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Group rings of finite groups over p-adic integers, some examples

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Abstract: A method to describe certain group rings of finite groups over p-adic integers is applied to calculate the group rings $\mathbb{Z}_2 J_1$ and $\mathbb{Z}_3 S_9$ up to Morita equivalence. The radical series are calculated.

Key words: p-adic group rings of fintie groups, radical series.

1 Introduction.

The p-adic integral group ring RG of a finite group G is the connecting link between the p-modular and the ordinary representation theory of G. Already R. Brauer (see e.g. [Bra56]) aimed to study the number theory of RG by investigating the factor algebra $RG/\pi RG$, where π is a prime element in R. If p divides the order of G, the group ring RG is not a maximal order, not even hereditary, but a much more complicated object. Therefore the first step to understand group rings is the calculation of examples.

Here it is always assumed that the p-modular and ordinary representation theory of G is known. In particular the decomposition matrix of RG is used. In his habilitation thesis ([Ple80], [Ple83]) Plesken develops methods to describe certain integral group rings. One gets a complete description if the p-decomposition numbers of G are ≤ 1. Plesken's methods aime to calculate a certain graduated overorder Γ (see Definition 2.1) of RG, canonically attached to RG. Γ has the same irreducible lattices as RG, which can be described combinatorially. As an application he describes among other single examples the blocks of cyclic defect ([Ple83, Chapter VIII], see also [Rog80], [Rog92] and [Lin96]) and the 2-adic group rings of $SL_2(p^f)$ for odd primes p. E. Kleinert ([Kle90], [Kle92]) refines this language to obtain a description of a smaller overorder of RG, which can be described as a multiple fibre product. Motivated by Plesken's ideas, H. Wingen, a student of W. Jehne, calculates p-adic integral group rings of certain Frobenius groups, where Gaussian and Jacobian sums appear as matrix entries ([Win93],[Win95]). A different approach to describe the integral group rings for the symmetric groups is developed by M. Künzer in his thesis and further articles ([Kün99], [Kün00], [Kün02]). He gives explicit homomorphisms between certain different Specht lattices modulo an integer m. This allows to identify a non-maximal overorder of $\mathbb{Z}S_n$.

My habilitation thesis refines Plesken's methods to obtain a complete description of the group ring RG for certain groups G via explicit generators for the Morita equivalent basic order. If the defect is small, then the calculation of RG is an easy application of these methods (see [Neb02], [Neb98]). Among other examples the group rings of $SL_2(p^f)$ over p-adic integers are described nearly up to

Morita equivalence (see [Neb00a], [Neb00b]) using [Kos94] and [Kos98].

Knowing these p-adic group rings, one can calculate many invariants, give a description of the unit group and the ideal theory or calculate automorphisms of these group rings as demonstrated in [HeN02].

The present paper is intended to present certain results describing p-adic group rings, which are either unpublished or only contained in [Neb99]. Therefore the language used to describe these group rings is briefly repeated in Section 2 before we give the group rings in Section 3. Here, proofs are omitted, if they can be obtained from [Neb99]. As an application, the radical series of the group rings are calculated. It is known that these series become periodic (see Section 4). The period length, however, is unknown in general.

2 Methods

Let G be a finite group and R the ring of integers in a finite extension field K of the p-adic numbers. Then the integral group ring RG provides a link between the ordinary representation theory of G over K and the p-modular representation theory of G over the residue field $k = R/(\pi)$. To calculate RG, information from the ordinary and p-modular representation theory of G is used. In particular, the decomposition matrix of RG should be known.

The first step to describe the order RG is to decompose this order (or more precisely an overorder that is canonically attached to RG) into smaller pieces using the central primitive idempotents $\epsilon_1, \ldots, \epsilon_s \in KG$ in the semisimple K-algebra KG and orthogonal idempotents $e_1, \ldots, e_h \in RG$ that are lifts of the central primitive idempotents of the biggest semisimple factor algebra $RG/J(RG)$ of RG. Note that the ϵ_i are unique and the idempotents e_i are unique up to conjugacy in RG. Clearly

$$RG = \bigoplus_{i,j=1}^{h} e_i RGe_j$$

where the e_iRGe_i are R-orders and e_iRGe_j are e_iRGe_i-e_jRGe_j-bimodules. The local rings e_iRGe_i and also the bimodules are in general still quite complicated. Therefore one first describes the projections of these rings and bimodules into the simple components of KG: Let

$$\Gamma := \bigoplus_{t=1}^{s} \epsilon_t RG.$$

Then the idempotents $e_i\epsilon_t$ are either 0 or map onto the central primitive idempotents of $\Gamma/J(\Gamma)$ and

$$\Gamma = \bigoplus_{t=1}^{s} \bigoplus_{i,j=1}^{h} \epsilon_t e_i RGe_j.$$

The rings $\epsilon_t e_i RGe_i$ are local R-orders in the simple K-algebra $\epsilon_t e_i KGe_i$ which are maximal orders, if the p-decomposition numbers of G are ≤ 1 and R is "big enough". In this case Γ is a so called **graduated order** (see [Ple83]) or **tiled order** ([Rog92]) and the order Γ can be described purely combinatorially using the language of **exponent matrices**.

2.1 Graduated orders.

Definition 2.1 *An R-order Γ in a semisimple K-algebra is called* **graduated**, *if Γ contains a system of orthogonal idempotents $e_1, \ldots, e_h$ with $e_i e_j = \delta_{ij} e_i$ and $1 = e_1 + \ldots + e_h$ such that $e_i \Gamma e_i$ are maximal R-orders in $e_i K\Gamma e_i$.*

If Γ is a graduated order, then Γ contains the central primitive idempotents of $K\Gamma$ and hence Γ is a direct sum of graduated orders in simple algebras. If Γ is a graduated order in the simple algebra $D^{n\times n}$ and $e_1, \ldots, e_h$ are lifts of the central primitive idempotents of $\Gamma/J(\Gamma)$, then the orders $e_i\Gamma e_i$ are maximal orders in $e_i D^{n\times n} e_i \cong D^{n_i \times n_i}$, hence of the form $\Omega^{n_i\times n_i}$ where Ω is the maximal order in D. The bimodules $e_i\Gamma e_j$ are isomorphic to $\Omega^{n_i\times n_j}$, i.e. of the form $(\wp^{m_{ij}})^{n_i\times n_j}$ for certain $m_{ij} \in \mathbb{Z}$ where $\wp$ denotes the maximal ideal of Ω. Hence Γ is conjugate to the graduated order

$$\Lambda(\Omega, n_1, \ldots, n_h, M) = \{X \in D^{n\times n} \mid X = (X_{ij}) \text{ and } X_{ij} \in (\wp^{m_{ij}})^{n_i\times n_j}\}$$

The matrix $M = (m_{ij})$ is called an **exponent matrix** of Γ.

Remark 2.2 *The entries in the exponent matrix satisfy:*

a) $m_{ii} = 0$ for all $i = 1, \ldots, h$.

b) $m_{ij} + m_{jk} \geq m_{ik}$ for all $i, j, k = 1, \ldots, h$.

c) $m_{ij} + m_{ji} > 0$ for all $1 \leq i \neq j \leq h$.

Proof. a) and b) follow from the fact that Γ contains 1 and is closed under multiplication. c) is a consequence that we have chosen the e_i to be lifts of the central primitive idempotents modulo the radical. □

In general, the exponent matrix M is not determined by Γ, but only the **structural invariants**

$$m_{ijk} := m_{ij} + m_{jk} - m_{ik}$$

Two graduated orders $\Lambda(\Omega, n_1, \ldots n_h, M)$ and $\Lambda(\Omega, n'_1, \ldots n'_{h'}, M')$ are isomorphic, if and only if $h = h'$ and there is a permutation σ of $\{1, \ldots, h\}$ such that $n_i = n'_{i\sigma}$ and $m_{ijk} = m'_{i\sigma, j\sigma, k\sigma}$ for all $i, j, k = 1, \ldots, h$ (see [Ple83, Proposition (II.6)]).

