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New simple Lie algebras over fields of characteristic 2

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1 Introduction

Lie algebras over fields of characteristic 0 or $p > 3$ were recently classified, but over field of characteristic 2 or 3 there are only partial results up to now. The main result on this matter was obtained by S. Skryabin [Sk]. He proved that any finite dimensional simple Lie algebra over a field of characteristic 2 has toroidal rank ≥ 2.

By definition a Lie algebra over a field of characteristic 2 is a 2-algebra if there exists a map $L \to L,\ x \to x^{[2]}$ such that $(x + x^{[2]})^{[2]} = x^{[2]} + x^{[4]}, x \in L$, $(x+y)^{[2]} = x^{[2]} + y^{[2]} + [x,y],\ \forall\, x,y \in L$.

Recall that the toroidal rank $t(L)$ of a Lie 2-algebra without center L over a field k of characteristic 2 is the maximal dimension of an abelian subalgebra with basis $\{t_1, ..., t_n\}$ such that $t_i^{[2]} = t_i, i = 1, ..., n$, where $n = t(L)$.

The next step in the classification of such Lie algebras was done in [GP] where the simple Lie 2-algebras of finite dimension over a field k of characteristic 2 and toroidal rank 2 were classified. The toroidal rank 3 case is much more difficult. For this case the following is still an open problem.

Problem. Classify the simple Lie algebras (or 2-algebras) over a field k of characteristic 2 and toroidal rank 3 which contains a Cartan subalgebra of dimension 3.

This Problem is easier than the classification of the simple Lie algebras over a field k of toroidal rank 3, but far away from being trivial. The main obstacle is the lack of examples.

In the first part of this work we construct an example of a simple Lie algebra of dimension 31 and of toroidal rank 3. We expect that this example will be useful for the construction of other simple Lie algebra of toroidal rank 3 containing a CSA of dimension 3. In the last section a series of new simple Lie algebras over k is constructed.

2 A First Example

We first recall the construction of a simple Lie 2-algebra L of dimension 31 which was made in [GP]. A basis of L has two parts W e V such that $|W| = 15,\ |V| = 16$

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and

$$\begin{aligned} W &= \{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4;\, t, h,\, m_{12}, m_{24}, m_2^3, m_1^3, m_2^4\} &(1)\\ V &= \{\sigma | \sigma \subseteq I = (1234)\}. &(2)\end{aligned}$$

The multiplication of these basis elements are given by the following formulae:

$$[t,h] = 0,\ [x,h] = 0,\ [x,t] = x,\ \text{for } x \in \{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\},$$

$$[x,t] = [x,h] = 0,\ \text{for } x \in T = \{m_{12}, m_{24}, m_2^3, m_1^3, m_2^4\},\ [T,T] = 0,$$

$$[y,h] = y,\ [y,t] = |y|y,\ \text{for } y \in V,$$

$$[e_i, e_j] = 0,\ [e_i, f_j] = \delta_{ij} h,\ \forall\, (ij) \neq (32),\ [e_3, f_2] = m_{12},$$

$$[f_i, f_j] = 0,\ \forall\, (ij) \neq (12),\ [f_1, f_2] = m_2^3.$$

The products $[T,V]$ e $[T,W]$ are given by

$$[f_i, m_i^j] = f_j,\ \text{if } i < j,\ [f_i, m_{ij}] = e_j,\ [f_j, m_{ij}] = e_i,\ [e_j, m_i^j] = e_i,\ \text{if } i < j,$$

$$[\sigma, m_i^j] = (\sigma \cup j) \setminus i,\ \text{for } i \in \sigma,\ j \notin \sigma,$$

$$[\sigma, m_{ij}] = \sigma \setminus (ij),\ \text{for } (ij) \subseteq \sigma$$

and the other products $[T,V], [T,W]$ are equal to zero.

