nabla n(t)|_\infty^2 \left| \frac{u_m}{\sqrt{n}} \right|^2 + \epsilon \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2
\end{aligned} \tag{3.4}$$

By taking  $\epsilon > 0$  such that  $c_1 = 2\mu - 2\epsilon > 0$ , we conclude that there is a positive constant  $C$  such that

$$\begin{aligned}
& \frac{d}{dt} \left| \frac{u_m}{\sqrt{n}} \right|^2 + c_1 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + 2\mu \left| \sqrt{\frac{F(n)}{n}} u_m \right|^2 \\
& \leq C \|g\|_{V'}^2 + C \left( |n'(t)|_\infty + |\nabla n(t)|_\infty^2 \right) \left| \frac{u_m}{\sqrt{n}} \right|^2
\end{aligned} \tag{3.5}$$

Hence, by using Gronwall's inequality (Lemma 2.5) and remembering that  $0 < n_0 \leq n(x, t)$ , we obtain that

$$\{u_m\}_1^\infty \text{ is uniformly bounded in } L^2(0, T; V) \cap L^\infty(0, T; H). \tag{3.6}$$

Now, a  $t$ -estimate, or at least some fractional  $t$ -derivative estimate, is necessary in order to be able use a compactness argument. For this, we proceed in a similar way as in the case of non-homogeneous fluids (see, for instance, Lions [6]).

We observe that (3.3) can be written as

$$\frac{d}{dt} \left( \frac{u_m}{n}, v \right) = (F_m(t), v), \quad \forall v \in V_m, \tag{3.7}$$

where

$$\begin{aligned}
(F_m(t), v) &= - \left( \frac{n'}{n^2} u_m, v \right) - \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), v \right) - \mu (\nabla u_m, \nabla \left( \frac{1}{n} \right) v) \\
&\quad - \mu (\nabla u_m, \frac{1}{n} \nabla v) - \mu \left( \frac{F(n)}{n} u_m, v \right) + \langle g, v \rangle.
\end{aligned} \tag{3.8}$$

Then, it is easy to verify that

$$|(F_m(t), v)| \leq C[k(t) + |\nabla u_m(t)|^2] |\nabla v|, \tag{3.9}$$

for a suitable  $k(\cdot) \in L^2(0, T)$ .

Now, we proceed similarly as in Lions [6]: being  $\delta$  a small positive number, for  $t \in [0, T - \delta]$ , we integrate (3.7) from  $t$  to  $t + \delta$  and take  $v = u_m(t + \delta) - u_m(t)$  in the result, to obtain

$$\left(\frac{u_m}{n}\right)(t + \delta) - \left(\frac{u_m}{n}\right)(t), u_m(t + \delta) - u_m(t) = \left(\int_t^{t+\delta} F_m(s) ds, u_m(t + \delta) - u_m(t)\right). \quad (3.10)$$

Now, let us define

$$\begin{aligned} X_m(t) &= \left(\frac{1}{n(t + \delta)}[u_m(t + \delta) - u_m(t)], u_m(t + \delta) - u_m(t)\right), \\ Y_m(t) &= \left(\left[\frac{1}{n(t + \delta)} - \frac{1}{n(t)}\right]u_m(t), u_m(t + \delta) - u_m(t)\right). \end{aligned}$$

Then (3.10) can be written as

$$X_m(t) = Y_m(t) + \left(\int_t^{t+\delta} F_m(s) ds, u_m(t + \delta) - u_m(t)\right). \quad (3.11)$$

Now, we observe that

$$\begin{aligned} |Y_m(t)| &= \left| \left( \int_t^{t+\delta} -\frac{n'(s)}{n(s)^2} ds, u_m(t), u_m(t + \delta) - u_m(t) \right) \right| \\ &\leq \frac{1}{n_0^2} \int_t^{t+\delta} |n'(s)|_{3/2} ds |u_m(t)|_6 (|u_m(t + \delta)|_6 + |u_m(t)|_6) \\ &\leq \frac{2C}{n_0^2} \delta^{1/2} \left( \int_t^{t+\delta} |n'(s)|_{3/2}^2 ds \right)^{1/2} (\|u_m(t + \delta)\|_V^2 + \|u_m(t)\|_V^2). \end{aligned}$$

Thus,

$$\int_0^{T-\delta} |Y_m(t)| dt \leq \frac{4C}{n_0^2} \delta^{1/2} \left( \int_0^T |n'(s)|_{3/2}^2 ds \right)^{1/2} \int_0^T \|u_m(t)\|_V^2 dt \leq C\delta^{1/2}.$$

