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Support Varieties and Nilpotent Elements for Simple Lie Algebras

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1. Support varieties for restricted Lie algebras

1.1. Let k be an algebraically closed field of characteristic $p > 0$. Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie algebra and $u(\mathfrak{g})$ be the restricted universal enveloping algebra. The algebra $u(\mathfrak{g})$ is a finite-dimensional cocommutative Hopf algebra. The cohomology ring $R = \mathrm{H}^{2\bullet}(u(\mathfrak{g}), k)$ is a finitely generated k-algebra. Let $M \in \mathrm{mod}(u(\mathfrak{g}))$ and $J_M = \mathrm{Ann}_R\ \mathrm{Ext}^{\bullet}_{u(\mathfrak{g})}(M, M)$. The *support variety* of M is defined as

$$V_{\mathfrak{g}}(M) = \mathrm{Maxspec}(R/J_M).$$

The support variety $V_{\mathfrak{g}}(M)$ is an affine homogeneous variety which is always contained in the support variety of the trivial module $V_{\mathfrak{g}}(k)$. A remarkable fact is that the complexity $c_{\mathfrak{g}}(M)$ (first defined by Alperin [Al]) equals $\dim V_{\mathfrak{g}}(M)$. This homological interpretation demonstrates that $V_{\mathfrak{g}}(M) = \{0\}$ if and only if M is a projective $u(\mathfrak{g})$-module. Support varieties also have the nice property that they behave naturally with respect to direct sums and tensor products:

$$\begin{aligned} V_{\mathfrak{g}}(M \oplus N) &= V_{\mathfrak{g}}(M) \cup V_{\mathfrak{g}}(N) \\ V_{\mathfrak{g}}(M \otimes N) &= V_{\mathfrak{g}}(M) \cap V_{\mathfrak{g}}(N). \end{aligned}$$

Jantzen [Jan1], Friedlander and Parshall [FP], Suslin, Friedlander and Bendel [SFB] showed that one can indeed identify the support variety of a module as a closed subvariety of $\mathfrak{g}$:

$$V_{\mathfrak{g}}(M) \cong \{x \in \mathfrak{g} : \ x^{[p]} = 0,\ M|_{u(<x>)} \text{ is not free}\} \cup \{0\}. \tag{1.1.1}$$

In particular one has

$$V_{\mathfrak{g}}(k) \cong \{x \in \mathfrak{g} : \ x^{[p]} = 0\} = \mathcal{N}_1(\mathfrak{g}). \tag{1.1.2}$$

The preceding identification indicates that "nilpotent elements" of $\mathfrak{g}$ will be an important ingredient in the theory. For this reason the variety $\mathcal{N}_1(\mathfrak{g})$ will be called the *restricted nullcone*. At the end of their paper Friedlander and Parshall [FP, (3.4)] posed the following question: when is $\mathcal{N}_1(\mathfrak{g})$ an irreducible variety? In general this is a difficult question to answer. Recent efforts [CLNP, UGA1, UGA2] have shown that when $\mathfrak{g} = \mathrm{Lie}\ G$, where G is a reductive algebraic group, $\mathcal{N}_1(\mathfrak{g})$ is always irreducible.

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1.2. We will now specialize the general theory to the case when $\mathfrak{g} = \text{Lie } G$ where G is a semisimple or reductive algebraic group. Let B be a Borel subgroup relative to a maximal torus T of G, and U be the unipotent radical of B. Set $\mathfrak{b} = \text{Lie } B$ and $\mathfrak{u} = \text{Lie } U$. With respect to (G, T), let Φ be the associated root system, Δ be the simple roots, and W be the Weyl group. If $\alpha \in \Phi$ then $\alpha^\vee$ will denote the coroot. Let ρ be the half sum of positive roots and h denote the Coxeter number.

Now let $X(T)$ be the set of integral weights, $X(T)_+$ be the dominant integral weights and $X_1(T)$ be the restricted weights. For $\lambda \in X(T)_+$, we can construct the induced module $H^0(\lambda)$ by considering λ as a one-dimensional character for B (by letting U act trivially) and inducing up to G: $H^0(\lambda) = \text{ind}_B^G \lambda$. If the field has characteristic zero then the modules $H^0(\lambda)$, $\lambda \in X(T)_+$, are precisely the simple finite-dimensional G-modules. Over fields of characteristic $p > 0$, this is far from being true, yet these modules still play a central role in the representation theory.

