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Irreducible conformal subalgebras of Cend_N and gc_N

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Abstract: We present classification of irreducible conformal subalgebras of the associative conformal algebra Cend_N, and discuss analogous problem for its adjoint Lie conformal algebra gc_N.

1 Introduction

The notion of conformal algebra appeared in the frame of mathematical physics (conformal field theory). Roughly speaking, conformal algebra is a "restriction" of vertex algebra: conformal operations encode singular part of operator product expansion (see [8] for details).

Algebraical definition of conformal algebra is a formalization of some machinery in the theory of local fields [9] (here the word "field" means formal distribution). This construction could be naturally generalized in the frame of pseudotensor categories [2]; the notion of pseudoalgebra [1] obtained has certain relations with quantum groups, Poisson algebras, etc.

The idea of this generalization is to replace a field $\mathbb{F}$ with a (cocommutative) Hopf algebra H, consider H-module C instead of vector space A over $\mathbb{F}$, and turn $\mathbb{F}$-linear multiplication $A\otimes A \to A$ into $(H\otimes H)$-linear map $C\otimes C \to (H\otimes H)\otimes_H C$. In particular, the case $H = \mathbb{F}[D]$ (with the canonical Hopf algebra structure) corresponds to conformal algebras.

The main object of our consideration is the conformal algebra Cend_N of conformal endomorphisms of free N-generated module V_N, $N \geq 1$ (see, e.g., [3, 4, 5, 9]). This is an analogue of endomorphism algebra $\mathrm{End}_N \simeq M_N(\mathbb{F})$ of N-dimensional vector space over a field $\mathbb{F}$. The important difference between usual and conformal algebras is that Cend_N is infinite (i.e., infinitely generated over $H = \mathbb{F}[D]$; that would mean that infinite dimensional algebra has finite dimensional faithful representation, in the language of usual algebras).

Moreover, Cend_N contains proper irreducible subalgebras in contrast to End_N (c.f. classical theorem by Burnside or, more generally, the Density theorem by Jacobson [7]).

Conformal analogue of Burnside theorem was conjectured by V. Kac [9]; it was partially proved in [3] (for finite subalgebras and for $N = 1$ as well). Here we prove this conjecture in general and develop structure theory of associative conformal algebras with finite faithful representation.

We preferably use the language of pseudoalgebras in order to simplify some

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observations in conformal algebras. In the last section, we translate another conjecture from [3]: on irreducible conformal subalgebras of Lie conformal algebra $\mathrm{gc}_N = \mathrm{Cend}_N^{(-)}$.

Throughout the paper, $\mathbb{F}$ be an algebraically closed field of zero characteristic, $\mathbb{Z}_+$ be the set of non-negative integers, and ∂_x denotes the usual derivation with respect to a variable x.

2 Pseudoalgebras and conformal algebras

Let H be a cocommutative Hopf algebra, and let C be a left H-module. An H-bilinear map

$$*: C \otimes C \to (H \otimes H) \otimes_H C$$

is called pseudoproduct (we mean $H \otimes H$ and higher tensor powers of H to be endowed with outer product module structure over H).

Definition 1. An H-module C endowed with pseudoproduct $*$ is called H-*pseudoalgebra*. Pseudoalgebra C is said to be *finite*, if C is a finitely generated H-module.

Since we prefer using the language of pseudoalgebras but avoid presenting formal definition of the corresponding pseudotensor category [1, 2], it is necessary to adduce the expression for composition of pseudoproducts. Let C be an H-pseudoalgebra with pseudoproduct $*$. Then the map $*$ could be extended as follows (c.f. [1]):

$$\begin{gathered} *: (H^{\otimes n} \otimes_H C) \otimes (H^{\otimes m} \otimes_H C) \to (H^{\otimes n+m} \otimes_H C), \\ (F \otimes_H a) * (G \otimes_H b) = (F \otimes G \otimes_H 1)(\Delta^{n-1} \otimes \Delta^{m-1} \otimes_H \mathrm{id})(a * b), \\ F \in H^{\otimes n},\ G \in H^{\otimes m}, \end{gathered}$$

where $\Delta^{k-1}(f) = f_{(1)} \otimes \cdots \otimes f_{(k)}$, $f \in H$, is the iterated coproduct.

