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Harish-Chandra categories and Kostant's theorem

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Abstract: We give a survey of the theory of Harish-Chandra categories and its applications based on joint results with Yu.Drozd, A.Molev and S.Ovsienko.

1 Introduction

We fix an algebraically closed field $\Bbbk$ of characteristic 0. Let U be an associative $\Bbbk$-algebra, $\Gamma \subset U$ its subalgebra. Denote by $Irr(U)$ the set of isomorphism classes of irreducible U-modules in some category $\mathcal{C}$. One of the important problems in the representation theory is the relation between the sets $Irr(\Gamma)$ and $Irr(U)$. First of all we restrict ourselves to the category $\mathcal{C}$ of Harish-Chandra U-modules with respect to Γ, i.e. those U-modules that decompose into a direct sum of irreducible Γ-modules. In some cases we can use $Irr(\Gamma)$ to parametrize $Irr(U)$ in $\mathcal{C}$. Hence the first problem is the following:

Question 1: Let $V \in Irr(\Gamma)$. Can it be lifted to a module in $Irr(U)$?

In particular, we have a positive answer to this question when U is free over Γ as left (right) module and thus

Question 2: When is U free over Γ?

As an example of such situation consider the case $U = U(\mathfrak{g})$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, and $\mathrm{Z} \subset U$ is the center of $U(\mathfrak{g})$. The famous result of Kostant [K] establishes that $U(\mathfrak{g})$ is free over Z as left (right) module. Another example is provided by the result of Bernstein and Luntz ([BL]) who showed that the polynomial algebra $P(\mathfrak{g}) = S(\mathfrak{g}^*)$ of the Lie algebra $\mathfrak{g}$ is free over the invariants $P(\mathfrak{g})^G$, where G is the corresponding Lie group.

Suppose now that any element of $Irr(\Gamma)$ admits a lifting to an irreducible U-module. Then we can get a first approximation of the classification (up to a finite indeterminacy) of $Irr(U)$ if the number of such liftings is finite. It leads to the following problem:

Question 3: Given $V \in Irr(\Gamma)$ is the number of liftings to $Irr(U)$ finite?

For example if $U = U(\mathfrak{g})$ and $\Gamma = \mathrm{Z}$ is the center of U then there exist infinitely many isomorphism classes of irreducible U-modules with a given character of Γ.

Finally, in some cases the elements of $Irr(\Gamma)$ can be used not only to parametrize the elements of $Irr(U)$ but also to parametrize the basis elements of modules in

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$Irr(U)$. This is possible when $\dim V = 1$ for any $V \in Irr(\Gamma)$ and the multiplicity $(W : V)$ of V in any $W \in Irr(U)$ is less or equal to 1. Therefore we have

Question 4: Given $V \in Irr(\Gamma)$ and $W \in Irr(U)$ is $(W : V)$ finite?

There is a hope for a positive answer to Question 4 when Γ is "sufficiently" large, i.e. GKdim is maximal.

Examples. 1. (Okunkov,Vershik, [OV]) Let $U = U_n = \mathbb{C}[S_n]$, $U_1 \subset \ldots \subset U_n$ is a chain of natural embeddings, Z_k the center of $\mathbb{C}[U_k]$ for all $k = 1, \ldots, n$. Then $\Gamma =< Z_1, \ldots, Z_n >$ is a maximal commutative subalgebra of U which is generated by the Jucys-Murphy elements $X_i = (1i) + \ldots + (i-1i)$, $i = 1, \ldots, n$. In this case the elements of $Specm\Gamma$ parametrize $Irr(S_n)$.

2. Let $U = U(\mathfrak{gl}(n))$, $U_k = U(\mathfrak{gl}(k))$, $U_n \supset U_{n-1} \supset \ldots \supset U_1$,. Let $Z_k \subset U_k$ be the center of U_k and $\Gamma =< Z_1, \ldots, Z_n >$ (the *Gelfand-Tsetlin subalgebra of* U). It was shown in [DFO] that generically the elements of $Irr(\Gamma)$ parametrize $Irr(U)$. Moreover, it was shown by Ovsienko [O] that any element of $Irr(\Gamma)$ admits only finitely many liftings to $Irr(U)$.

3. Let $P(\mathfrak{g})$ be the polynomial algebra of a simple finite-dimensional Lie algebra $\mathfrak{g}$. We can view $P(\mathfrak{g})$ as a Lie algebra with respect to the Poisson bracket. Fomenko, Mischenko and Vinberg ([FM], [V]) constructed a series of commutative subalgebras in $P(\mathfrak{g})$ of maximal GKdim which leads to a family of maximal commutative subalgebras in $U(\mathfrak{g})$.