On the other hand, using (3.9) one verifies that

$$\int_0^{T-\delta} \left| \left( \int_0^{T-\delta} F_m(s) ds, u_m(t + \delta) - u_m(t) \right) \right| dt \leq C\delta^{1/2},$$

and so, with the above estimates, (3.11) implies that

$$\int_0^{T-\delta} |X_m(t)| dt \leq C\delta^{1/2}.$$

Since  $n(x, t) \geq n_0 > 0$ , this last inequality implies that for any small  $\delta > 0$  there holds

$$\int_0^{T-\delta} |u_m(t + \delta) - u_m(t)|^2 \leq C\delta^{1/2}. \quad (3.12)$$

Thus, (3.6) and (3.12) imply that we can apply Lemma 2.4 to the sequence  $\{u_m\}_1^\infty$ , to conclude that

$$\begin{aligned} \{u_m\}_1^\infty & \text{ remains in a compact set of } L^p(0, T; L^q(\Omega)) \\ \text{for any } p \in [2, \infty), q \in [2, 6), & \text{ such that } \frac{1}{p} + \frac{3}{2q} > \frac{3}{4}. \end{aligned} \quad (3.13)$$

Therefore, there is a function  $u$  and a subsequence, which we still denote  $\{u_m\}_1^\infty$ , such that

$$\begin{aligned} u_m & \rightarrow u \text{ weakly-}^* \text{ in } L^\infty(0, T; H), \\ u_m & \rightarrow u \text{ weakly in } L^2(0, T; V), \\ u_m & \rightarrow u \text{ strongly in } L^p(0, T; L^q(\Omega)), \text{ with } p \text{ and } q \text{ as in (3.13)}. \end{aligned} \quad (3.14)$$

Now, we take the inner product in  $L^2(Q_T)$  of (3.3) with  $\phi(x, t) = \psi(t)v(x)$ , where  $\psi \in C^1([0, T])$  is such that  $\psi(T) = 0$  and  $v \in V_m$ . By doing the same integrations by parts as those done in the formal derivation of (3.2), we are left with

$$\begin{aligned} & \left\{ \begin{aligned} & - \int_0^T \left( \frac{u_m}{n}, \frac{\partial \phi}{\partial t} \right) dt + \int_0^T \left( \frac{n'}{n^2} u_m, \phi \right) dt \\ & - \sum_{i,j=1}^d \int_{Q_T} \frac{(u_m)_j}{n} \frac{(u_m)_i}{n} \frac{\partial \phi_i}{\partial x_j} dx dt + \sum_{i,j=1}^d \int_{Q_T} \frac{\partial}{\partial x_j} \left( \frac{1}{n} \right) (u_m)_j \frac{(u_m)_i}{n} \phi_i dx dt \\ & + \mu \int_0^T (\nabla u_m, \nabla \left( \frac{1}{n} \right) \phi) dt + \mu \int_0^T (\nabla u_m, \frac{1}{n} \nabla \phi) dt \\ & + \mu \int_0^T \left( \frac{F(n)}{n} u_m, \phi \right) dt = \int_0^T \langle g, v \rangle dt + \left( \frac{u_{m0}}{n(\cdot, 0)}, \phi(0) \right) dt. \end{aligned} \right. \end{aligned} \quad (3.15)$$

Now, as in Lions [6] or [7], the convergences in (3.14) are enough to pass the limit as  $m \rightarrow +\infty$  in (3.15) and obtain that (3.2) is true for any  $\phi$  that is a finite linear combinations of functions of the above form. Since these functions form a dense set in  $\Upsilon$  (see the definition of  $\Upsilon$  in (3.1)), we conclude  $u$  satisfies (3.2) with any  $\phi \in \Upsilon$ . 2

## 4 Strong Solutions

In this section we will consider the existence of solutions of (1.1) that are more regular than those considered in the previous section. Such solutions will correspond to data that are also more regular than the ones in Theorem 3.1 (in particular, we will need more regular initial conditions), and they will be called *strong solutions* of (1.1).