Remark 2.3 *Let Λ be an R-order in a semisimple K-algebra $K\Lambda$ and let $\epsilon_1, \ldots, \epsilon_s$ be the central primitive idempotents of $K\Lambda$. Then $\Gamma := \oplus_{t=1}^{s} \epsilon_t \Lambda$ is a graduated order if and only if the decomposition numbers of Λ are ≤ 1 and $Z(\Gamma)$ is a maximal order in $Z(K\Lambda)$.*

2.2 Symmetric orders.

Definition 2.4 *An R-order Λ in the semisimple algebra $A = K\Lambda$ is called* symmetric *if there is a nondegenerate symmetric associative K-bilinear form $\Phi : A \times A \to K$ such that Λ is self dual with respect to Φ, i.e. $\Lambda = \Lambda^{\#} = \{a \in A \mid \Phi(\Lambda, a) \subset R\}$.*

One easily shows that the nondegenerate symmetric associative K-bilinear forms on the separable K-algebra A are precisely the forms

$$Tr_u : A \times A \to K, (a, b) \mapsto \sum_{t=1}^{s} tr_{red}(au\epsilon_t b)$$

where $u \in Z(A)^*$ and tr_{red} denotes the reduced trace of $A\epsilon_t$ to K.

The most important examples of symmetric orders are blocks of group rings of finite groups. Let G be a finite group. Then RG is a symmetric order in $A = KG$ with respect to $|G|^{-1}$ times the regular trace bilinear form. If $\chi_t(1)$ denotes the dimension of an absolutely irreducible constituent of the simple $KG\epsilon_t$-module, then this associative symmetric bilinear form equals Tr_u, where $u = |G|^{-1} \sum_{t=1}^{s} \chi_t(1)\epsilon_t$.

Lemma 2.5 *([Thé95], Proposition (1.6.2)) If Λ is a symmetric R-order with respect to Φ and e, f are idempotents in Λ then $\Phi_{|(e\Lambda f)\times(f\Lambda e)}$ is a nondegenerate R-bilinear pairing. In particular $e\Lambda e$ is a symmetric order.*

One important means to deal with symmetric orders is Jacobinski's conductor formula (see [Jac81]).

Theorem 2.6 *Let Λ be a symmetric order and Γ be an overorder of Λ contained in $K\Lambda$. Then the dual $\Gamma^{\#}$ of Γ is the biggest Γ-ideal that is contained in Λ.*

The dual of a graduated order can also be described by exponent matrices.

Theorem 2.7 *(cf. [Ple83, Theorem (III.8)]) Let Λ be a symmetric R-order with respect to Tr_u such that $\Gamma = \oplus_{t=1}^{s} \epsilon_t \Lambda$ is a graduated order*

$$\Gamma = \bigoplus_{t=1}^{s} \Lambda(\Omega_t, n_t, M^{(t)}).$$

Then the biggest Γ-ideal in Λ is

$$\Gamma^{\#} = \oplus_{t=1}^{s} (\epsilon_t \Lambda \cap \Lambda).$$

Here

$$\Gamma^{\#} = \oplus_{t=1}^{s} \Lambda(\Omega_t, \kappa_t J_{n_t} - (M^{(t)})^{tr})$$

where $J_{n_t} \in \{1\}^{n_t \times n_t}$ denotes the all-ones-matrix, $\kappa_t = \mu_t - \delta_t$, such that $\wp_t^{-\delta_t}$ is the inverse different of the maximal R-order Ω_t, and $\wp_t^{\mu_t} = (u\epsilon_t)^{-1}\Omega_t$.

Since $\Gamma^{\#}$ is contained in Γ this gives an upper bound on the entries of the exponent matrix:

$$m_{ij}^{(t)} + m_{ji}^{(t)} \leq \kappa_t.$$

2.3 Glueing.

By Theorem 2.7, one has the following inclusions:

$$\Gamma = \bigoplus_{t=1}^{s} \epsilon_t \Lambda \supseteq \Lambda \supseteq \bigoplus_{t=1}^{s} (\epsilon_t \Lambda \cap \Lambda) = \Gamma^{\#}$$

In this section we use the fact that this inclusion describes Λ as an amalgamation of the orders $\Lambda\epsilon_t$, where the amalgamating factor is known from the symmetrizing form, to derive further combinatorial conditions on the entries $m_{ij}^{(t)}$ of the exponent matrices.

For $1 \leq i \leq h$ let

$$c_i := \{t \mid d_{ti} = 1\}$$

and denote by $\tilde{e}_i$ a primitive idempotent in Λ with $\tilde{e}_i e_i = \tilde{e}_i$. Then $P_i := \tilde{e}_i \Lambda$ $(1 \leq i \leq h)$ are the projective indecomposable Λ modules. Multiplying the inclusion above with $\tilde{e}_i$, one sees that for $t \in c_i \cap c_j$ the number $\kappa_t - m_{ij}^{(t)} - m_{ji}^{(t)}$ is the multiplicity of the simple Λ-module S_j (with $S_j e_j = S_j$) in $\epsilon_t P_i / (\epsilon_t P_i \cap P_i)$.

Definition 2.8 *Let $1 \leq i \leq h$. Then the* amalgamation matrix $A(P_i) \in (\mathbb{Z}_{\geq 0} \cap \{.\})^{|c_i| \times h}$ *is the matrix of which the rows are labeled by the elements of c_i and the columns by $\{1, \ldots, h\}$ such that*

$$A(P_i)_{t,j} = \begin{cases} \kappa_t - m_{ij}^{(t)} - m_{ji}^{(t)} & \text{if } t \in c_j \cap c_j \\ . & \text{else} \end{cases}$$

Remark 2.9 *The entries in the amalgamation matrices satisfy:*

(i) $A(P_i)_{t,j} = A(P_j)_{t,i}$ *for all* $1 \leq i, j \leq h$, $t \in c_i \cap c_j$.

(ii) $A(P_i)_{t,i} = \kappa_t$.

(iii) If $c_i \cap c_j = \{t\}$, *then* $A(P_i)_{t,j} = 0$.

(iv) ([Ple83, Corollary (IV.7)]) If $c_i \cap c_j = \{t, l\}$ *then* $A(P_i)_{t,j} = A(P_i)_{l,j}$. *More generally the two maximal entries in any column of* $A(P_i)$ *are equal.*

2.4 A language to describe RG.

To describe the suborder RG of the graduated order $\Gamma = \oplus_{t=1}^{s} \epsilon_t RG$ it remains to calculate the rings $e_i RG e_i$, the bimodules $e_i RG e_j$ and the multiplication $e_i RG e_j \times e_j RG e_k \to e_i RG e_k$. The idempotents ϵ_t yield a "canonical basis" of $e_i KRG e_j$ where the coefficients of the "basis" element that corresponds to ϵ_t lie in D_t.

We now assume that $k = R/\pi R$ is a splitting field for kG and that the division algebras D_t are commutative. This can be achieved by replacing K by a suitable unramified extension. Let $P_1, \dots, P_h$ represent the isomorphism classes of projective indecomposable RG right modules.

Then RG is Morita equivalent to

$$\Lambda := \mathrm{End}_{RG}(P_1 \oplus \ldots \oplus P_h) = \bigoplus_{i,j=1}^{h} \mathrm{Hom}_{RG}(P_i, P_j)$$

and Λ is a basic order in the sense that the simple Λ-modules are one dimensional vector spaces over k.

Since there is an idempotent $e \in RG$ such that $\Lambda \cong eRGe$, Lemma 2.5 shows that Λ is symmetric. Note that the module categories of RG and Λ are equivalent. In particular the decomposition numbers of RG and Λ are equal. We assume that for $1 \leq i \leq h$ the endomorphism rings $\mathrm{End}_{RG}(P_i)$ are commutative which is equivalent to say that the decomposition numbers of RG are ≤ 1.