Besides we have

$[\emptyset, f_1] = 1,$	$[\emptyset, f_2] = 2,$	$[\emptyset, f_3] = 3,$	$[\emptyset, f_4] = 4$
$[1, f_1] = 0,$	$[1, f_2] = 12,$	$[1, f_3] = 13,$	$[1, f_4] = 14,$
$[2, f_1] = 12,$	$[2, f_2] = 0,$	$[2, f_3] = 23,$	$[2, f_4] = 24,$
$[3, f_1] = 13,$	$[3, f_2] = 23,$	$[3, f_3] = 0,$	$[3, f_4] = 34,$
$[4, f_1] = 14,$	$[4, f_2] = 24,$	$[4, f_3] = 34,$	$[4, f_4] = 0,$
$[12, f_1] = 0,$	$[12, f_2] = 3,$	$[12, f_3] = 123,$	$[12, f_4] = 124,$
$[13, f_1] = 0,$	$[13, f_2] = 123,$	$[13, f_3] = 0,$	$[13, f_4] = 134,$
$[14, f_1] = 0,$	$[14, f_2] = 124,$	$[14, f_3] = 134,$	$[14, f_4] = 0,$
$[23, f_1] = 123,$	$[23, f_2] = 0,$	$[23, f_3] = 0,$	$[23, f_4] = 234,$
$[24, f_1] = 124,$	$[24, f_2] = 0,$	$[24, f_3] = 234,$	$[24, f_4] = 0,$
$[34, f_1] = 134,$	$[34, f_2] = 234,$	$[34, f_3] = 0,$	$[34, f_4] = 0,$
$[123, f_1] = 0,$	$[123, f_2] = 0,$	$[123, f_3] = 0,$	$[123, f_4] = I,$
$[124, f_1] = 0,$	$[124, f_2] = 34,$	$[124, f_3] = I,$	$[124, f_4] = 0,$
$[134, f_1] = 0,$	$[134, f_2] = I,$	$[134, f_3] = 0,$	$[134, f_4] = 0,$
$[234, f_1] = I,$	$[234, f_2] = 0,$	$[234, f_3] = 0,$	$[234, f_4] = 0$
$[I, f_1] = 0,$	$[I, f_2] = 0,$	$[I, f_3] = 0,$	$[I, f_4] = 0.$

$$[\sigma, e_i] = \sigma \setminus i,\ \text{for } i \in \sigma;\ [\sigma, e_i] = 0,\ \text{for } i \notin \sigma.$$

$$\pi \cdot \psi = \begin{cases} f_i, & \pi \cap \psi = i,\ \pi \cup \psi = I; \\ e_i, & \pi \cap \psi = \emptyset,\ \pi \cup \psi = I \setminus i; \\ h + |\pi| t, & \pi \cap \psi = \emptyset,\ \pi \cup \psi = I. \end{cases}$$

$$[12, 24] = m_{12},\ [I, 12] = m_2^3,\ [12, 124] = e_2,$$

$$[2, 124] = m_{12},\ [123, 124] = m_2^3,$$

and the other products are $[\sigma, \mu] = 0$, for $\sigma,\ \mu \subseteq I$.

It is easy to see that $\dim L = 31$ and $\dim L^2 = 28$. Now we define a 2-operation on the algebra L given by

$$f_2^{[2]} = m_1^3,\ (12)^{[2]} = m_{24},\ (124)^{[2]} = m_2^4,\ t^{[2]} = t,\ h^{[2]} = h,$$

and $a^{[2]} = 0$ for all other $a \in V \cup W$.

The algebra L has a subalgebra K with a basis $\{t, h, m_{12}, m_{24}, m_2^4, m_2^3, m_1^3\}$. This Cartan subalgebra is not toroidal and has toroidal rank 2. On the other hand, the algebra L has another Cartan subalgebra H with basis $\{x, y = x^{[2]}, z = x^{[4]}\}$, where $x = t + m_1^3 + (12) + (124)$. It is an easy calculation to prove that $z^{[2]} = x^{[8]} = z + x$. We note that $H \cap L^2 = 0$, whence $L = H \oplus L^2$.