Our first result is a local existence theorem for such solutions:

**Theorem 4.1** *Let  $\Omega \subset \mathbb{R}^d$ , with  $d = 2$  or  $3$ , be an open and bounded set with boundary of class  $C^3$ . Let  $u_0 \in V$ ,  $g \in L^\infty(0, T, L^2(\Omega))$  and  $F : (0, 1] \rightarrow \mathbb{R}$  a*

continuous function. Suppose also that the porosity  $n : Q_T = \Omega \times (0, T) \rightarrow (0, 1]$  is such that  $n \in L^\infty(Q_T)$ ,  $\nabla n \in L^\infty(0, T, L^6(\Omega)) \subset L^\infty(0, T, L^3(\Omega))$  and  $n' \in L^\infty(Q_T)$ . If there is  $0 < \bar{n} < 1$  such that  $\|n - \bar{n}\|_{L^\infty(Q_T)}$ ,  $\|\nabla n\|_{L^\infty(0, T, L^3(\Omega))}$  and  $\|n'\|_{L^\infty(Q_T)}$  are small enough, then there is  $\bar{T} = \bar{T}(T, \Omega, \|u_0\|_V) \in (0, T]$  such that the solution  $u$  of (3.2) satisfies  $u \in L^\infty(0, \bar{T}, V) \cap L^2(0, \bar{T}, (H^2(\Omega)))^d$ .

**Proof:**

We start by observing that if  $\|n(x, t) - \bar{n}\|_{L^\infty(Q_T)}$  is small enough, there is  $n_0 > 0$  such that

$$0 < n_0 \leq n(x, t) \leq 1 \quad (4.1)$$

almost everywhere in  $Q_T$ . In what follows, we will assume that this holds.

Then, the above hypotheses imply those of Theorem 3.1, we already know that there is a solution in  $L^2(0, T, V) \cap L^\infty(0, T, H)$ . We have to prove that there is  $0 < \bar{T} \leq T$  such that actually  $u \in L^\infty(0, \bar{T}, V) \cap L^2(0, \bar{T}, H^2(\Omega))$ . For this, it is enough to obtain estimates in  $L^\infty(0, \bar{T}, V)$  and  $L^2(0, \bar{T}, H^2(\Omega))$  holding uniformly for approximate solutions  $u_m$  built as in the previous section.

In order to this, we will consider approximate solutions built using the spectral basis,  $\{e_i\}_1^\infty$  associated to the Stokes operator  $A$ . That is, we will work with the basis formed by the eigenfunctions of  $A$  (see their properties in Section 1.) As previously, we define

$$\begin{aligned} V_m &= \text{span} [e_1, \dots, e_m] \\ u_m(x, t) &= \sum_{i=1}^m c_{i,m}(t) e_i(x) \end{aligned}$$

Now, the weak formulation can then be written as:

$$\begin{cases} (\frac{u'_m}{n}, w) + (\frac{u_m}{n} \cdot \nabla (\frac{u_m}{n}), w) - \mu(\Delta u_m, \frac{w}{n}) \\ + \mu(\frac{F(n)}{n} u_m, w) = (g, w), \quad \forall w \in V_m, \\ u_m(0) = u_{0m} \in V, \end{cases} \quad (4.2)$$

where now  $u_{0m} \rightarrow u_0$  in  $V$ , and thus for any  $m \in \mathbb{N}$  we have  $\|u_{0m}\|_V \leq C\|u_0\|_V$  for a fixed positive constant  $C$ .

As in the previous section, we have that these approximate solution is defined on  $[0, T]$  and that (3.6) hold for them.

Now, we observe that by our choice of basis, for  $v \in V_m$ , we have that  $Av \in V_m$ . Thus, we can take  $w = Au_m$  in equation (4.2), to obtain, after an integration by parts in the third term,

$$\begin{cases} (\frac{u'_m}{n}, Au_m) + (\frac{u_m}{n} \cdot \nabla (\frac{u_m}{n}), Au_m) - \mu(\Delta u_m, \frac{Au_m}{n}) \\ + \mu(\frac{F(n)}{n} u_m, Au_m) = (g, Au_m) \end{cases} \quad (4.3)$$

To estimate the terms in the above equation, we recall that (2.3) implies that for  $t \in [0, T]$  there exists  $\psi_m(t) \in H^1(\Omega)$  such that

$$Au_m(t) = -\Delta u_m(t) - \nabla \psi_m(t) \quad (4.4)$$

and

$$|\nabla \psi_m(t)| \leq C|Au_m(t)|, \quad (4.5)$$

with a constant  $C > 0$  independent of  $t \in [0, T]$  and  $m \in \mathbb{N}$ .