The main new idea for describing the order Λ is to embed the R-lattices $\mathrm{Hom}_{RG}(P_i, P_j)$ simultaneously for all $1 \leq i, j \leq h$ into a commutative finite-dimensional K-algebra E such that the multiplication
$\mathrm{Hom}_{RG}(P_i, P_j) \times \mathrm{Hom}_{RG}(P_j, P_l) \to \mathrm{Hom}_{RG}(P_i, P_l)$ can be performed in E.

To this purpose let

$$V := \bigoplus_{t=1}^{s} V_t$$

be the sum over a system of representatives of the isomorphism classes of simple KG-modules and

$$E := \mathrm{End}_{KG}(V) \cong \bigoplus_{t=1}^{s} D_t \cong Z(KG).$$

Let $1 \leq j \leq h$. Since $\mathrm{End}_{RG}(P_j)$ is commutative, the KG-module V has a unique KG-submodule isomorphic to $K \otimes_R P_j$ and up to isomorphism a unique RG-sublattice isomorphic to P_j. For all $1 \leq j \leq h$ choose an embedding

$$\iota_j : P_j \hookrightarrow V.$$

Let Q_j be the unique KG-invariant complement of $K \otimes_R \iota_j(P_j)$ in V,

$$V = (K \otimes_R \iota_j(P_j)) \oplus Q_j.$$

Then the RG-homomorphisms $\varphi \in \mathrm{Hom}_{RG}(P_j, P_i)$ for $1 \leq i, j \leq h$ are considered as elements of E by letting

$$\varphi_{|Q_j} = 0.$$

Definition 2.10 *For $i = 1, \ldots, h$ let $\pi_i : V \to K \otimes_R P_i$ be the projection onto $K \otimes_R P_i$ with $\pi_i(Q_i) = 0$ and $\iota_i \pi_i = id_{P_i}$. Then for $1 \le i, j \le h$ there are embeddings*

$$\mathrm{Hom}_{RG}(P_i, P_j) \hookrightarrow E, \ \varphi \mapsto \pi_i \varphi \iota_j.$$

Via these embeddings $\mathrm{Hom}_{RG}(P_i, P_j)$ *is viewed as a subset*

$$\Lambda_{ij} := \pi_i \mathrm{Hom}_{RG}(P_i, P_j)\iota_j \subset E.$$

Remark 2.11 *For $1 \le i \ne j \le h$ the endomorphism ring $\mathrm{End}_{RG}(P_j)$ is canonically (i.e. independent of the choice of ι_j) embedded into E, whereas the embedding $\mathrm{Hom}_{RG}(P_i, P_j) \hookrightarrow E$ depends on the choice of ι_i and ι_j.*

In the examples only the Morita equivalent basic order $\mathrm{End}_\Lambda(\oplus_{i=1}^h P_i)$ is described. To this aim we choose suitable embeddings $\iota_i : P_i \hookrightarrow V$ and give generators of $\Lambda_{ij} \subset E$, by using the canonical "basis" of E, formed by the central primitive idempotents ϵ_t. The elements in Λ_{ij} are linear combinations

$$\sum_{t \in c_i \cap c_j} a_t \epsilon_t$$

where $a_t \in D_t$ (more precisely in $\wp_t^{m_{ij}^{(t)}}$). A basis matrix of Λ_{ij} is a $\sum_{t \in c_i \cap c_j} \dim_K(D_t) \times |c_i \cap c_j|$ matrix of which the lines are the coefficients a_t of an R-basis of Λ_{ij}.

3 Some examples of group rings.

After the leading example to illustrate the notation, this section describes the principal blocks of $\mathbb{Z}_2 J_1$ (defect 3) and of $\mathbb{Z}_3 S_9$ (defect 4).

3.1 The group ring $\mathbb{Z}_2 S_4$.

As a first simple example, the group ring $\mathbb{Z}_2 S_4$ is described to illustrate the notation. The decomposition matrix of $\mathbb{Z}_2 S_4$ is:

	1	2
1	1	0
1′	1	0
2	0	1
3	1	1
3′	1	1

Here $1'$ denotes the sign character, $3+1$ is the natural permutation character and $3' = 3 \cdot 1'$. The graduated hull of $\Lambda = \mathbb{Z}_2 S_4$ is

$$\Gamma := \epsilon_1 \Lambda \oplus \epsilon_{1'} \Lambda \oplus \epsilon_2 \Lambda \oplus \epsilon_3 \Lambda \oplus \epsilon_{3'} \Lambda$$

where $\epsilon_1\Lambda \cong \epsilon_{1'}\Lambda \cong \mathbb{Z}_2 = \Lambda(\mathbb{Z}_2, 1, (0))$, $\epsilon_2\Lambda \cong \mathbb{Z}_2^{2\times 2} = \Lambda(\mathbb{Z}_2, 2, (0))$, and $\epsilon_3\Lambda \cong \epsilon_{3'}\Lambda \cong \Lambda(\mathbb{Z}_2, 1, 2, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix})$.

If Λ' denotes the basic order of Λ and $e_1 \in \Lambda'$ and $e_2 \in \Lambda'$ are lifts of the central primitive idempotents and $B_{i,j}$ denotes a basis matrix of $e_i\Lambda' e_j$ then:

$$B_{1,1} = \begin{array}{cccc} 1 & 1' & 3 & 3' \\ \hline 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{array} \quad B_{2,2} = \begin{array}{ccc} 2 & 3 & 3' \\ \hline 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{array} \quad B_{1,2} = \begin{array}{cc} 3 & 3' \\ \hline 4 & 4 \\ 0 & 8 \end{array} \quad B_{2,1} = \begin{array}{cc} 3 & 3' \\ \hline 1 & 1 \\ 0 & 2 \end{array}$$

The eleven generators described in these basis matrices correspond to following generators of Λ' (as $\mathbb{Z}_2$-lattice):

$$(1,1,0,\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}),\quad (0,2,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}),$$

$$(0,0,0,\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}),\quad (0,0,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}),$$

$$(0,0,1,\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}),\quad (0,0,0,\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}),$$

$$(0,0,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix}),\quad (0,0,0,\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}),$$

$$(0,0,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 8 \\ 0 & 0 \end{pmatrix}),\quad (0,0,0,\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}),$$

$$(0,0,0,\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}).$$

Proof. The exponent matrices are easily obtained, e.g. with [Ple83, Theorem (VI.1)]. This also gives the form of the bimodules $e_i\Lambda' e_j$ for $i \neq j$. From the central characters of S_4, one gets that $e_2 Z(\Lambda')$ is symmetric and hence $e_2 Z(\Lambda') = e_2\Lambda' e_2$. Similarly one finds that $e_1 Z(\Lambda')$ is the suborder of index 2 spanned by the lines of $\begin{array}{cccc} 1 & 1' & 3 & 3' \\ \hline 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 8 \end{array}$ of $e_1\Lambda' e_1$. The latter is contained in the dual $(e_1 Z(\Lambda'))^{\#}$. It remains to decide which of the 3 non-trivial classes of $(e_1 Z(\Lambda'))^{\#}/e_1 Z(\Lambda')$ (represented by $(0,2,0,2), (0,2,2,0), (0,0,2,2)$) lies in $e_1\Lambda' e_1$. Since 1 and 1′ are not congruent modulo 4, the last class is impossible. To decide between the other two classes, we use the one-box shift morphism given by M. Künzer.