Let F be the splitting field of the polynomial $p(s) = s^7 + s^3 + 1$ over F_2, the field of two elements. It is clear that $|F| = 2^7$. Denote by $\Lambda = \{\lambda_1, ..., \lambda_7\}$ the set of all roots of $p(s)$. Then $\Lambda \cup \{0\}$ is an additive group isomorphic to $\mathbf{Z}_2^3$.

The first goal is to find a Cartan decomposition of the algebra L in relation to the subalgebra H. For this we consider the adjoint action of x on L and calculate the eigenspaces $A_i = \{v \in L \,/\, [v, x] = \lambda_i v\}$. The table below shows the action of x on the basis elements.

v	$[v, x]$	v	$[v, x]$
e_1	$e_1 + (2) + (24)$	(2)	$(2) + m_{12}$
e_2	$e_2 + (1) + (14)$	(3)	$(3) + e_4 + t + h$
e_3	$e_1 + e_3$	(4)	$(4) + e_3$
e_4	$e_4 + (12)$	(12)	$(23) + e_2$
f_1	$f_1 + f_3$	(13)	f_1
f_2	$f_2 + (3) + (34)$	(14)	(34)
f_3	$f_3 + (123) + (1234)$	(23)	f_2
f_4	$f_4 + (124)$	(24)	m_{12}
h	$(12) + (124)$	(34)	$t + f_4$
t	(124)	(123)	$(123) + m_2^3$
m_2^3	$(13) + (134)$	(124)	$(234) + (124) + e_2$
m_{12}	$\emptyset + (4)$	(134)	$(134) + f_1$
$\emptyset$	e_3	(234)	$(234) + f_2$
(1)	$(1) + (3)$	(1234)	m_2^3

If $v = \alpha_i e_i + \sum_{j=1}^{4} \beta_j f_j + \theta h + \epsilon t + \eta m_2^3 + \delta m_{12} + \sum_{\sigma \subseteq \{1,2,3,4\}} d_\sigma \sigma$ is a generic element of L then, for each $\lambda_i \in F$, the eigenspace A_i has the following basis (here $\lambda = \lambda_i$):

$$\begin{aligned}
\omega_1^i &= \lambda^2(\lambda+1)e_1 + \lambda^2(\lambda+1)^2 e_3 + m_{12} + \lambda^{-1}\emptyset + \lambda^2(2) + (\lambda+1)^{-1}(4) \\
&+ \lambda(\lambda+1)(24), \\
\omega_2^i &= \lambda^2(\lambda+1)^2 f_1 + \lambda^2(\lambda+1) f_3 + m_2^3 + \lambda^{-1}(13) + \lambda^2(123) + (\lambda+1)^{-1}(134) \\
&+ \lambda(\lambda+1)(1234), \\
\omega_3^i &= \lambda^2(\lambda+1)^2 e_2 + \lambda(\lambda+1)^{-1} e_4 + \lambda(\lambda+1)^2 f_2 + t + h + \lambda^2(\lambda+1)(1) \\
&+ \lambda(3) + ((\lambda+1)\lambda)^{-1}(12) + \lambda(\lambda+1)^2(14) + (\lambda+1)^3\lambda(23), \\
\omega_4^i &= (\lambda+1)\lambda^3 e_2 + \lambda^3 f_2 + \lambda(\lambda+1)^{-1} f_4 + t + \lambda^3(1) + (\lambda+1)\lambda^2(14) \\
&+ \lambda(34) + (\lambda+1)^{-2}(124) + (\lambda+1)\lambda^3(234).
\end{aligned} \tag{3}$$