Let us consider two simplest cases: $H = \mathbb{F}$ (one-dimensional Hopf algebra), and $H = \mathbb{F}[D]$ (universal enveloping of one-dimensional Lie algebra). In the first case, the definition of pseudoalgebra coincides with usual definition of an algebra over the field $\mathbb{F}$. In the second case, we actually get the definition of conformal algebra as it was stated in [8] (see [1]).

The last sentence should be explained. Indeed, for any two elements $a, b \in C$ their pseudoproduct could be uniquely expressed as follows [1]:

$$a * b = \sum_{n \geq 0} \frac{(-D)^n}{n!} \otimes 1 \otimes_H c_n, \quad c_n \in C;$$

and the coefficients of this distribution are just the n-products of conformal algebra: $a \,{}_n\, b = c_n$.

The operations $(\cdot\,_n\,\cdot) : C \otimes C \to C$ satisfy the relations

(C1) $a\,_n\,b = 0\ ,\ n \gg 0$;

(C2) $Da\,_n\,b = -na\,_{n-1}\,b,\ n \in \mathbb{Z}_+$;

(C3) $a\,_n\,Db = D(a\,_n\,b) + na\,_{n-1}\,b,\ n \in \mathbb{Z}_+$.

These conditions are usually recognized to be the axioms of conformal algebra [8]. So, Definition 1 for $H = \mathbb{F}[D]$ is equivalent to the following

Definition 2. *Conformal algebra* is a vector space C over $\mathbb{F}$ endowed with a linear operator D and with a family of $\mathbb{F}$-bilinear maps $(\cdot\,_n\,\cdot)$, $n \in \mathbb{Z}_+$, satisfying the axioms (C1)–(C3).

From now on, H denotes the polynomial Hopf algebra $\mathbb{F}[D]$.

There is a powerful technique (invented in [9]) for processing calculations in conformal algebras: the notion of λ-product. Namely, for any two elements a, b of a conformal algebra C one may consider the generating function $(a\,_\lambda\,b)$ of the sequence $\{a\,_n\,b \mid n \in \mathbb{Z}_+\}$ with respect to a formal variable λ. Then

$$(a\,_\lambda\,b) = \sum_{n\geq 0} \frac{\lambda^n}{n!} \otimes_H (a\,_n\,b) \in \mathbb{F}[\lambda, D] \otimes_H C$$

is a polynomial in λ, called λ-product of a and b. Axioms (C1)–(C3) could be easily rewritten in terms of this λ-product, and this operation could be considered as a foundation of the definition of conformal algebra (see, e.g., [3, 6]). In fact, λ-product coincides with pseudoproduct, if one (informally) identifies λ with $-D\otimes 1$, see [1].

The "conformal versions" of associativity, commutativity and Jacobi identity could be easily expressed in terms of pseudoproduct [1] as well as (in a more complicated form) via n-products [11]:

$$\underline{\text{Associativity}}:\quad a * (b * c) = (a * b) * c,$$

$$a\,_n\,(b\,_m\,c) = \sum_{s\geq 0} \binom{n}{s} (a\,_{n-s}\,b)\,_{m+s}\,c;$$

$$\underline{\text{(Anti)Commutativity}}:\quad a * b = (\tau_{12} \otimes_H \mathrm{id})(b * a),$$

$$a\,_n\,b = \pm(-1)^n \sum_{s\geq 0} \frac{1}{s!}(-D)^s (b\,_{n+s}\,a);$$

$$\underline{\text{Jacobi}}:\quad a * (b * c) - (\tau_{12} \otimes_H \mathrm{id})(b * (a * c)) = (a * b) * c,$$

$$a\,_n\,(b\,_m\,c) - b\,_m\,(a\,_n\,c) = \sum_{s\geq 0} \binom{n}{s} (a\,_{n-s}\,b)\,_{m+s}\,c$$

(here τ_{12} means the permutation of corresponding tensor factors). One may see that these identities are similar to the "classical" ones.