Let $\lambda = \ydiagram{4}$ and $\mu = \ydiagram{3,1}$. Then λ is the partition belonging to the trivial character 1 and μ belongs to the character 3. μ is obtained from λ by shifting the most right box in λ to the lower left box in μ. The shift length is 4. Therefore [Kün00, Theorem 3.1] implies that there is a homomorphism of $e_1\Lambda'\epsilon_1$ to $e_1\Lambda'\epsilon_3/4e_1\Lambda'\epsilon_3$ of order 4. Therefore all elements of $e_1\Lambda' e_1(\epsilon_1+\epsilon_3)$ are congruent to 0 or 1 modulo $4\epsilon_1\Lambda' \oplus 4\epsilon_3\Lambda'$. □

Corollary 3.1 *The Jacobson radical J of $\mathbb{Z}_2S_4$ satisfies $J^4 = 2J^3$. The exponent matrices of J^3 are* $(3),(3),(3),\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix},\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, *hence J^3 is conjugate to* 8 *times the maximal order in $\mathbb{Q}_2S_4$.*

3.2 The group ring $\mathbb{Z}_2[\zeta_3]J_1$.

Let $R := \mathbb{Z}_2[\zeta_3]$ be the ring of integers in the unramified extension of degree 2 of $\mathbb{Q}_2$. This section gives generators for the Morita equivalent basic order of the principal block of RJ_1. The proofs are included for several reasons: First, this calculation is not contained in [Neb99], second, to demonstrate the methods developed in [Neb99], third to show the limit of these methods, since here serious computer calculations are involved to decide the last subtle question, and forth, to correct the error in the Loewy series of the projective indecomposable $\mathbb{F}_4J_1$-modules as given in [LaM78] and also in [Mic89].

The group ring RJ_1 has 7 blocks, 6 of which have defect ≤ 1. The defect group of the principal block Λ is elementary abelian of order 2^3. A decomposition matrix of Λ is as follows:

	1	20	$56a$	$56b$	76
1	1	.	.	.	.
77	1	.	.	.	1
$77a$	1	1	1	.	.
$77b$	1	1	.	1	.
133	1	1	1	1	.
$133a$	1	.	.	1	1
$133b$	1	.	1	.	1
209	1	1	1	1	1

Theorem 3.2 *The graduated hull Γ of Λ is*

$$\Gamma = \Lambda_1 \oplus \Lambda_{77} \oplus \Lambda_{77a} \oplus \Lambda_{77b} \oplus \Lambda_{133} \oplus \Lambda_{133a} \oplus \Lambda_{133b} \oplus \Lambda_{209}$$

with $\Lambda_1 = \Lambda(R,1,(0))$, $\Lambda_{77} = \Lambda(R,1,76,\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})$,
$\Lambda_{77a} = \Lambda(R,20,56a,1,M)$, $\Lambda_{77b} = \Lambda(R,20,56b,1,M)$,

$$\Lambda_{133} = \Lambda(R, 20, 56a, 56b, 1, \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}), \ \ \Lambda_{133a} = \Lambda(R, 76, 1, 56b, M),$$

$$\Lambda_{133b} = \Lambda(R, 76, 1, 56a, M),$$

$$\Lambda_{209} = \Lambda(R, 76, 1, 56a, 56b, 20, \begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}), \textit{ where } M := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. By choosing suitable bases, we may assume that the first columns of the exponent matrices consist of 0 only. Since the Cartan-invariant $c_{76,20}$ is 1, the corresponding entry of the exponent matrix N of Λ_{209} is 3 (see 2.9). The non trivial Galois automorphism σ of R over $\mathbb{Z}_2$ induces a ring automorphism of RJ_1. (by $\sum a_g g \mapsto \sum a_g^\sigma g$). Therefore $N_{76,56a} = N_{76,56b}$ by [Ple83, Proposition (IV.1)]. Since all 2-Brauer-characters of J_1 are real, one has $N_{56b,56a} = N_{56a,56b} \geq 1$ by [Neb99, Folgerung 5.2.12]. Hence $N_{56a,56b} = N_{56b,56a} = 1$ because $N_{56b,56a} + N_{56a,56b} \leq 3 = \nu_2(|J_1|)$. Similarly $O_{56a,56b} = O_{56b,56a} = 1$, where O is the exponent matrix of Λ_{133}.

Let P_{76} be the projective indecomposable Λ-lattice with head 76. With the condensation given in [Lux97] one calculates the Loewy-series of $P_{76}/2P_{76}$:

$$\begin{array}{ccc} & 76 & \\ 1 & 76 & \\ 56a & & 56b \\ 1 & 20 & 1 \\ 56a & 76 & 56b \\ & 1 & \\ & 76 & \end{array}.$$

(Note that this Loewy-series is given incorrectly in both [LaM78] and [Mic89].)

In particular 20 occurs only in the 4-th layer of the Loewy-series of $P_{76}/2P_{76}$. Therefore the Loewy-length of $\epsilon_{209}P_{76}/2\epsilon_{209}P_{76}$ is at least 4. Therefore one can apply [Neb99, Folgerung 5.2.12] to find the exponent matrix N of Λ_{209} as in [Neb99, Satz 5.6.1].

The Cartan-invariants $c_{76,56a} = c_{76,56b}$ are 2. Since the corresponding entrys in N are 2, also the entrys in the exponent matrices of Λ_{133b} and Λ_{133a} are 2 by [Ple83, Corollary (IV.7)] (see 2.9). Again, the form of the Loewy-series of $P_{76}/2P_{76}$, implies that the exponent matrices of Λ_{133b} and Λ_{133a} are as given in the theorem.

With [Lux97] one calculates the Loewy-series of $P_{20}/2P_{20}$ as

$$\begin{array}{ccc} & 20 & \\ 56a & 56b & \\ 1 & 20 & 1 \\ 56a & 76 & 56b \\ 1 & 20 & 1 \\ 56a & 56b & \\ & 20 & \end{array}$$

Therefore the exponent matrices O resp. M of Λ_{133} resp. Λ_{77a} and Λ_{77b} are of the form

$$O = \begin{pmatrix} 0 & x & x & y \\ 0 & 0 & 1 & y-x \\ 0 & 1 & 0 & y-x \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & a & b \\ 0 & 0 & b-a \\ 0 & 0 & 0 \end{pmatrix}$$

with $1 \leq x < y \leq 3$ and $1 \leq a < b \leq 3$.

The amalgamation matrix (see 2.9) of P_{20} is hence given as

P_{20}	1	20	$56a$	$56b$	76
$77a$	$3-b$	3	$3-a$	.	.
$77b$	$3-b$	3	.	$3-a$	. .
133	$3-y$	3	$3-x$	$3-x$	.
209	1	3	2	2	0

By [Ple83, Corollary (IV.7)(iii)] (see 2.9) the two maximal entries in the columns of the amalgamation matrix are equal. Therefore $a = 1$ or $x = 1$. Since the sum $3 - a + 3 - x + 2$ over a column of the amalgamation matrix is even by [Neb99, Lemma 5.3.14] one gets $a = x = 1$. From the first column of the amalgamation matrix above one concludes that either $b = 2$ or $y = 2$. The amalgamation matrix of P_{56a} is

P_{56a}	1	20	$56a$	$56b$	76
$77a$	$4-b$	2	3	.	.
133	$4-y$	2	3	1	. .
$133b$	2	.	3	.	1
209	2	2	3	1	1

Since the sum over the first column is even, one gets $b \equiv y \pmod 2$ and hence $b = y = 2$. Therefore we know all exponent matrices except of the one of Λ_{77}. This exponent matrix if of the form $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ for some $1 \leq c \leq 3$ Since the sum over the first column of the amalgamation matrix of P_{76} is even, one gets the c is odd, hence $c = 1$ or $c = 3$.