With the help of (4.4), we can rewrite the first term in (4.3) as

$$\begin{aligned} \left(\frac{u'_m}{n}, Au_m\right) &= -\left(\frac{u'_m}{n}, \Delta u_m\right) - \left(\frac{u'_m}{n}, \nabla \psi_m\right) \\ &= \left(\nabla \left(\frac{u'_m}{n}\right), \nabla u_m\right) - \left(\frac{u'_m}{n}, \nabla \psi_m\right) \\ &= \left(\frac{\nabla u'_m}{n}, \nabla u_m\right) - \left(\frac{\nabla n}{n^2} u'_m, \nabla u_m\right) - \left(\frac{u'_m}{n}, \nabla \psi_m\right). \end{aligned} \quad (4.6)$$

Now, we observe that

$$\begin{aligned} \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2(t) &= \frac{d}{dt} \int_{\Omega} \frac{|\nabla u_m(x, t)|^2}{n(x, t)} dx \\ &= 2 \left( \frac{\nabla u'_m(t)}{n(t)}, \nabla u_m(t) \right) - \left( \nabla u_m(t), \frac{n'}{n^2} \nabla u_m(t) \right). \end{aligned} \quad (4.7)$$

Using this observation in (4.6), we obtain

$$\begin{aligned} \left(\frac{u'_m}{n}, Au_m\right) &= \frac{1}{2} \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + \frac{1}{2} \left( \frac{n'}{n^2} \nabla u_m, \nabla u_m \right) \\ &\quad - \left( \frac{\nabla n}{n^2} u'_m, \nabla u_m \right) - \left( \frac{u'_m}{n}, \nabla \psi_m \right). \end{aligned}$$

Again using (4.4), the third term in (4.3) can be rewritten as

$$-(\Delta u_m, \frac{Au_m}{n}) = (Au_m + \nabla \psi_m, \frac{Au_m}{n}) = \left| \frac{Au_m}{\sqrt{n}} \right|^2 + \left( \frac{\nabla \psi_m}{n}, Au_m \right)$$

Substituting the last two identities in (4.3) and collecting the terms, leave us with

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + \mu \left| \frac{Au_m}{\sqrt{n}} \right|^2 &= -\frac{1}{2} \left( \frac{n'}{n^2} \nabla u_m, \nabla u_m \right) + \left( \frac{\nabla n}{n^2} u'_m, \nabla u_m \right) \\ &\quad + \left( \frac{u'_m}{n}, \nabla \psi_m \right) - \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), Au_m \right) - \mu \left( \frac{\nabla \psi_m}{n}, Au_m \right) \\ &\quad - \mu \left( \frac{F(n)}{n} u_m, Au_m \right) + (g, Au_m) \end{aligned} \quad (4.8)$$

Now, we can take  $w = u'_m$  in (4.2) to obtain

$$\begin{cases} \left( \frac{u'_m}{n}, u'_m \right) + \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), u'_m \right) + \mu \left( \nabla u_m, \frac{\nabla u'_m}{n} \right) \\ + \mu \left( \nabla u_m, \nabla \left( \frac{1}{n} \right) u'_m \right) + \mu \left( \frac{F(n)}{n} u_m, u'_m \right) = (g, u'_m) \end{cases}$$

By using (4.7) in the third term of the above equation, we obtain

$$\begin{cases} \left| \frac{u'_m}{\sqrt{n}} \right|^2 + \frac{\mu}{2} \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 = -\frac{\mu}{2} \left( \frac{n'}{n^2} \nabla u_m, \nabla u_m \right) - \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), u'_m \right) \\ - \mu \left( \nabla u_m, \nabla \left( \frac{1}{n} \right) u'_m \right) - \mu \left( \frac{F(n)}{n} u_m, u'_m \right) + (g, u'_m) \end{cases} \quad (4.9)$$

By adding (4.8) and (4.9), we are left with

$$\begin{cases} \left( \frac{1+\mu}{2} \right) \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + \mu \left| \frac{Au_m}{\sqrt{n}} \right|^2 + \left| \frac{u'_m}{\sqrt{n}} \right|^2 = -\frac{1+\mu}{2} \left( \frac{n'}{n^2} \nabla u_m, \nabla u_m \right) \\ + (1+\mu) \left( \frac{\nabla n}{n^2} u'_m, \nabla u_m \right) + \left( \frac{u'_m}{n}, \nabla \psi_m \right) - \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), Au_m \right) \\ - \mu \left( \frac{\nabla \psi_m}{n}, Au_m \right) - \mu \left( \frac{F(n)}{n} u_m, Au_m \right) + (g, Au_m) \\ - \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), u'_m \right) - \mu \left( \frac{F(n)}{n} u_m, u'_m \right) + (g, u'_m). \end{cases} \quad (4.10)$$

Now, we have to estimate the absolute value every one of the terms appearing in the right-hand side of equation (4.10). In the following, we will use the letters  $a$  and  $b$  to denote small positive constants that will be chosen later on.