Remark 1. It is easy to construct "conformal analogue" of any homogeneous polylinear identity $f = f(x_1, \ldots, x_n)$. Namely, it is sufficient to replace any monomial $\alpha_{i_1,\dots,i_n}[x_{i_1}\dots x_{i_n}]$ from f with expression $\alpha_{i_1,\dots,i_n}(\sigma\otimes_H \mathrm{id}_C)[x_{i_1}*\dots*x_{i_n}]$, where $\mathbb{S}_n\ni\sigma: k\mapsto i_k$, and $[\dots]$ means the same bracketing as in the initial monomial.

These identities have a very natural motivation: associativity (resp., commutativity, Lie, etc.) of conformal algebra C means that C could be embedded into the space of formal distributions $A[[z,z^{-1}]]$ over associative (resp., commutative, Lie, etc.) algebra A.

Various elementary properties of usual algebras hold for conformal algebras. In particular, if C is an associative conformal algebra, then the same H-module endowed with new operation

$$[a*b]=(a*b)-(\tau_{12}\otimes_H \mathrm{id})(b*a)$$

is a Lie conformal algebra denoted by $C^{(-)}$, as usual.

The notion of representation of an associative (or Lie) conformal algebra is also very natural [1, 4, 9].

Definition 3. Let C be an associative or Lie conformal algebra, and let V be a left H-module ($H=\mathbb{F}[D]$ as before). Then V is said to be *conformal C-module*, if an H-bilinear map

$$*: C\otimes V\to (H\otimes H)\otimes_H V,$$

is defined, such that the usual relations hold:

$$a*(b*x)=(a*b)*x,\quad \text{in associative case;}$$
$$a*(b*x)-(\tau_{12}\otimes_H \mathrm{id})(b*(a*x))=(a*b)*x,\quad \text{in Lie case;}$$
$$(a,b\in C,\quad x\in V).$$

This notion could also be expressed in terms of bilinear "n-actions", $n\in\mathbb{Z}_+$, such that the analogues of axioms (C1)–(C3) hold, as well as associativity (or Jacobi identity, for Lie case).

Representation is said to be *finite*, if V is a finitely generated H-module.

3 Conformal algebras Cend_N and gc_N: explicit construction

Let us adduce a general construction of various examples of conformal algebras, one of these examples would be the main object of our study.

Let A be an associative algebra, and let $A[v]$ be the algebra of polynomials over A on a commuting variable v. Denote by $\partial=\partial_v$ the usual derivation of $A[v]$ with respect to v. Consider free H-module $\mathfrak{D}(A):=H\otimes A[v]$ endowed with pseudoproduct $*$ given by

$$a*b=b_{(-1)}\otimes 1\otimes_H ab_{(2)}\equiv\sum_{n\geq 0}\frac{(-D)^n}{n!}\otimes 1\otimes_H a\partial^n(b),\quad a,b\in A[v]$$

(here we use Sweedler's notation for $A[v]$ endowed with natural H-comodule structure). In terms of n-products, the last expression means $a\ {}_n\ b = a\partial^n(b)$ for $a, b \in A[v]$.

Proposition 1. *$\mathfrak{D}(A)$ is an associative conformal algebra.*

There exists another construction appearing directly from consideration of formal distribution algebras [9]: by $\mathfrak{D}^\circ(A)$ we denote the same H-module $H\otimes A[v]$ endowed with

$$a \odot b = 1 \otimes a_{(1)} \otimes_H a_{(2)} b \equiv \sum_{n\geq 0} 1 \otimes \frac{D^n}{n!} \otimes_H \partial^n(a)b, \quad a, b \in A[v].$$

Proposition 2. *$\mathfrak{D}^\circ(A)$ is an associative conformal algebra.*

Proposition 3. *$\mathfrak{D}^\circ(A) \simeq \mathfrak{D}(A)$, the isomorphism is given by*

$$\mathfrak{D}^\circ(A) \ni a \equiv 1 \otimes a(v) \longleftrightarrow a_{(-1)} \otimes a_{(2)} \equiv a(v-D) \in \mathfrak{D}(A).$$

Example 1. Conformal subalgebra $\mathrm{Cur}\, A = H\otimes A \subset \mathfrak{D}(A)$ generated by constant polynomials, called *current conformal algebra* over A.