Either by explicit calculation of the R-lattices in the 77-dimensional representation of J_1 or from the proof of the next theorem one finds $c = 1$. □

Theorem 3.3 *Let $I := \{1, 20, 56a, 56b, 76\}$ denote the indices of the simple Λ-modules. For $i \in I$ let P_i be the corresponding projective indecomposable Λ-module and V be the sum of representatives of all simple $K\Lambda$-modules and $E := \mathrm{End}_{K\Lambda}(V) \cong Z(K\Lambda)$. Then there are embeddings $\iota_i : P_i \to V$, $i \in I$, with $\iota_{56b} = \iota_{56a}^{\sigma}$ for the non trivial Galois automorphism σ of R over $\mathbb{Z}_2$, such that the following elements $a_{il} \in \Lambda_{i,l} := \pi_i \mathrm{Hom}_\Lambda(P_i, P_l)\iota_l \subset E$ together with the primitive idempotents $id_{P_i} \in \mathrm{End}_\Lambda(P_i) \subset E$ (for all $i \in I$) generate the basic order*

$$\Lambda_0 := \mathrm{End}_\Lambda(\oplus_{i \in I} P_i) \sim \Lambda$$

as R-order:

$$a_{20,56a} = 2\epsilon_{77a} + 2\epsilon_{133} + \epsilon_{209}$$
$$a_{20,56b} = 2\epsilon_{77b} + 2\epsilon_{133} + \epsilon_{209}$$
$$a_{1,56a} = \epsilon_{77a} + \epsilon_{133} + 2\epsilon_{133b} + 2\zeta_3^2\epsilon_{209}$$
$$a_{1,56b} = \epsilon_{77b} + \epsilon_{133} + 2\epsilon_{133a} + 2\zeta_3\epsilon_{209}$$
$$a_{1,76} = 2\epsilon_{77} + \epsilon_{133a} + \epsilon_{133b} + \epsilon_{209}$$
$$a_{76,76} = 2\zeta_3^2\epsilon_{133a} + 2\zeta_3\epsilon_{133b} + 2\epsilon_{209}$$
$$a_{56a,20} = \zeta_3^2\epsilon_{77a} + \zeta_3\epsilon_{133} + 2\epsilon_{209}$$
$$a_{56b,20} = \zeta_3\epsilon_{77b} + \zeta_3^2\epsilon_{133} + 2\epsilon_{209}$$
$$a_{56a,1} = 2\epsilon_{77a} + 2\epsilon_{133} + \epsilon_{133b} + \zeta_3\epsilon_{209}$$
$$a_{56b,1} = 2\epsilon_{77b} + 2\epsilon_{133} + \epsilon_{133a} + \zeta_3^2\epsilon_{209}$$
$$a_{76,1} = \epsilon_{77} + 2\epsilon_{133a} + 2\epsilon_{133b} + 2\epsilon_{209}$$

The lines of the following basis matrices $B_{i,j}$ are the coefficients of basis elements of $\Lambda_{i,j}$ with respect to the canonical basis given by the central primitive idempotents. Only those matrices $B_{i,j}$ are displayed, where $|c_i \cap c_j| > 2$,

$B_{1,1} =$	1	77	$77a$	$77b$	133	$133a$	$133b$	209
	1	1	1	1	1	1	1	1
	0	2	0	0	0	2	2	2
	0	0	$2\zeta_3$	$2\zeta_3^2$	2	$2\zeta_3^2$	$2\zeta_3$	2
	0	0	2	2	0	2	2	0
	0	0	0	0	4	0	0	4
	0	0	0	0	0	4	4	0
	0	0	0	0	0	$4\zeta_3$	$4\zeta_3^2$	4
	0	0	0	0	0	0	0	8

$B_{20,20} =$	$77a$	$77b$	133	209
	1	1	1	1
	$2\zeta_3$	$2\zeta_3^2$	0	2
	$4\zeta_3^2$	$4\zeta_3$	0	4
	0	0	0	8

$B_{76,76} =$	77	$133a$	$133b$	209
	1	1	1	1
	0	$2\zeta_3^2$	$2\zeta_3$	2
	0	0	4	4
	0	0	0	8

$B_{56a,56a} =$	$77a$	133	$133b$	209
	1	1	1	1
	$2\zeta_3^2$	$2\zeta_3$	0	2
	0	0	4	4
	0	0	0	8

$B_{56b,56b} =$	$77b$	133	$133a$	209
	1	1	1	1
	$2\zeta_3$	$2\zeta_3^2$	0	2
	0	0	4	4
	0	0	0	8

$B_{20,56a} =$	$77a$	133	209
	2	2	1
	0	4	$2\zeta_3$
	0	0	4

$B_{20,56b} =$	$77b$	133	209
	2	2	1
	0	4	$2\zeta_3^2$
	0	0	4

$B_{56a,20} =$	$77a$	133	209
	ζ_3^2	ζ_3	2
	2	2	0
	4	0	0

$B_{56b,20} =$	$77b$	133	209
	ζ_3	ζ_3^2	2
	2	2	0
	4	0	0

$B_{1,56a} =$	$77a$	133	$133b$	209
	1	1	2	$2\zeta_3^2$
	2	0	0	$4\zeta_3^2$
	2	2	0	0
	4	0	0	0

$B_{1,56b} =$	$77b$	133	$133a$	209
	1	1	2	$2\zeta_3$
	2	0	0	$4\zeta_3$
	2	2	0	0
	4	0	0	0

$B_{56a,1} =$	$77a$	133	$133b$	209
	2	2	1	ζ_3
	0	4	2	0
	0	0	2	$2\zeta_3$
	0	0	0	$4\zeta_3$

$B_{56b,1} =$	$77b$	133	$133a$	209
	2	2	1	ζ_3^2
	0	4	2	0
	0	0	2	$2\zeta_3^2$
	0	0	0	$4\zeta_3^2$

$B_{1,20} =$	$77a$	$77b$	133	209
	1	1	1	0
	ζ_3^2	ζ_3	1	4
	0	0	2	0
	0	0	0	8

$B_{1,76} =$	77	$133a$	$133b$	209
	2	1	1	1
	0	2	2	0
	0	$2\zeta_3$	$2\zeta_3^2$	2
	0	0	0	4

$B_{20,1} =$	$77a$	$77b$	133	209
	4	4	0	1
	$4\zeta_3$	$4\zeta_3^2$	4	0
	0	0	8	0
	0	0	0	2

$B_{76,1} =$	77	$133a$	$133b$	209
	1	2	2	2
	0	4	4	0
	0	$4\zeta_3$	$4\zeta_3^2$	4
	0	0	0	8

Proof. If $n, m \in c_i = \{t \in \{1, \ldots, s\} \mid d_{ti} = 1\}$, then $4(\epsilon_n + \epsilon_m)e_i \in \Lambda_{i,i} = \Lambda^{\#}_{i,i}$. Therefore it remains to find for Λ_{11} three and for the other Λ_{ii} one additional generator to determine the orders Λ_{ii}.

W.l.o.g. one may assume that

$$\begin{array}{c|ccc} B_{20,56a} = & 77a & 133 & 209 \\ \hline & 2 & 2 & 1 \\ & 0 & 4 & 2w \\ & 0 & 0 & 4 \end{array}$$

for some $w \in \{1, \zeta_3, \zeta_3^2\}$. Dualizing one gets

$$\begin{array}{c|ccc} B_{56a,20} = & 77a & 133 & 209 \\ \hline & w-1 & -w & 2 \\ & -2 & 2 & 0 \\ & 4 & 0 & 0 \end{array}$$

. Since the projection onto the first component of $\Lambda_{56a,20}$ is R it follows that $w \neq 1$. Applying the Galois automorphism σ, one may assume that $w = \zeta_3$. Applying σ to these matrices one gets $B_{20,56b}$ and $B_{56b,20}$. Multiplying these generators one finds the missing generators for the orders $\Lambda_{20,20}$, $\Lambda_{56a,56a}$ and $\Lambda_{56b,56b}$, as well as bases for the bimodules $\Lambda_{56a,56b}$ and $\Lambda_{56b,56a}$.