Under some conditions the positive answer to Question 2 gives also positive answers to Questions 3 and 4. This is related to the study of Harish-Chandra categories.

2 Harish-Chandra categories

Consider an associative algebra A and a subalgebra $\Gamma \subset A$. Let $\operatorname{cfs}\Gamma$ be the *cofinite spectrum* of Γ,

$$\operatorname{cfs}\Gamma = \{\mathbf{m} \in Specm\Gamma | \ \dim \Gamma/\mathbf{m} < \infty\}.$$

If $\mathbf{m} \in \operatorname{cfs}\Gamma$ then $\Gamma/\mathbf{m} \simeq M_{l(\mathbf{m})}(\mathbb{k})$. Denote by $L_{\mathbf{m}}$ the corresponding simple Γ-module, $\dim L_{\mathbf{m}} = l(\mathbf{m})$.

Let A-mod denote the categoryy of finitely generated left modules over an associative algebra A. The *Harish-Chandra category* $H(A,\Gamma)$ associated with the pair (A,Γ) is a full abelian subcategory in U-mod which consists of finitely generated A-modules M such that

$$M = \bigoplus_{\mathbf{m}\in \mathrm{cfs}\,\Gamma} M(\mathbf{m}),$$

where

$$M(\mathbf{m}) \;=\; \{x \in M \mid \exists k,\ \mathbf{m}^k x = 0\}.$$

The objects of the category $H(A,\Gamma)$ are called Harish-Chandra modules for the pair (A,Γ). Harish-Chandra modules play a central role in the classical representation theory (cf. [Di]). In particular, weight modules over a semisimple Lie algebra $\mathfrak{g}$ are Harish-Chandra modules with respect to the pair $(U(\mathfrak{g}), U(H))$, where $U(H)$ is the universal enveloping algebra of a Cartan subalgebra H. Another important example is given by the Gelfand–Tsetlin modules [DFO1] over the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$. These modules are Harish-Chandra modules with respect to the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_n)$.

A Harish-Chandra module M is called *weight* if the following condition holds: for all $\mathbf{m} \in \mathrm{cfs}\,\Gamma$ and all $x \in M(\mathbf{m})$ one has $\mathbf{m}\,x = 0$. The full subcategory of $H(A,\Gamma)$ consisting of weight modules will be denoted $HW(A,\Gamma)$. The *support* of a Harish-Chandra module M is the subset $\mathrm{Supp}\, M \subseteq \mathrm{cfs}\,\Gamma$ which consists of those $\mathbf{m}$ which have the property $M(\mathbf{m}) \neq 0$. If for a given $\mathbf{m}$ there exists an irreducible Harish-Chandra module M with $M(\mathbf{m}) \neq 0$ then we say that $\mathbf{m}$ *extends* to M. A central problem in the theory of Harish-Chandra modules is the existence and uniqueness of such extension. In the case when the extension is unique, the irreducible Harish-Chandra modules are parametrized by the equivalence classes of the elements of $\mathrm{cfs}\,\Gamma$. A theory of Harish-Chandra modules for a general pair (U,Γ) was developed in [DFO].

An effective tool for the study of the category $\mathbb{H}(A,\Gamma)$ is the notion of a Harish-Chandra subalgebra introduced in [DFO]. A subalgebra Γ is *quasi-commutative* if $\forall \mathbf{m} \neq \mathbf{n}$,

$$Ext^1_\Gamma(L_{\mathbf{m}}, L_{\mathbf{n}}) = 0,$$

and it is *quasi-central* if $\forall a \in A$, $\Gamma a \Gamma$ is finitely generated as left (right) Γ-module. A subalgebra Γ is called *Harish-Chandra subalgebra* if Γ is quasi-commutative and quasi-central.

Example. 1) Let $A = U(\mathfrak{g})$, $\Gamma = U(\mathfrak{a})$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{a}$ its reductive Lie subalgebra. Then Γ is a Harish-Chandra subalgebra in A. 2) Let $A = U(\mathfrak{gl}_n)$ and let Γ be the Gelfand-Tsetlin subalgebra of A. Then Γ is a Harish-Chandra subalgebra in A.