Theorem 2.1. *The algebra L described above has the following Cartan decomposition*

$$L = H \oplus \sum_{i=1}^{7} \oplus A_i,$$

where $A_i = \{v \in L | [v, x] = \lambda_i v\}$ has a basis $\{\omega_1^i, \omega_2^i, \omega_3^i, \omega_4^i\}$ given by (3). Moreover, if $\lambda_i + \lambda_j = \lambda_k$, then the basis elements multiply as follows

$$\begin{aligned}
[\omega_1^i, \omega_2^j] &= \lambda_i^2\lambda_j^2\lambda_k^3(\lambda_k+1)\,\omega_3^k + \frac{\lambda_i(\lambda_i+1)\lambda_j(\lambda_j+1)}{\lambda_k^2(\lambda_k+1)}\,\omega_4^k \in F(\omega_3^k,\, \omega_4^k), \\
[\omega_1^i, \omega_3^j] &= \lambda_i(\lambda_i+1)\lambda_j(\lambda_j+1)^2\lambda_k(\lambda_k+1)\,\omega_1^k \in F(\omega_1^k), \\
[\omega_1^i, \omega_4^j] &= \lambda_i^2\lambda_j^3\lambda_k^2\,\omega_1^k \in F(\omega_1^k), \\
[\omega_2^i, \omega_3^j] &= \lambda_i(\lambda_i+1)^2\lambda_j(\lambda_j+1)\lambda_k(\lambda_k+1)\,\omega_2^k \in F(\omega_2^k), \\
[\omega_2^i, \omega_4^j] &= \lambda_i^2\lambda_j^3\lambda_k^2\,\omega_2^k \in F(\omega_2^k), \\
[\omega_3^i, \omega_4^j] &= \tfrac{\lambda_i\lambda_j^3\lambda_k^3(\lambda_k+1)}{\lambda_i+1}\omega_3^k + \frac{\lambda_i\lambda_j^6(\lambda_j+1)^3}{(\lambda_i+1)(\lambda_k+1)}\,\omega_4^k \in F(\omega_3^k,\, \omega_4^k), \\
[\omega_1^i, \omega_1^j] &= [\omega_2^i, \omega_2^j] = 0, \\
[\omega_3^i, \omega_3^j] &= \lambda_k^2(\lambda_k+1)^2\lambda_i^3(\lambda_i+1)^2\omega_3^k + \frac{(\lambda_i+1)[(\lambda_j+1)^3 + \lambda_i^2\lambda_k^2]}{\lambda_i\lambda_k^3}\,\omega_4^k \in F(\omega_3^k,\, \omega_4^k), \\
[\omega_4^i, \omega_4^j] &= \lambda_i^3\lambda_j^3\lambda_k\,\omega_4^k \in F(\omega_4^k).
\end{aligned}$$

Proof: Note that $[A_i, A_i] = 0$, as the nilradical of H is zero because H has toroidal rank 3. The proof goes through easy but lengthy calculations with the basis elements, verifying that the identities listed above hold. □

Note that the basis $\{\omega_1^i, \omega_2^i, \omega_3^i, \omega_4^i\}$ of each subspace A_i is not defined over the field $\mathbf{Z}_2$, but over F. By Theorem 13 [J] (p. 192) the Cartan subalgebra H has a toroidal basis $\{t_1, t_2, t_3\}$, that is, $t_i^{[2]} = t_i$, for $i = 1, 2, 3$. Hence, for each $v \in A_i$, we have $[v, t_j] = a\,v$, where $a \in \mathbf{Z}_2$ and it does not depend on v, only on i e j. To find such a $\mathbf{Z}_2$-basis is not and easy task.

It is also easy to prove that

$$(\omega_1^i)^{[2]} = (\omega_2^i)^{[2]} = 0, \qquad [\omega_1^i, \omega_2^j]^{[2]}, \quad (\omega_3^i)^{[2]}, \quad (\omega_4^i)^{[2]} \in H,$$

hence $A_i^{[2]} \subseteq H$ and $A_i^{[2]} = \varphi_i(A_i)$ where $\varphi_i : A_i \longrightarrow H$ is such that $y \longmapsto y^{[2]}$ and $ker\, \varphi_i = < \omega_1^i, \omega_2^i >$, hence $\dim \varphi_i(A_i) = 2$.