The first term is estimated as follows:

$$\left| \left( \frac{n'}{n^2} \nabla u_m, \nabla u_m \right) \right| \leq C |n'|_\infty \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2.$$

The second term in right-hand side of (4.10) is estimated as:

$$\begin{aligned} \left| \left( \frac{\nabla n}{n^2} \nabla u_m, u'_m \right) \right| &\leq \frac{C}{a} |\nabla n|_3^2 |\nabla u_m|_6^2 + a |u'_m|^2 \\ &\leq \frac{C}{a} |\nabla n|_3^2 \left| \frac{Au_m}{\sqrt{n}} \right|^2 + a \left| \frac{u'_m}{\sqrt{n}} \right|^2. \end{aligned}$$

Before we estimate the third term in the right-hand side of (4.10), we observe that if we define the function  $r(x, t)$  by

$$\frac{1}{n(x, t)} = \frac{1}{\bar{n}} + r(x, t), \quad (4.11)$$

then it satisfies

$$\|r\|_{L^\infty(Q_T)} \leq C \|n - \bar{n}\|_{L^\infty(Q_T)} \quad (4.12)$$

with the constant  $C = 1/(n_0 \bar{n})$ .

Then, by using the above, (4.5) and (2.2), the third term in the right-hand side of (4.10) is estimated as follows:

$$\begin{aligned} \left| \left( \frac{u'_m}{n}, \nabla \psi_m \right) \right| &= \left| \frac{1}{\bar{n}} (u'_m, \nabla \psi_m) + (r u'_m, \nabla \psi_m) \right| \\ &\leq 0 + \|r\|_{L^\infty(Q_T)} c \|u'_m\| |\nabla \psi_m| \\ &\leq \frac{C}{a} \|n - \bar{n}\|_{L^\infty(Q_T)}^2 \left| \frac{u'_m}{\sqrt{n}} \right|^2 + a \left| \frac{A u_m}{\sqrt{n}} \right|^2. \end{aligned}$$

The fourth term in the right-hand side of (4.10) is estimated as follows:

$$\left| \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), A u_m \right) \right| \leq \left| \left( \frac{u_m}{n} \cdot \frac{\nabla u_m}{n}, A u_m \right) \right| + \left| \left( \frac{u_m}{n} \cdot \nabla \left( \frac{1}{n} \right) u_m, A u_m \right) \right|.$$

The first of the above terms is estimated by using (2.1):

$$\begin{aligned} \left| \left( \frac{u_m}{n} \cdot \frac{\nabla u_m}{n}, A u_m \right) \right| &\leq |u_m|_4 |\nabla u_m|_4 |A u_m| \\ &\leq |u_m|^{\frac{1}{4}} |\nabla u_m|^{\frac{3}{4}} |\nabla u_m|^{\frac{1}{4}} \|u_m\|_{H^2(\Omega)}^{\frac{3}{4}} |A u_m| \\ &\leq C |\nabla u_m|^{\frac{5}{4}} |A u_m|^{\frac{7}{4}} \leq \frac{C}{a} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^{10} + a \left| \frac{A u_m}{\sqrt{n}} \right|^2. \end{aligned} \quad (4.13)$$

The second of the two terms is estimated as

$$\begin{aligned} \left| \left( \frac{u_m}{n} \cdot \nabla \left( \frac{1}{n} \right) u_m, A u_m \right) \right| &\leq C |\nabla n|_6 |u_m|_6^2 |A u_m| \\ &\leq \frac{C}{a} |\nabla n|_6^2 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^4 + a \left| \frac{A u_m}{\sqrt{n}} \right|^2. \end{aligned} \quad (4.14)$$

The fifth term in the right-hand side of (4.10) is estimated again with the help of (4.11), (4.12) and (2.2):

$$\begin{aligned} \left| \left( \frac{\nabla \psi_m}{n}, A u_m \right) \right| &= \left| \frac{1}{\bar{n}} (\nabla \psi_m, A u_m) + (r \nabla \psi_m, A u_m) \right| \\ &\leq 0 + C \|n - \bar{n}\|_{L^\infty(Q_T)} \left| \frac{A u_m}{\sqrt{n}} \right|^2. \end{aligned}$$