Define a category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ with the set of objects $\mathrm{Ob}\,\mathcal{A} = \mathrm{cfs}\,\Gamma$ and with the space of morphisms $\mathcal{A}(\mathbf{m},\mathbf{n})$ from $\mathbf{m}$ to $\mathbf{n}$,

$$\mathcal{A}(\mathbf{m},\mathbf{n}) \;=\; \lim_{\leftarrow n,m} U/(\mathbf{n}^n U + U\mathbf{m}^m).$$

The category $\mathcal{A}$ is endowed with the topology of the inverse limit while the category of $\Bbbk$-vector spaces ($\Bbbk$-mod) is endowed with the discrete topology. Consider the category $\mathcal{A}$-mod$_d$ of continuous functors (*discrete modules*) $M : \mathcal{A} \to \Bbbk$-mod.

Theorem 2.1. *([DFO], Theorem 17) The categories $\mathcal{A}$-mod$_d$ and $H(A,\Gamma)$ are equivalent.*

Now suppose that the subalgebra Γ of A is commutative. In this case $\mathrm{cfs}\,\Gamma$ coincides with the set $\mathrm{Specm}\,\Gamma$ of all maximal ideals in Γ. We endow this set with the Zariski topology. For $\mathbf{m} \in \mathrm{Specm}\,\Gamma$ denote by $\hat{\mathbf{m}}$ the completion of $\mathbf{m}$. Clearly, $\hat{\mathbf{m}} \subseteq \Gamma_{\mathbf{m}}$, where $\Gamma_{\mathbf{m}} = \lim_{\leftarrow k} \Gamma/\mathbf{m}^k$. Consider the two-sided ideal $I \subseteq \mathcal{A}$ generated by the completions $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \mathrm{Specm}\,\Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$. Theorem 2.1 implies the following.

Corollary 2.2. *The categories $\mathbb{HW}(U,\Gamma)$ and $\mathcal{A}_W$-mod are equivalent.*

Let $Irr(\mathbf{m})$ be the set of isomorphism classes of simple modules L in $H(A,\Gamma)$ with $L(\mathbf{m}) \neq 0$. Then the functor $F_{\mathbf{m}} : H(A,\Gamma) \to \mathcal{A}(\mathbf{m},\mathbf{m}) - mod$, $M \rightarrowtail M(\mathbf{m})$, induces a bijection between $Irr(\mathbf{m})$ and simple $\mathcal{A}(\mathbf{m},\mathbf{m})$-modules.

The subalgebra Γ is called *big in* $\mathbf{m} \in \mathrm{Specm}\,\Gamma$ if $\mathcal{A}(\mathbf{m},\mathbf{m})$ is finitely generated as a left (or, equivalently, right) $\Gamma_{\mathbf{m}}$-module.

Theorem 2.3. *([DFO], Corollary 19.) If Γ is big in $\mathbf{m} \in \mathrm{Specm}\,\Gamma$ then there exist finitely many non-isomorphic irreducible Harish-Chandra U-modules M such that $M(\mathbf{m}) \neq 0$. For any such module $\dim M(\mathbf{m}) < \infty$.*

This theorem gives sufficient conditions for positive answer to Question 3. Hence the general strategy in Harish-Chandra categories $H(A,\Gamma)$ with a commutative Γ should be the following:

- check whether A is free over Γ as a left (right) module. It gives a positive answer to Question 1.

- check whether Γ is a Harish-Chandra subalgebra and whether $\mathcal{A}(\mathbf{m},\mathbf{m})$ is finitely generated as $\Gamma_{\mathbf{m}}$-module. This gives a positive answer to Question 3.

Examples. 1) $A = U(\mathfrak{g})$, $\Gamma = U(H)$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, $H \subset \mathfrak{g}$ is a Cartan subalgebra. Then $\mathcal{A} = \oplus_{\xi \in H^*/Q} W_\xi$, where W_ξ is a ξ-weight lattice and Q is the root lattice.

2) Let G a finite group, $N < G$ a normal subgroup, $U = \Bbbk[G], \Gamma = \Bbbk[N]$, $\hat{N}$ the space of characters of N. Then G acts on N by conjugations and on $\hat{N}$ as follows: $\chi \mapsto \chi^g$, $\chi^g(x) = \chi(gxg^{-1})$, $x \in N$. Denote by $Y = Y(G, N)$ the groupoid with objects $\mathrm{Ob}Y = \hat{N}$ and morphisms $Y(\chi_1, \chi_2) = \{g \in G | \chi_1 = \chi_2^g\}$, $\chi_1, \chi_2 \in \hat{N}$. Let $X = Y/N$ and $\Bbbk X$ its linear envelope. Then $\mathcal{A} \simeq \Bbbk X$.