From now on we use the following notation: $d^{\alpha}_{\alpha+\beta} = [\omega_1^{\alpha}, \omega_2^{\beta}]$. Note that $d^{\alpha}_{\alpha+\beta} = d^{\beta}_{\alpha+\beta}$ and consider the algebra

$$S = < d^{\alpha}_{\alpha+\beta} \,/\, \alpha, \beta \in \{\lambda_i \,|\, i = 1, \ldots, 7\} >$$

where the generators satisfy the following relations

$$[d^{\beta}_{\alpha}, d^{\alpha}_{\lambda}] = \begin{cases} d^{\alpha}_{\alpha+\lambda} & \text{if } \lambda \notin \{\alpha, \beta, \alpha+\beta\} \\ 0 & \text{if } \lambda \in \{\alpha, \beta, \alpha+\beta\} \end{cases}$$

and if $\{\alpha, \beta, \lambda\}$ and $\{\alpha, \tau, \lambda\}$ are linearly independent sets, then

$$[d^{\beta}_{\alpha}, d^{\tau}_{\lambda}] = \begin{cases} d^{\beta}_{\alpha+\lambda} & \text{if } \tau = \beta \text{ or } \beta = \lambda \\ d^{\beta+\alpha}_{\alpha+\lambda} & \text{if } \tau = \alpha+\beta \text{ or } \tau = \alpha+\beta+\lambda\,. \end{cases}$$

Proposition 2.1. *The algebra S described above is a simple Lie algebra defined over a field of two elements.*

Note that S is not a new simple Lie algebra, it is a special Lie algebra of Cartan type.

3 A more generic construction

On the construction of the algebra made in the first section, a pattern was identified which motivated a construction of more generic algebras as we describe in this section.

Let F_n be the finite field of 2^n elements and $U = F_n^3$. Define a "determinant form"(anti-symmetric and trilinear) $(\ \ ) : U \wedge U \wedge U \longrightarrow F_2$ by $a \wedge b \wedge c \longmapsto det(a, b, c)$.

Let V and W be vector spaces over k with bases $B = \{a \,|\, a \in U^*\}$ and $\overline{B} = \{\overline{a} \,|\, a \in U^*\}$, respectively, where $U^* = U \setminus \{0\}$. Note that $\dim V = \dim W = 2^{3n} - 1$. Let A_n be the algebra generated by the transformations of $V \oplus W$ defined on the basis $B \cup \overline{B}$ by

$$v\, d_a^b = (a \wedge b \wedge v)\,(v + a) \qquad \text{and} \qquad \overline{v}\, d_a^b = (a \wedge b \wedge \overline{v})\, \overline{v + a}\,.$$

Lemma 3.1. *For $a, b, c, g \in B$, with $a + c \neq 0$, there exists $s \in B$ such that*

$$[d_a^b, d_c^g] = d^s_{a+c} = d_a^b\, d_c^g + d_c^g\, d_a^b \tag{4}$$

Proof: For all $y \in B$, we have on one hand

$$\begin{aligned}(y\,d_a^b)\,d_c^g + (y\,d_c^g)\,d_a^b &= (y\wedge a\wedge b)\,(y+a)\,d_c^g + (y\wedge c\wedge g)\,(y+c)\,d_a^b\\ &= (y\wedge a\wedge b)\,((y+a)\wedge c\wedge g)\,(y+a+c)\\ &\quad+(y\wedge c\wedge g)\,((y+c)\wedge a\wedge b)\,(y+a+c)\\ &= [(y\wedge a\wedge b)\,(a\wedge c\wedge g) + (y\wedge c\wedge g)\,(c\wedge a\wedge b)]\,(y+a+c).\end{aligned}$$