The sixth term in the right-hand side of (4.10) can be estimated as

$$\left| \left( \frac{F(n)}{n} u_m, A u_m \right) \right| \leq \frac{C}{a} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + a \left| \frac{A u_m}{\sqrt{n}} \right|^2.$$

The seventh and the last term in the right-hand side of (4.10) are estimated in similar way:

$$|(g, A u_m)| \leq \frac{C}{a} |g|^2 + a \left| \frac{A u_m}{\sqrt{n}} \right|^2,$$

$$|(g, u'_m)| \leq \frac{C}{a} |g|^2 + a \left| \frac{u'_m}{\sqrt{n}} \right|^2.$$

The eighth term in the right-hand side of (4.10) is estimated as the fourth:

$$\left| \left( \frac{u_m}{n} \cdot \nabla \left( \frac{u_m}{n} \right), u'_m \right) \right| = \left| \left( \frac{u_m}{n} \cdot \frac{\nabla u_m}{n}, u'_m \right) + \left( \frac{u_m}{n} \cdot \nabla \left( \frac{1}{n} \right) u_m, u'_m \right) \right|.$$

The first of these two terms, for any positive  $a$  and  $b$ , can be estimated by

$$\begin{aligned} \left| \left( \frac{u_m}{n} \cdot \frac{\nabla u_m}{n}, u'_m \right) \right| &\leq |u_m|_4 |\nabla u_m|_4 |u'_m| \leq |u_m|^{\frac{1}{4}} |\nabla u_m|^{\frac{3}{4}} |\nabla u_m|^{\frac{1}{4}} \|u_m\|_{H^2(\Omega)}^{\frac{3}{4}} |u'_m| \\ &\leq C |\nabla u_m|^{\frac{5}{4}} |Au_m|^{\frac{3}{4}} |u'_m| \leq \frac{C}{a} |\nabla u_m|^{\frac{5}{2}} |Au_m|^{\frac{3}{2}} + a |u'_m|^2 \\ &\leq \frac{C}{ab} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^{10} + b \left| \frac{Au_m}{\sqrt{n}} \right|^2 + a \left| \frac{u'_m}{\sqrt{n}} \right|^2. \end{aligned}$$

The second of the above terms can be estimated similarly as in (4.14):

$$\begin{aligned} \left| \left( \frac{u_m}{n} \cdot \nabla \left( \frac{1}{n} \right) u_m, u'_m \right) \right| &\leq C |\nabla n|_6 |u_m|_6^2 |u'_m| \\ &\leq C |\nabla n|_6 |\nabla u_m|^2 |u'_m| \leq \frac{C}{a} |\nabla n|_6^2 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^4 + a \left| \frac{u'_m}{\sqrt{n}} \right|^2. \end{aligned}$$

Finally, the ninth term in the right-hand side of (4.10) is estimated as

$$\left| \left( \frac{F(n)}{n} u_m, u'_m \right) \right| \leq \frac{C}{a} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + a \left| \frac{u'_m}{\sqrt{n}} \right|^2. \quad (4.15)$$

Substituting all the previous estimates in (4.10) and collecting the terms that are similar, we obtain

$$\begin{aligned} &\left\{ \frac{1+\mu}{2} \frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + (\mu - 5a - b - C \|n - \bar{n}\|_{L^\infty(Q_T)} - 2 \frac{C}{a} \|\nabla n\|_3^2) \left| \frac{Au_m}{\sqrt{n}} \right|^2 \right. \\ &\quad \left. + (1 - 5a - \frac{C}{a} \|n - \bar{n}\|_{L^\infty(Q_T)}^2) \left| \frac{u'_m}{\sqrt{n}} \right|^2 \leq \bar{C} (1 + \|n'\|_{L^\infty(\Omega)}^2) \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 \right. \\ &\quad \left. + \bar{C} |\nabla n|_6^2 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^4 + \bar{C} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^{10} + \bar{C} |g|^2. \right. \end{aligned} \quad (4.16)$$