3 Kostant's Theorem

This section is based on the paper [FO]. Let U be an associative algebra with an increasing filtration $\{U_i\}_{i\in\mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, $U_iU_j \subseteq U_{i+j}$. Let $\overline{U} = \mathrm{gr}\ U = \overline{U} = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}$. Assume that U is *almost commutative*, i.e. there exists an epimorphism $\phi : \Bbbk[X_1, \dots, X_n] \to \overline{U}$. Besides, assume that $\mathrm{Ker}\,\phi$ is generated by a regular sequence and $\overline{U}$ has no nilpotent elements. Such algebra U is called *special filtered* in [FO].

Theorem 3.1. *([FO], Theorem 1) Let U be a special filtered associative algebra, $\Gamma = < g_1, \dots, g_k >$ a commutative subalgebra such that $\overline{g}_1, \dots, \overline{g}_k$ form a regular sequence in $\overline{U}$. Then U is free as a left (right) Γ-module.*

This theorem can be used to give an easy proof of Kostant's theorem [K].

3.1 *Case of* $\mathfrak{gl}_n$

Consider the full lineal Lie algebra $\mathfrak{gl}_n$ with the standard basis $E_{ij}, i, j = 1, \dots, n$. Let $U = U(\mathfrak{gl}(n))$ and $\Gamma = \Bbbk[z_1, \dots, z_n]$ is the center of U, where

$$z_m = \sum_{(i_1,\dots,i_m)\in\{1,\dots,n\}^m} E_{i_1i_2}E_{i_2i_3}\dots E_{i_mi_1}, \tag{3.1}$$

$m = 1, \dots, n$. Then U is filtered by the PBW theorem and $\overline{U} = S(\mathfrak{gl}(n)) = \Bbbk[x_{ij}, i, j = 1, \dots, n]$. Hence U is special filtered algebra. Consider an $n \times n$ matrix (x_{ij}) with entries x_{ij} and consider its characteristic polynomial

$$det(\lambda I - (x_{ij})) = \lambda^n + \bar{z}_1\lambda^{n-1} + \dots + \bar{z}_n.$$

Then the variety $V(\bar{z}_1, \dots, \bar{z}_n) \subset \Bbbk^{n^2}$ is a variety of nilpotent matrices by the Cayley-Hamilton theorem, where $\bar{z}_1, \dots, \bar{z}_n$ are the graded images of the generators of Γ. Since this variety is irreducible and $dimV = n^2 - n$ we immediately conclude that the sequence $\bar{z}_1, \dots, \bar{z}_n$ is regular and, hence, U is free over Γ by Theorem 3.1.

3.2 *Classical Kostant's theorem*

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra, $U = U(\mathfrak{g})$, Γ the center of U. Then U is special filtered. Let $I(\mathfrak{g})$ be the space of $\mathfrak{g}$-invariants in $S(\mathfrak{g})$.

Then $I(\mathfrak{g}) =< f_1, \ldots, f_d >$, where f_i are homogeneous algebraically independent elements. Since the variety $V(f_1, \ldots, f_d)$ is irreducible of dimension $dim\mathfrak{g} - d$ then the sequence $f_1, \ldots, f_d$ is regular. Let $\phi : S(\mathfrak{g}) \to U(\mathfrak{g})$ be a symmetrization map,

$$x \otimes y \rightarrowtail 1/2(x \otimes y + y \otimes x).$$

Then $\phi(I(\mathfrak{g})) \simeq \Gamma$ and $\Gamma = < \phi(f_1), \ldots, \phi(f_d) >$. Using the fact that $\overline{\phi(f_i)} = f_i$ and applying Theorem 3.1, we conclude that U is free as a module over Γ.

3.3 *Gelfand-Tsetlin modules*

Let $U = U(\mathfrak{gl}(n))$, $U_k = U(\mathfrak{gl}(k))$, $k = 1, \ldots, n$, $Z_k \subset U_k$ the center of U_k and $\Gamma =< Z_1, \ldots, Z_n >$ is the *Gelfand-Tsetlin* subalgebra. Then the Harish-Chandra category $H(U, \Gamma)$ is the category of Gelfand-Tsetlin modules. Note that Z_m is a polynomial algebra in m variables $\{c_{mk} \mid k = 1, \ldots, m\}$,

$$c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}, \tag{3.2}$$

and Γ is a polynomial algebra in $\dfrac{n(n+1)}{2}$ variables $\{c_{ij} \mid 1 \leqslant j \leqslant i \leqslant n\}$.