From the Ext-quiver of Λ and the exponent matrices, one sees that the bimodule $\Lambda_{1,56a}$ is generated (as a bimodule) by one element which may be chosen of the form $\epsilon_{77a}+\epsilon_{133}+2\epsilon_{133b}+2a\epsilon_{209}\in E$ by choosing the embeddings ι_1 and ι_{56a} suitably where $a\in R^*$. Multiplying by $\Lambda_{56a,56a}$ one finds a basis matrix

$B_{1,56a}=$	$77a$	133	$133b$	209
	1	1	2	$2a$
	$2(\zeta_3-1)$	$-2\zeta_3$	0	$4a$
	$2x$	2	0	0
	4	0	0	0

for $\Lambda_{1,56a}$ where $x=1,\zeta_3$ or ζ_3^2. Dualizing yields the basis matrix

$B_{56a,1}=$	$77a$	133	$133b$	209
	-2	$2x$	$2-x-\zeta_3-x\zeta_3$	$(-3\zeta_3+3-3x\zeta_3)a^{-1}$
	0	0	2	$2a^{-1}$
	0	0	0	$4a^{-1}$
	0	-4	2	0

of $\Lambda_{56a,1}$. Therefore $x+\zeta_3+x\zeta_3\not\equiv 0 \pmod 2$ and $1+\zeta_3+x\zeta_3\not\equiv 0 \pmod 2$ hence $x=1$.

Multiplying these generators one finds the element $\alpha:=-2\epsilon_{77a}+2\epsilon_{133}+2\epsilon_{133b}-2\epsilon_{209}\in\Lambda_{1,1}$.

The bimodules $\Lambda_{1,56b}$ and $\Lambda_{56b,1}$ are obtained applying σ. Since $\Lambda_{1,56a}\Lambda_{56a,20}\Lambda_{20,56b}\subset\Lambda_{1,56b}$, one has that $-2\zeta_3\epsilon_{133}+4a\epsilon_{209}\in\Lambda_{1,56b}$. Since $2\epsilon_{133}-4a^\sigma\epsilon_{209}\in\Lambda_{1,56b}$ one gets $a^\sigma=a\zeta_3^2$, hence $a=\zeta_3^2$.
In particular $\beta:=2\epsilon_{77b}+2\epsilon_{133}+2\epsilon_{133a}+2\epsilon_{209}\in\Lambda_{1,1}$.

The suborder of $\Lambda_{1,1}$ generated by e_1,α,β and the elements $4(\epsilon_n+\epsilon_m)$ $(n,m\in c_1)$ (of index 2) acts on $\Lambda_{1,76}$ as the order generated by the lines of the matrix

77	$133a$	$133b$	209
1	1	1	1
0	2	0	2
0	0	2	2
0	0	0	4

Therefore one finds that c in the proof above is $c=1$. Otherwise $c=3$ and $(1,0,0,0)\in\Lambda_{76,1}$ and therefore $(0,2,0,0)\in\Lambda_{76,1}$ which contradicts the fact that ϵ_{133a} is amalgamated in $\Lambda_{76,1}$ with depth 2.

Also $\Lambda_{1,76}$ is generated as a bimodule by one element as one sees by the Ext-quiver of Λ. For suitable embeddings one finds that

$$\begin{array}{rrrrr} B_{1,76}= & 77 & 133a & 133b & 209 \\ \hline v := & 2 & 1 & 1 & 1 \\ v\alpha = & 0 & 0 & 2 & -2 \\ v\beta = & 0 & 2 & 0 & 2 \\ & 0 & 0 & 0 & 4 \end{array} \quad \text{and dually} \quad \begin{array}{rrrrr} B_{76,1}= & 77 & 133a & 133b & 209 \\ \hline & 1 & 2 & -2 & -2 \\ & 0 & 4 & 0 & 4 \\ & 0 & 0 & 4 & 4 \\ & 0 & 0 & 0 & 8 \end{array},$$

from which one gets the third missing generator $\gamma = 2\epsilon_{77} + 2\epsilon_{133a} - 2\epsilon_{133b} - 2\epsilon_{209}$ of $\Lambda_{1,1}$.

It remains to find the missing generator δ of $\Lambda_{76,76}$. For this we have to perform several explicit computer calculations:

Calculating certain 276-dimensional J_1-lattices one finds that $(\epsilon_{77}+\epsilon_{209})\Lambda_{76,76} = \{(x,y) \in R \oplus R \mid x \equiv y \pmod 2\}$. Therefore one may choose δ as $\delta := 2a\epsilon_{133a} + 2b\epsilon_{133b} + 2\epsilon_{209}$ for certain $a, b \in R^*$. Since the scalar product of δ with 1 is integral, one finds that $a + b + 1 \equiv 0 \pmod 4$. Moreover one may choose δ to be fix under the Galois automorphism σ from which one gets that $a = \zeta$ and $b = \zeta^2$, where ζ is one of ζ_3 or ζ_3^2. The value of ζ will be decided by explicit calculations modulo 2.

The radical $J(\Lambda_{1,1})$ contains the (Galois invariant) element $\alpha' := 2\epsilon_1 + 2\zeta_3\epsilon_{77a} + 2\zeta_3^2\epsilon_{77b} + 2\zeta_3\epsilon_{133a} + 2\zeta_3^2\epsilon_{133b} + 2\epsilon_{209}$, which is not contained in $J(\Lambda_{1,1})^2$. The element $\beta' := 2\zeta_3\epsilon_{77a} + 2\zeta_3^2\epsilon_{77b} + 2\epsilon_{209}$ lies in $J(\Lambda_{20,20}) - J(\Lambda_{20,20})^2$. Both elements α' and β' act equally on $\Lambda_{1,20}$. If $\zeta = \zeta_3$ then $\delta = 2\zeta_3\epsilon_{133a} + 2\zeta_3^2\epsilon_{133b} + 2\epsilon_{209}$ is an element in $J(\Lambda_{76,76}) - J(\Lambda_{76,76})^2$ that acts like α' on $\Lambda_{1,76}$.

Condensation of $\mathbb{F}_2\Lambda$ does not only give the Loewy-series of the projective indecomposable $\mathbb{F}_2\Lambda$-modules, but also a tool to calculate the endomorphism rings and homomorphism bimodules of these modules. Explicit calculations yield that $J(\mathrm{End}_{\mathbb{F}_2G}(\overline{P_{20}}))$ acts as a one dimensional space on $\mathrm{Hom}_{\mathbb{F}_2G}(\overline{P_1}, \overline{P_{20}})$. The kernel of this action is $J(\mathrm{End}_{\mathbb{F}_2G}(\overline{P_{20}}))^2$. The same holds for the action of $J(\mathrm{End}_{\mathbb{F}_2G}(\overline{P_{76}}))$ on $\mathrm{Hom}_{\mathbb{F}_2G}(\overline{P_1}, \overline{P_{76}})$. Therefore we know the image of β and γ under these actions. But there is no element of $J(\mathrm{End}_{\mathbb{F}_2G}(\overline{P_1}))$ that acts on $\mathrm{Hom}_{\mathbb{F}_2G}(\overline{P_1}, \overline{P_{20}})$ like β' and on $\mathrm{Hom}_{\mathbb{F}_2G}(\overline{P_1}, \overline{P_{76}})$ like δ. Therefore $\zeta = \zeta_3^2$. □

Corollary 3.4 *Let $J = J(\Lambda)$ be the Jacobson radical Λ. Then*

$$J^7 = J^5(2 + 2\epsilon_1).$$

The Loewy series of the projective indecomposable Λ-lattices are given as follows, *where the last two layers are repeated periodically:*

P_1			P_{20}		P_{76}		P_{56a}			P_{56b}		
1			20		76		$56a$			$56b$		
1	76	$56ab$		$56ab$	1	76	1	20		1	20	
1^4	20^2		1^2	20^2	76	$56ab$	76	$56a^2$	$56b$	76	$56b^2$	$56a$
1	76^3	$56a^3b^3$	76	$56a^2b^2$	1^3	20	1^3	20^2		1^3	20^2	
1^7	20^4		1^4	20^3	76^3	$56a^2b^2$	76^2	$56a^3$	$56b^3$	76^2	$56b^3$	$56a^3$
1	76^4	$56a^4b^4$	76	$56a^3b^3$	1^4	20	1^4	20^2		1^4	20^2	
1^8	20^4		1^4	20^4	76^4	$56a^2b^2$	76^2	$56a^4$	$56b^3$	76^2	$56b^4$	$56a^3$

From Theorem 3.3 one obtains the principal block of $\mathbb{Z}_2 J_1$ by Galois descent.