Here, the constant  $\bar{C} = \bar{C}(\Omega, \mu, n_0, a, b)$ , while the constant  $C = C(\Omega, \mu, n_0)$ , but it does not depend on  $a$  and  $b$ . After choosing  $a$  and  $b$  such that  $\mu - 5a - b > 0$  e  $1 - 5a > 0$ , we can take  $|\varepsilon|_{L^\infty(Q_T)}$  and  $|\nabla n|_{L^\infty(0,T;L^3(\Omega))}$  small enough such that the constants in the second and third terms in the inequality above are positive. Thus, we can use Lemma 2.6 to conclude that there is  $\bar{T} \in (0, T]$  and continuous



functions  $G(\cdot)$  and  $\tilde{G}(\cdot)$  defined on  $[0, \bar{T}]$  such that for all  $t \in [0, \bar{T}]$

$$\begin{cases} \left| \nabla u_m(t) / \sqrt{n(t)} \right|^2 \leq G(t), \\ \int_0^t \left| u'_m(t) / \sqrt{n(t)} \right|^2 dt \leq \tilde{G}(t), \\ \int_0^t \left| Au_m(t) / \sqrt{n(t)} \right|^2 dt \leq \tilde{G}(t). \end{cases}$$

The fact that  $n_0 \leq n(x, t)$  and the ellipticity of the Stokes operator  $A$ , together with the above estimates, furnish the required uniform in  $m \in \mathbb{N}$  estimates for  $u_m$ , and the theorem is proved. 2

Now, we are going to use to show conditions guaranteeing the existence of global in time strong solutions:

**Theorem 4.2** *Suppose that when the spatial dimension is  $d = 2$ , we have the conditions stated in Theorem 4.1. Then, the strong solution obtained there is global (that is, we can take  $\bar{T} = T$ ).*

*Suppose that when the spatial dimension is  $d = 3$ , we have the conditions stated in Theorem 4.1 and, moreover, that the porosity  $n$  satisfies  $n' \in L^\infty(Q_T)$  and  $\nabla n \in L^\infty(Q_T)$ , with small enough norms. Then, the strong solution of Theorem 4.1 is global (that is, we can take  $\bar{T} = T$ ).*

**Remark 4.1** In this theorem  $T$  could be  $+\infty$

**Proof:** By the above hypotheses, the existence of a solution in  $L^2(0, T, V) \cap L^\infty(0, T, H)$  is guaranteed by Theorem 3.1 in any of the above cases.

In both cases, it is enough to prove the corresponding estimates for the Faedo-Galerkin approximations  $u_m$ .

Let us consider first the simpler case in which the spatial dimension is  $d = 2$ .

In this situation, we must reconsider the derivation of (4.16). In fact, the only term that must be reestimated is (4.13). This is done as follows:

$$\begin{aligned} \left| \left( \frac{u_m}{n} \cdot \frac{\nabla u_m}{n}, Au_m \right) \right| &\leq |u_m|_4 |\nabla u_m|_4 |Au_m| \\ &\leq |u_m|^{\frac{1}{2}} |\nabla u_m|^{\frac{1}{2}} |\nabla u_m|^{\frac{1}{2}} \|u_m\|_{H^2(\Omega)}^{\frac{1}{2}} |Au_m| \leq \frac{C}{a} |u_m| |\nabla u_m|^4 + a \left| \frac{Au_m}{\sqrt{n}} \right|^2. \end{aligned}$$

Now, we observe that, since  $u_m \in L^2(0, T, V) \cap L^\infty(0, T, H)$ , the function

$$\sigma(t) = \int_0^t |u_m(s)| |\nabla u_m(s)|^2 ds$$

is well-defined.

With this remark and the above estimate, working as before, we end up with a differential inequality similar to (4.16), but now in a situation in which we can work as in the two dimensional case for the classical Navier-Stokes equations. Then, we can apply directly Gronwall's Lemma in this new inequality, and we obtain the required estimates without imposing further restrictions on the data.

This finishes the proof in the case  $d = 2$ .

Now, let us consider the case  $d = 3$ .

We will show that  $\sup_{0 \leq t \leq T} |\nabla u_m(\cdot, t)|_2 \leq C$  under the supplementary conditions that  $\|u_0\|_V$  and  $\|g\|_{L^\infty(0, T; L^2(\Omega))}$  are small enough.