On the other hand, $y\,d_{a+c}^s = (y\wedge(a+c)\wedge s)\,(y+a+c)$. Note that both scalars (operators) in front of the vector $(y+a+c)$ are linear on y and $a+c$ belongs to both kernels and the images of the other basis vectors are the same. Besides note that s is not unique as $s+a+c$ also satisfies (4). □

Corollary 3.1. *The algebra S_n of transformations $< d_a^b \,|\, a, b \in B >$ is a simple Lie algebra over k of dimension $2(2^{3n}-1)$.*

Consider $L_n = V \oplus A \oplus W$ and define the operations $[a,\overline{b}] = d_{a+b}^a = [\overline{a}, b]$ for all $a,\, b \in B, \overline{a},\, \overline{b} \in \overline{B},\ v \in V,\ w \in W$. Moreover, $V^2 = W^2 = 0$, that is, $[v_1, v_2] = 0$ and $[w_1, w_2] = 0$, for all $v_i \in V,\ w_i \in W$.

Lemma 3.2. *For the algebra A and the vector spaces V and W described above, we have*

$$[V,\,W]\cdot A = [V\cdot A,\,W] + [V,\,W\cdot A]. \tag{5}$$

Proof: To prove (5) we will show that

$$[[v,\,d_a^b],\,w] + [[d_a^b, w],\,v] + [[w,\,v],\,d_a^b] = 0. \tag{6}$$

The left hand side of (6) is equal to $(v\wedge a\wedge b)[v+a,\,w] + (a\wedge b\wedge w)[a+w,\,v] + [d_{v+w}^v,\,d_a^b]$ which applied to a vector $u \in V$ gives us (below $X = u+v+a+w$)
$(v\wedge a\wedge b)\,u\,d_{v+a+w}^w + (a\wedge b\wedge w)\,u\,d_{a+w+v}^v + (u\,d_{v+w}^v)\,d_a^b + (u\,d_a^b)\,d_{v+w}^v =$
$(v\wedge a\wedge b)\,(u\wedge(v+a+w)\wedge w)\,X + (a\wedge b\wedge w)\,(u\wedge(a+w+v)\wedge v)\,X +$
$(u\wedge(v+w)\wedge v)\,(u+v+w)\,d_a^b + (u\wedge a\wedge b)\,(u+a)\,d_{v+w}^v =$
$(v\wedge a\wedge b)\,(u\wedge(v+a)\wedge w)\,X + (a\wedge b\wedge w)\,(u\wedge(a+w)\wedge v)\,X +$
$(u\wedge w\wedge v)\,((u+v+w)\wedge a\wedge b)\,X + (u\wedge a\wedge b)\,((u+a)\wedge w\wedge v)\,X$

Now using linearity and anti-symmetry we can reduce the coefficient of X to

$$\underbrace{(v\wedge a\wedge b)\,(u\wedge a\wedge w)}_{(i)} + \underbrace{(a\wedge b\wedge w)\,(u\wedge a\wedge v)}_{(ii)} + \underbrace{(u\wedge a\wedge b)\,(a\wedge w\wedge v)}_{(iii)}. \tag{7}$$

Now if $v \in < a,\, b >$ then (7) is equal to zero, so we can suppose that $v \notin < a,\, b >$ and in this case $(v\wedge a\wedge b) = 1$. Hence we need to prove that

$$(u\wedge a\wedge w) = (a\wedge b\wedge w)\,(u\wedge a\wedge v) + (u\wedge a\wedge b)\,(a\wedge w\wedge v). \tag{8}$$

Note that both sides of (8) are linear on w, therefore, as $\{a,\, v,\, b\}$ is a basis of V it is enough to prove (8) for this basis, what is trivial. □

As a corollary of this lemma we get:

Theorem 3.1. *The algebra L_n together with the operations described above is a simple Lie algebra of dimension $4(2^{3n}-1)$, with a basis given by the union of the bases of V, W and A_n. The toroidal rank of L_n is $3n$ and L_1 is isomorphic to the Lie algebra of dimension 28 from the beginning of this paper.*

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