Theorem 3.2. *([O]) U is free over Γ.*

Let $X = (x_{ij})_{i,j=1}^n$ and $X_i = X_{1\ldots i}^{1\ldots i}$ is a principal matrix, $i = 1, \ldots, n$. The graded images $\bar{c}_{ij}$, $i \geq j$, of the generators c_{ij} are the coefficients of the polynomial $det(\lambda I_i - X_i) = 0$. Consider the variety $V = V(\bar{c}_{ij})$. It was shown in [O] that this variety is equidimensional and dimension of every irreducible component of V is $n(n-1)/2$. Hence the sequence $\bar{c}_{ij}, 1 \leqslant j \leqslant i \leqslant n$ is regular. By Theorem 3.1 U is free over Γ. It also follows from [O] that Γ is a a Harish-Chandra subalgebra. Moreover, Question 3 has a positive answer in the category $H(U, \Gamma)$.

Examples. 1) Let $n = 2$, $V = V(x_{11}, x_{22}, x_{12}x_{21})$. In this case V has two irreducible components of dim $= 1$.

2) Consider $T_n = \{(x_{ij}) | x_{ij} = 0, i \geq j\}$, $T_n \subset \mathbb{A}^{n^2}$, $\dim T_n = n(n-1)/2$. Then $\sigma T_n \sigma^{-1} \subset V$ for all $\sigma \in S_n \hookrightarrow GL_n$. Assume $n = 3$. In this case V has a component which not of type $\sigma T_n \sigma^{-1}$:

$$V(x_{11}, x_{22}, x_{33}, x_{12}, x_{21}, x_{31}x_{13} + x_{32}x_{23}).$$

4 Yangians

In this section we closely follow the work [FMO]. The *Yangian* $Y(n) = Y(\mathfrak{gl}(n))$ of $\mathfrak{gl}_n$ is a unital associative algebra over $\Bbbk$ with countably many generators

$t_{ij}^{(1)}$, $t_{ij}^{(2)}$, ... where $1 \le i, j \le n$, and the defining relations

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u), \tag{4.3}$$

where $t_{ij}(u) = \sum_{k=0}^{\infty} t_{ij}^{(k)} u^{-k} \in Y_n[[u^{-1}]]$ and u, v are formal variables. This algebra originally appeared in the works of Takhtajan-Faddeev [TF] and Kulish–Sklyanin [KS]. The term "Yangian" and generalizations of $\mathrm{Y}(\mathfrak{gl}_n)$ to an arbitrary simple Lie algebra were introduced by Drinfeld [D1]. Note that the universal enveloping algebra $U(\mathfrak{gl}_n)$ is a subalgebra of $Y(n)$ and also a homomorphic image of $Y(n)$.

Drinfeld generators ([D2])

$$a_i(u), \quad i = 1, \dots, n, \qquad b_i(u), \;\; c_i(u), \quad i = 1, \dots, n-1$$

of the algebra $Y(n)$ are defined as certain *quantum minors* of the matrix $\big(t_{ij}(u)\big)$. The coefficients of the series $a_i(u)$, $i = 1, \dots, n$ form a commutative subalgebra of $Y(n)$ and can be viewed as an analogue of the Gelfand–Tsetlin subalgebra of $U(\mathfrak{gl}_n)$. The Harish-Chandra modules for $Y(n)$ with respect to this subalgebra are natural analogues of Gelfand–Tsetlin modules for $\mathfrak{gl}_n$. In the theory of finite-dimensional representations of $Y(n)$ the important role is played by the following restricted version of the Yangian. Let p be an integer. The restricted Yangian $Y_p(\mathfrak{gl}_n)$, of level p has the generators $t_{ij}^{(k)}$, $i, j = 1, \dots, n$, $k = 1, \dots, p$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] =$$

$$\frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)),$$

where

$$T_{ij}(u) = \delta_{ij}\, u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\, u^{p-k}.$$

We can define a filtration on $Y_p(\mathfrak{gl}_n)$ by assigning $deg\, t_{ij}^{(k)} = k$. We also have the following analogue of the PBW theorem for the algebra $\mathrm{Y}_p(\mathfrak{gl}_n)$ which shows that $\mathrm{Y}_p(\mathfrak{gl}_n)$ is a special filtered algebra.