For this, we first observe that by using the fact that  $\left| \frac{u_m}{\sqrt{n}} \right| \leq C \left| \frac{\nabla u_m}{\sqrt{n}} \right|$ , when  $\|n'\|_{L^\infty(Q_T)} + \|\nabla n\|_{L^\infty(Q_T)}^2$  is small enough, (3.5) implies that

$$\frac{d}{dt} \left| \frac{u_m}{\sqrt{n}} \right|^2 + C \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 \leq C|g(t)|^2.$$

This implies that there is  $C > 0$ , independent of  $m \in \mathbb{N}$ , such that

$$\left\| \frac{u_m}{\sqrt{n}} \right\|_{L^\infty(0, T; L^2(\Omega))}^2 \leq C(|u_0|^2 + \|g\|_{L^\infty(0, T; L^2(\Omega))}^2). \quad (4.17)$$

Now, we observe that, in the derivation of (4.16), if we had estimated the term  $(\frac{F(n)}{n} u_m, u'_m)$  in (4.10) as

$$(\frac{F(n)}{n} u_m, u'_m) \leq Ca^{-1}|u_m|^2 + a|u'_m|^2,$$

with any  $a > 0$ , after choosing suitable small  $a$  and  $b$  as before, (4.16) would become the following inequality:

$$\frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 + C \left| \frac{Au_m}{\sqrt{n}} \right|^2 \leq C \left| \frac{u_m}{\sqrt{n}} \right|^2 + C_2 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^4 + C_3 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^{10} + C|g(t)|^2.$$

Now, we use  $|\nabla u_m| \leq C|Au_m|$  in the second term in the left hand-side of this last inequality, as well as the estimate (4.17), to obtain

$$\frac{d}{dt} \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2 \leq C_1 S(u_0, g) + C_2 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^4 + C_3 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^{10} - C_4 \left| \frac{\nabla u_m}{\sqrt{n}} \right|^2,$$

where  $S(u_0, g) = C(|u_0|^2 + \|g\|_{L^\infty(0, T; L^2(\Omega))}^2)$ .

By calling  $\phi(t) = \left| \frac{\nabla u_m(\cdot, t)}{\sqrt{n(\cdot, t)}} \right|_2^2$ , the last inequality is written as:

$$\phi' \leq C_1 S(u_0, g) + C_2 \phi^2 + C_3 \phi^5 - C_4 \phi,$$

with fixed positive constants  $C_i$ , for  $i = 1, \dots, 4$ .

Now, we choose  $N = N(C_2, C_3, C_4)$  large enough in order that:

$$C_2 N^3 + C_3 < \frac{C_4}{2} N^4,$$

and, once such  $N$  is fixed, take  $u_0$  and  $g$  such that  $S(u_0, g) \leq \frac{1}{N} \frac{C_4}{2C_1}$ . Under the these conditions, we have

$$\phi' \leq \frac{C_4}{2N} + C_2 \phi^2 + C_3 \phi^5 - C_4 \phi.$$

Now, the continuous function  $H(\cdot)$  defined by  $H(z) = \frac{C_4}{2N} + C_2 z^2 + C_3 z^5 - C_4 z$  satisfies  $H(0) > 0$  and

$$H\left(\frac{1}{N}\right) = \frac{C_4}{2N} + \frac{C_2}{N^2} + \frac{C_3}{N^5} - \frac{C_4}{N} = \frac{C_2 N^3 + C_3 - (C_4/2)N^4}{N^5} < 0$$

Thus, there exists a root of  $H$ ,  $R$ , such that  $0 < R < \frac{1}{N}$ .

Consider now the following scalar initial value problem:

$$\begin{aligned} z' &= H(z) = \frac{C_4}{2N} + C_2 z^2 + C_3 z^5 - C_4 z, \\ z(0) &= \frac{|\nabla u_0|^2}{n_0}. \end{aligned}$$

Since  $F(z) > 0$  for  $z \in [0, R)$  and  $H(R) = 0$ , for any initial condition  $z(0) \in (0, R)$  we have  $z(\cdot)$  globally defined and  $z(t) < R$ , for all  $t \in [0, \infty)$ .

Now, because  $\phi(0) \leq z(0)$ , we can apply results on differential inequalities (see for instance Hale [3]) for  $u_0$  and  $g$  satisfying at same time

$$\begin{cases} |u_0|^2 + \|g\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \frac{1}{N} \frac{C_4}{2C_1}, \\ |\nabla u_0|^2 < R n_0. \end{cases}$$

We conclude that  $\phi(\cdot)$  is globally defined and that  $0 \leq \phi(t) \leq z(t) \leq R$  for all  $t \in [0, \infty)$ .

With the above information, returning to (4), we obtain all the required uniform in  $m \in \mathbb{N}$  estimates. Thus, the theorem is proved. 2

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