Corollary 3.5 *Let Λ_0 denote the principal block of $\mathbb{Z}_2 J_1$. Then*

$$\epsilon_1\Lambda_0 \sim \mathbb{Z}_2,\ \epsilon_{77}\Lambda_0 \sim \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix},$$

$$(\epsilon_{77a}+\epsilon_{77b})\Lambda_0 \sim \begin{pmatrix} O & 2R & 4R \\ R & R & 2R \\ R & R & O \end{pmatrix},\ \epsilon_{133}\Lambda_0 \sim \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2^{1\times 2} & 4\mathbb{Z}_2 \\ \mathbb{Z}_2^{2\times 1} & S & 2\mathbb{Z}_2^{2\times 1} \\ \mathbb{Z}_2 & \mathbb{Z}_2^{1\times 2} & \mathbb{Z}_2 \end{pmatrix},$$

$$(\epsilon_{133a}+\epsilon_{133b})\Lambda_0 \sim \begin{pmatrix} O & 2O & 4R \\ O & O & 2R \\ R & R & R \end{pmatrix},$$

$$\epsilon_{209}\Lambda_0 \sim \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 & 4\mathbb{Z}_2^{1\times 2} & 8\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 & 2\mathbb{Z}_2^{1\times 2} & 4\mathbb{Z}_2 \\ \mathbb{Z}_2^{2\times 1} & \mathbb{Z}_2^{2\times 1} & S & 2\mathbb{Z}_2^{2\times 1} \\ \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2^{1\times 2} & \mathbb{Z}_2 \end{pmatrix},$$

where $\sim$ means "is Morita equivalent to" and $R := \mathbb{Z}_2[\zeta_3]$, $O = \mathbb{Z}_2 + 2\mathbb{Z}_2[\zeta_3] \leq R$, and $S := \mathbb{Z}_2[\zeta_3] + 2\mathbb{Z}_2^{2\times 2} \leq \mathbb{Z}_2^{2\times 2}$.

3.3 The group ring of $\mathbb{Z}_3 S_9$.

This section gives generators of the Morita equivalent basic order of the principal block of $\mathbb{Z}_3 S_9$. The proof can be obtained from [Neb99, Section 5.6.4]. The group ring $\mathbb{Z}_3 S_9$ has 5 blocks, 4 of which have defect ≤ 1. (see [Jam78]). A decomposition matrix for the principal block Λ of $\mathbb{Z}_3 S_9$ is given in [Jam78]:

	1	1′	7	7′	21	21′	35	35′	41	41′
1	1	.	.	.	.	.	.	.	.	.
1′	.	1	.	.	.	.	.	.	.	.
8	1	.	1	.	.	.	.	.	.	.
8′	.	1	.	1	.	.	.	.	.	.
42a	.	.	.	.	1	1	.	.	.	.
28	.	.	1	.	1	.	.	.	.	.
28′	.	.	.	1	.	1	.	.	.	.
70	.	.	.	.	.	.	1	1	.	.
42	.	1	.	.	.	.	.	.	1	.
42′	1	.	.	.	.	.	.	.	.	1
48	.	.	1	.	.	.	.	.	1	.
48′	.	.	.	1	.	.	.	.	.	1
56	.	.	.	.	1	.	1	.	.	.
56′	.	.	.	.	.	1	.	1	.	.
84	.	1	.	1	.	.	1	.	1	.
84′	1	.	1	.	.	.	.	1	.	1
105	1	.	1	.	1	.	1	.	1	.
105′	.	1	.	1	.	1	.	1	.	1
120	1	1	1	.	.	.	1	1	1	.
120′	1	1	.	1	.	.	1	1	.	1
168	.	1	1	1	1	1	1	1	1	.
168′	1	.	1	1	1	1	1	1	.	1

Here χ' denotes the tensor product of the character χ with the sign character $1'$.

Theorem 3.6 *The graduated hull $\Gamma = \oplus_{\chi \in Irr(\Lambda)} \Lambda \epsilon_\chi$ is as follows, where only one of the two summands $\Lambda\epsilon_\chi$ and $\Lambda\epsilon_{\chi'}$ is given.*

$$\Lambda_1(1,(0)),\ \Lambda_8(1,7,\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}),\ \Lambda_{42a}(21,21',\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}),$$

$$\Lambda_{28}(7,21,\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}),$$

$$\Lambda_{70}(35,35',\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}),\ \Lambda_{42}(1',41,\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}),\ \Lambda_{48}(7,41,\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}),$$

$$\Lambda_{56}(21,35,\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}),\ \Lambda_{84}(7',1',35,41,\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}),$$

$$\Lambda_{105}(35,1,7,21,41,\begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}),$$

$$\Lambda_{120}(35', 1, 1', 7, 35, 41, \begin{pmatrix} 0 & 2 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}),$$

$$\Lambda_{168}(21', 1', 7, 7', 21, 35, 35', 41, \begin{pmatrix} 0 & 2 & 2 & 1 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}).$$

Theorem 3.7 *Let $I := \{1, 1', 7, 7', 21, 21', 35, 35', 41, 41'\}$ denote the indices of the simple Λ-modules. For $i \in I$ let P_i be the corresponding projective indecomposable Λ-module and V be the sum of representatives of all simple $\mathbb{Q}_3\Lambda$-modules and $E := \mathrm{End}_{\mathbb{Q}_3\Lambda}(V) \cong Z(\mathbb{Q}_3\Lambda)$. Then there are embeddings $\iota_i : P_i \to V$, $i \in I$, such that the following elements $a_{il} \in \Lambda_{i,l} := \pi_i \mathrm{Hom}_\Lambda(P_i, P_l)\iota_l \subset E$ together with the primitive idempotents $id_{P_i} \in \mathrm{End}_\Lambda(P_i) \subset E$ (for all $i \in I$) generate the basic order*

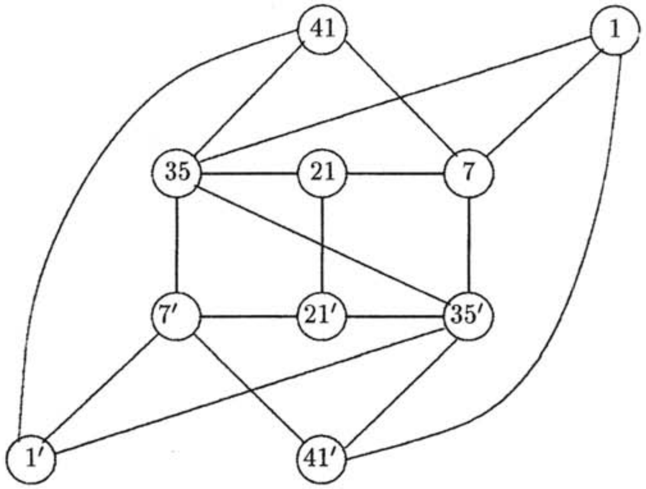
$$\Lambda_0 := \mathrm{End}_\Lambda(\oplus_{i\in I} P_i) \sim \Lambda$$

as $\mathbb{Z}_3$-order:

$a_{1,7} = 9\epsilon_8 + 3\epsilon_{84'} + 3\epsilon_{105} + \epsilon_{120} + \epsilon_{168'}$
$a_{1',7'} = 9\epsilon_{8'} + 3\epsilon_{84} + 3\epsilon_{105'} + \epsilon_{120'} + \epsilon_{168}$
$a_{1,35} = \epsilon_{105} + \epsilon_{120} + \epsilon_{120'} - \epsilon_{168'}$
$a_{1',35'} = \epsilon_{105'} + \epsilon_{120} + \epsilon_{120'} - \epsilon_{168}$
$a_{1,41'} = 3\epsilon_{42'} + 3\epsilon_{84'} + 3\epsilon_{120'} - 3\epsilon_{168'}$
$a_{1',41} = 3\epsilon_{42} + 3\epsilon_{84} + 3\epsilon_{120} - 3\epsilon_{168}$
$a_{7,21} = 3\epsilon_{28} + \epsilon_{105} + \epsilon_{168} + \epsilon_{168'}$
$a_{7',21'} = 3\epsilon_{28'} + \epsilon_{105'} + \epsilon_{168} + \epsilon_{168'}$
$a_{7,35'} = 3\epsilon_{84'} + \epsilon_{120} + \epsilon_{168} + 3\epsilon_{168'}$
$a_{7',35} = 3\epsilon_{84} + \epsilon_{120'} + 3\epsilon_{168} + \epsilon_{168'}$
$a_{7,41} = 3\epsilon_{48} + \epsilon_{105} + 3\epsilon_{120} + 3\epsilon_{168}$
$a_{7',41'} = 3\epsilon_{48'} + \epsilon_{105'} + 3\epsilon_{120'} + 3\epsilon_{168'}$
$a_{21,21'} = 3\epsilon_{42a} + \epsilon_{168} - 3\epsilon_{168'}$
$a_{21,35} = 9\epsilon_{56} + \epsilon_{105} + 3\epsilon_{168} + 3\epsilon_{168'}$
$a_{21',35'} = 9\epsilon_{56'} + \epsilon_{105'} + 3\epsilon_{168} + 3\epsilon_{168'}$
$a_{35,35'} = 9\epsilon_{70} - \epsilon_{120} - 3\epsilon_{120'} + \epsilon_{168} - 12\epsilon_{168'}$
$a_{35,41} = \epsilon_{84} + 3\epsilon_{105} + 3\epsilon_{120} + 3\epsilon_{168}$
$a_{35',41'} = \epsilon_{84'} + 3\epsilon_{105'} + 3\epsilon_{120'} + 3\epsilon_{168'}$

$a_{7,1} = \epsilon_8 + \epsilon_{84'} - \epsilon_{105} + 3\epsilon_{120} + 6\epsilon_{168'}$
$a_{7',1'} = \epsilon_{8'} + \epsilon_{84} - \epsilon_{105'} + 3\epsilon_{120'} + 6\epsilon_{168}$
$a_{35,1} = 3\epsilon_{105} + 3\epsilon_{120} + 3\epsilon_{120'} - 3\epsilon_{168'}$
$a_{35',1'} = 3\epsilon_{105'} + 3\epsilon_{120} + 3\epsilon_{120'} - 3\epsilon_{168}$
$a_{41',1} = \epsilon_{42'} + \epsilon_{84'} - \epsilon_{120'} + \epsilon_{168'}$
$a_{41,1'} = \epsilon_{42} + \epsilon_{84} - \epsilon_{120} + \epsilon_{168}$
$a_{21,7} = 3\epsilon_{28} + 3\epsilon_{105} + 3\epsilon_{168} - 3\epsilon_{168'}$
$a_{21',7'} = 3\epsilon_{28'} + 3\epsilon_{105'} - 3\epsilon_{168} + 3\epsilon_{168'}$
$a_{35',7} = \epsilon_{84'} + 3\epsilon_{120} + 3\epsilon_{168} + \epsilon_{168'}$
$a_{35,7'} = \epsilon_{84} + 3\epsilon_{120'} + \epsilon_{168} + 3\epsilon_{168'}$
$a_{41,7} = \epsilon_{48} + 3\epsilon_{105} - \epsilon_{120} - \epsilon_{168}$
$a_{41',7'} = \epsilon_{48'} + 3\epsilon_{105'} - \epsilon_{120'} - \epsilon_{168'}$
$a_{21',21} = \epsilon_{42a} + 3\epsilon_{168} - \epsilon_{168'}$
$a_{35,21} = \epsilon_{56} - 3\epsilon_{105} - \epsilon_{168} + 4\epsilon_{168'}$
$a_{35',21'} = \epsilon_{56'} - 3\epsilon_{105'} + 4\epsilon_{168} - \epsilon_{168'}$
$a_{35',35} = \epsilon_{70} + 3\epsilon_{120} + \epsilon_{120'} + 3\epsilon_{168} - \epsilon_{168'}$
$a_{41,35} = -3\epsilon_{84} + \epsilon_{105} + \epsilon_{120} - \epsilon_{168}$
$a_{41',35'} = -3\epsilon_{84'} + \epsilon_{105'} + \epsilon_{120'} - \epsilon_{168'}$

The Ext-quiver of $\Lambda/3\Lambda$ is obtained from the following graph, by replacing all edges by arrows in both directions:



A calculation of the radical series $(J^n)_{n\in\mathbb{N}}$, where $J = J(\Lambda)$ is the Jacobson radical of Λ shows the following:

Theorem 3.8

$$Z := 9(\epsilon_1 + \epsilon_{1'} + \epsilon_8 + \epsilon_{8'} + \epsilon_{28} + \epsilon_{28'} + \epsilon_{70} + \epsilon_{56} + \epsilon_{56'})$$

$$+3(\epsilon_{42a} + \epsilon_{42} + \epsilon_{42'} + \epsilon_{48} + \epsilon_{48'} + \epsilon_{84} + \epsilon_{84'} + \epsilon_{105} + \epsilon_{105'} + \epsilon_{120} + \epsilon_{120'} + \epsilon_{168} + \epsilon_{168'}),$$

then the Jacobson radical $J := J(\Lambda)$ satisfies

$$J^5 Z = J^7.$$

During the calculation of the radical series we obtain the Loewy structure of the projective indecomposable Λ-lattices, as well as the Loewy series of the projective indecomposable $\Lambda/3\Lambda$-modules ([Neb99], see also [Tan00]).

4 The radical series.

Let Λ be an R-order in the K-algebra $A = K\Lambda$ and $J := J(\Lambda)$ be the Jacobson radical of Λ.

Theorem 4.1 *(cf. [Kün99, Remark E.2.1]) The radical series $(J^n)_{n\in\mathbb{N}}$ becomes periodic, i.e. there are $m > n \in \mathbb{N}$ and a unit $a \in Z(A)$, such that $J^m = aJ^n$.*

Proof. The radical J and all its powers are two-sided ideals of Λ, hence $\Lambda - \Lambda$-bimodules. The $\Lambda - \Lambda$-bimodules in A are the sublattices for the order $\Gamma :=$

$\Lambda^{op} \otimes_{Z(\Lambda)} \Lambda$. By the Theorem of Jordan and Zassenhaus (cf. [Rei75, Theorem 26.4]) there are only finitely many isomorphism classes of Γ-lattices in A. Therefore there are $m > n \in \mathbb{N}$ such that $J^m \cong J^n$ as Γ-lattices. All Γ-isomorphisms are given by multiplication with units in $Z(A)$, hence there is $a \in Z(A)$ such that $J^m = aJ^n$. □

There are many open questions about this radical series.

Which isomorphism classes of bimodules occur as powers of J?

What is the period of the radical series (determine a)?

What is the period length?

In the examples in Section 3, the period length is either 1 ($\mathbb{Z}_2 S_4$) or 2. For the principal block of $\mathbb{Z}_3 M_{11}$, which has simple modules, that are not self dual, the period length is 3 (see [Neb99, 5.6.3]).

What is the pre-period length?

Give bounds for these lengths.

Is there an exponent n_0, such that $\epsilon_t J^n \subset J^n$ for all $1 \leq t \leq s$ and $n \geq n_0$?

In the examples in Section 3, there is such an exponent n_0. An example, where there is no such n_0, is the group ring $\Lambda := \mathbb{Z}_2[i]C_4$. The Jacobson radical $J := J(\Lambda)$ satisfies

$$J^3 = \{(a,b,c,d) \in 2(1+i)\mathbb{Z}_2[i]^4 \mid a \equiv b \pmod 4,\ c \equiv d \pmod 4\}$$

and $J^4 = (1+i)J^3$. Therefore J^n is indecomposable for all n.

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