Example 2. Conformal algebra $\mathcal{W} = \mathfrak{D}(\mathbb{F})$ is called *conformal Weyl algebra.*

Example 3. The main objects of our study could be constructed as matrix algebra over $\mathcal{W}$:

$$\mathrm{Cend}_N = M_N(\mathcal{W}) = \mathfrak{D}(M_N(\mathbb{F}));$$
$$\mathrm{gc}_N = \mathrm{Cend}_N^{(-)}.$$

The last example provides us with the explicit construction of Cend_N. One may see that Cend_N is just a conformal analogue of matrix algebra $M_N(\mathbb{F})$. There is another source of similarity: by the initial definition (see, e.g., [9]) Cend_N is the conformal algebra of conformal endomorphisms of free N-generated H-module (see [1, 5]).

It would be natural to expect a great degree of similarity between algebraical properties of $M_N(\mathbb{F})$ and Cend_N. But one of the main algebraical features distinguishes these algebras: Cend_N is not finite, although $M_N(\mathbb{F})$ is finite-dimensional.

By the very definition, Cend_N has finite faithful representation. The canonical module could be constructed as follows: define the family of n-actions

$$(\cdot\ {}_n\ \cdot) : \mathrm{Cend}_N \otimes V_N \to V_N$$

on the free N-generated H-module $V_N = H \otimes \mathbb{F}^N$ by

$$a(v)\ {}_n\ u = a(D)\frac{d^n}{dD^n}(u), \quad a(v) \in M_N(\mathbb{F}[v]),\ u \in V_N.$$

These operations satisfy (C1)–(C3) and associativity. The action of Cend_N (gc_N) on V_N is clearly faithful and irreducible.

The following definition is motivated by the classical theorem by Burnside.

Definition 4. Conformal subalgebra C of Cend_N (or gc_N) is called *irreducible*, if there are no non-trivial C-submodules of V_N.

Example 4. Left (conformal) ideal $\mathrm{Cend}_{N,Q} = \mathrm{Cend}_N(Q_{(-1)} \otimes Q_{(2)})$ generated by an element of the form

$$Q_{(-1)} \otimes Q_{(2)} := \sum_{s\geq 0} \frac{1}{s!}(-D)^s \otimes \partial^s(Q),$$

where $Q \in M_N(\mathbb{F}[v])$, is irreducible iff $\det Q(v) \neq 0$.

Example 5. Current subalgebra $\mathrm{Cur}_N \equiv \mathrm{Cur}\, M_N(\mathbb{F}) \subset \mathrm{Cend}_N$ is irreducible as well as its conjugates $\mathrm{Cur}_N^Q = Q^{-1}\,\mathrm{Cur}_N(Q_{(-1)} \otimes Q_{(2)})$, $\deg_v \det Q = 0$ (see [3] for details).

Remark 2. We prefer using the isomorphic structure $\mathfrak{D}(M_N(\mathbb{F}))$ for Cend_N, instead of $\mathfrak{D}^\circ(M_N(\mathbb{F}))$ used in [3, 5, 9] et al. One may apply the isomorphism from Proposition 3 in order to translate all these constructions into the language of formal distributions.

4 Subalgebras of Cend_N

The following statement (conformal version of Burnside theorem) has been conjectured in [9], now we can conclude that it is true.

Theorem 1. *Examples 4 and 5 exhaust all irreducible conformal subalgebras of* Cend_N.