Proposition 4.1. *([C], [M]) The associated graded algebra $\overline{\mathrm{Y}}_p(\mathfrak{gl}_n) = \mathrm{gr}\mathrm{Y}_p(\mathfrak{gl}_n)$ is a polynomial algebra in variables $\bar{t}_{ij}^{(k)}$, $i, j = 1, \dots, n$, $k = 1, \dots, p$.*

Set $\mathrm{T}(u) = (T_{ij}(u))_{i,j=1}^{n}$ and consider the following element in $\mathrm{Y}_p(\mathfrak{gl}_n)[u]$, called *quantum determinant*

$$\operatorname{qdet} \mathrm{T}(u) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, T_{1\sigma(1)}(u) T_{2\sigma(2)}(u-1) \dots T_{n\sigma(n)}(u-n+1). \tag{4.4}$$

The coefficients d_s of the powers u^{np-s}, $s = 1, \dots, np$ of $\operatorname{qdet} \mathrm{T}(u)$ are algebraically independent generators of the center Γ of $\mathrm{Y}_p(\mathfrak{gl}_n)$ ([C], [M]).

Theorem 4.2. *([FO]) For all $n \geq 2, p \geq 1$ the restricted Yangian $\mathrm{Y}_p(\mathfrak{gl}_n)$ is a free module over its center.*

Remark. 1. If $p = 1$ then Theorem 4.2 is the classical Kostant's theorem;

2. If $p = \infty$ then the statement of the theorem was proved by Molev, Nazarov and Olshansky in [MNO];

3. When $p = 2$ the result of the theorem is due to Geoffriau [G].

4.1 Yangian $Y_p(\mathfrak{gl}_2)$

Consider now the Yangian $Y_p(\mathfrak{gl}_2)$. Drinfeld generators $a_1(u)$ and $a_2(u)$ are defined in this case by

$$a_1(u) = t_{11}(u)\, t_{22}(u-1) - t_{21}(u)\, t_{12}(u-1), \qquad a_2(u) = t_{22}(u). \tag{4.5}$$

Denote $\Gamma = < d_i, t_{22}^{(k)} >$, $i = 1, \ldots, 2p$, $k = 1, \ldots, p$.

Theorem 4.3. *([FMO]) 1. $Y_p(\mathfrak{gl}_2)$ is a free module over the subalgebra Γ;*

2. Γ is a Harish-Chandra subalgebra;

3. Γ is a maximal commutative subalgebra of $Y_p(\mathfrak{gl}_2)$;

4. Every character of Γ defines finitely many irreducible Harish-Chandra $Y_p(\mathfrak{gl}_2)$-modules.

Theorem 4.2 gives positive answers to Questions 1 and 3 of the Introduction in the category $H(U, \Gamma)$.

Example. If $p = 2$ then the variety

$$V = V(\bar{d}_i, \bar{t}_{22}^{(k)}, i = 1, \ldots, 2p, k = 1, \ldots, p)$$

has three irreducible components of dim $= 3$:

$$x_{11}^{(1)} = x_{11}^{(2)} = x_{22}^{(1)} = x_{21}^{(1)} = x_{21}^{(2)} = 0,$$

$$x_{11}^{(1)} = x_{11}^{(2)} = x_{22}^{(1)} = x_{12}^{(1)} = x_{12}^{(2)} = 0,$$

$$x_{11}^{(1)} = x_{11}^{(2)} = x_{22}^{(1)} = x_{12}^{(2)} = x_{21}^{(2)} = 0,$$

where $x_{ij} = \bar{t}_{ij}$.

Finally, we can establish sufficient conditions that guarantee positive answer to Question 4 of the Introduction for modules in $H(U, \Gamma)$.

Suppose $\mathbf{m} \in \operatorname{Specm}\Gamma$ is generated by $t_{22}(u) - \beta(u)$ and $D(u) - \gamma(u)$, where

$$\beta(u) = \prod_{i=1}^{p}(u + \beta_i), \gamma(u) = \prod_{i=1}^{2p}(u + \gamma_i).$$

We say that $\mathbf{m}$ is *generic* if $\beta_i - \beta_j \notin \mathrm{Z}$.

Theorem 4.4. (*[FMO]*) *If* $\mathbf{m}$ *is generic then* $\mathcal{A}(\mathbf{m},\mathbf{m}) = \Gamma_{\mathbf{m}}$ *and hence there exists a unique irreducible* $V \in H(U,\Gamma)$ *with* $V(\mathbf{m}) \neq 0$. *Moreover,* $\dim V(\mathbf{m}) = 1$.

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