Sketch of the proof. Let $C \subsetneq \mathrm{Cend}_N$ be an irreducible subalgebra. For every $a \in C$ consider linear operators $a(n) : V_N \to V_N$, $a(n)u := a \,{}_n\, u$, $n \in \mathbb{Z}_+$, $u \in V_N$. The set $S(C) = \{a(n) \mid a \in C, n \in \mathbb{Z}_+\} \subset \mathrm{End}_{\mathbb{F}} V_N$ is a subalgebra of $M_N(W)$, where $W = \mathbb{F}\langle D, \partial_D \mid [\partial_D, D] = 1\rangle$ is the first Weyl algebra. Moreover, $[D, S(C)] \subseteq S(C)$. Hence, $S_1 = \mathbb{F}[D]S(C)$ is also a subalgebra, and this subalgebra must be dense in $M_N(W)$ with respect to the finite topology [7].

The density of S_1 allows to conclude that $\mathbb{F}[v]C$ is a left ideal of Cend_N, so it is of the form $\mathrm{Cend}_{N,Q}$, where $\det Q \neq 0$ [3]. Therefore, we have

$$\mathbb{F}[v]C = C + vC + v^2C + \cdots = \mathrm{Cend}_{N,Q} \qquad (*)$$

There are three cases:

(1) the sum $(*)$ is direct;

(2) $C \cap vC \neq 0$;

(3) $C \cap vC = 0$ but $(*)$ is not direct.

It is possible to show that (1) corresponds to finite subalgebra $C \simeq \mathrm{Cur}_N$, condition (2) implies $C = \mathrm{Cend}_{N,Q}$, and (3) is impossible.

Corollary 1. *Let C be an associative conformal algebra with finite faithful irreducible representation. Then $C \simeq \mathrm{Cend}_{N,Q}$ or $C \simeq \mathrm{Cur}_N$, for some $N \geq 1$, $\det Q \neq 0$. All these algebras are simple.*

Corollary 2 [5, 9]. *Simple finite associative conformal algebra is isomorphic to* Cur_N, *for some* $N \geq 1$.

These results allow to get some advance in structure theory of associative conformal algebras with finite faithful representation.

Theorem 2. *Semisimple associative conformal algebra with finite faithful representation is a direct sum of those described by Theorem 1.*

Theorem 3. *For every associative conformal algebra C with finite faithful representation there exists nilpotent ideal I of C such that C/I is semisimple.*

5 Some subalgebras of gc_N

In the previous section, we have presented structure theory of associative conformal algebras with finite faithful representation, in particular, Theorem 1 completely describes simple ones. In the case of Lie algebras, the picture seems to be more complicated.

Finite irreducible subalgebras of gc_N (i.e., finite Lie conformal algebras with faithful finite irreducible representation) were described in [5] (note that they are not necessarily simple). It was proved in [3] that the following subalgebras of Lie conformal algebra gc_N act irreducibly on the canonical module ($\det Q(v) \neq 0$ everywhere):

$$\begin{aligned}
&\mathrm{gc}_{N,Q} = \mathrm{Cend}_{N,Q}^{(-)};\\
&\mathrm{oc}_{N,Q} = \{a(Q_{(-1)} \otimes Q_{(2)}) \mid a \in \mathrm{Cend}_N,\, \sigma(a) = -a\},\ Q^t(-v) = Q(v);\\
&\mathrm{spc}_{N,Q} = \{a(Q_{(-1)} \otimes Q_{(2)}) \mid a \in \mathrm{Cend}_N,\, \sigma(a) = a\},\ Q^t(-v) = -Q(v),
\end{aligned}$$

where σ is the anti-involution of Cend_N defined by the rule $\sigma(a(D,v)) = a^t(D, D-v)$.

Conjecture [3]. Conformal subalgebras $\mathrm{gc}_{N,Q}$, $\mathrm{oc}_{N,Q}$, $\mathrm{spc}_{N,Q}$ and their conjugates (with respect to automorphisms of Cend_N) exhaust all infinite irreducible subalgebras of gc_N.

This conjecture was partially proved in [6]; another result could be found in [12].

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