Let G be a reductive algebraic group where char $k = p > 0$. If M is a G-module then we can regard M as a $u(\mathfrak{g})$-module. Moreover, the support variety $V_{\mathfrak{g}}(M)$ is stable under conjugation by elements of G. Observe that

$$V_{\mathfrak{g}}(M) \subseteq \mathcal{N}_1(\mathfrak{g}) \subseteq \mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} :\ x \text{ is nilpotent}\}.$$

The variety of nilpotent elements $\mathcal{N}(\mathfrak{g})$ is usually called the (ordinary) nullcone of $\mathfrak{g}$. Note that for $p \geq h$, $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$. There are only finitely many G-orbits in $\mathcal{N}$ and the G-orbits are classified. Since $V_{\mathfrak{g}}(M)$ is G-stable one has

$$V_{\mathfrak{g}}(M) = \overline{G \cdot x_1} \cup \overline{G \cdot x_2} \cup \cdots \cup \overline{G \cdot x_t}$$

where $G \cdot x_j$, $j = 1, 2, \ldots, t$ are certain G-orbits in $\mathcal{N}(\mathfrak{g})$.

1.3. The Good and the Bad (Primes): A fundamental part of conjugacy class/nilpotent orbit theory involves distinguishing between between fields of good and bad characteristic. Let k be a field of characteristic $p > 0$. A prime p relative to Φ is said to be *good* in the following cases:

- $\Phi = A_n$: all primes
- $\Phi = B_n, C_n, D_n$: $p > 2$
- $\Phi = E_6, E_7, F_4, G_2$: $p > 3$
- $\Phi = E_8$: $p > 5$.

In all other cases, the prime p is called *bad* relative to Φ.

2. Support varieties for induced modules: good primes

2.1. Jantzen conjecture: For $\lambda \in X(T)$, let

$$\Phi_\lambda = \{\alpha \in \Phi :\ (\lambda + \rho, \alpha^\vee) \in p\mathbb{Z}\}.$$

When p is good, Φ_λ is a subroot system of Φ and there exists $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for some $J \subseteq \Delta$. Here Φ_J is the subroot system generated by the simple roots in J. If $J \subseteq \Delta$ then one can express $\mathfrak{g}$ via a Levi decomposition:

$$\mathfrak{g} = \mathfrak{u}_J^+ \oplus \mathfrak{l}_J \oplus \mathfrak{u}_J.$$

The following theorem proved by Nakano, Parshall and Vella [NPV] determines the support varieties for induced modules over fields of good characteristic.

Theorem. *Let G be a reductive algebraic group over k with p good. Let $\lambda \in X(T)_+$ and $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for some $J \subseteq \Delta$. Then*

$$V_{\mathfrak{g}}(H^0(\lambda)) = G \cdot \mathfrak{u}_J.$$

The statement of this theorem was conjectured by Jantzen [Jan2] in 1987. Jantzen verified this conjecture when $\Phi = A_n$. The irreducible variety $G \cdot \mathfrak{u}_J$ is the closure of an orbit. Orbits which arise in this way are called Richardson orbits. When $\Phi = A_n$ all orbits are Richardson, but this is not true for other types.

2.2. The irreducibility of the restricted nullcone: As an immediate application of the preceding theorem one has the following corollary [NPV, (6.3.1) Cor.].

Corollary. *Let G be a reductive algebraic group and p be a good prime. Then $\mathcal{N}_1(\mathfrak{g})$ is irreducible. Furthermore, it is the closure of a Richardson orbit.*

Proof. This follows by the verification of the Jantzen conjecture because the trivial module k is of the form $H^0(\lambda)$ where λ is the zero weight and

$$\mathcal{N}_1(\mathfrak{g}) = V_{\mathfrak{g}}(k) = G \cdot \mathfrak{u}_J$$

for some $J \subseteq \Delta$. □

2.3. The computation of the restricted nullcone: Carlson, Lin, Nakano, and Parshall [CLNP] determined explicitly $\mathcal{N}_1(\mathfrak{g})$ as the closure of a Richardson orbit for all good primes using results in [NPV]. The following example outlines a basic argument used in the paper for the exceptional Lie algebras.

Example: Let $\Phi = E_8$ and assume that $p = 11$. Observe that

$$\dim \mathcal{N}_1(\mathfrak{g}) = \dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_0|.$$

But, $\Phi_0 = \{\alpha \in \Phi : \ (\rho, \alpha^\vee) \in p\mathbb{Z}\}$. By considering roots which have height divisible by p, one shows that

$$\Phi_0 \cong A_2 \times A_2 \times A_1 \times A_1.$$

Consequently,

$$\dim \mathcal{N}_1(\mathfrak{g}) = |\Phi| - |\Phi_0| = 240 - (6 + 6 + 2 + 2) = 224.$$

The only orbits of dimension 224 are $\mathcal{O}(E_8(a_6))$ and $\mathcal{O}(E_7(a_2))$ (which is not Richardson) [Car, CM]. Here we are using the Bala-Carter labelling of orbits. Hence,

$$\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(E_8(a_6))}.$$

The following tables [CLNP] describe the restricted nullcone for exceptional Lie algebras for good primes. The simple roots are enumerated using the Bourbaki labelling. When $p = h - 1$ it is interesting to note that the restricted nullcone is the closure of the subregular orbit.

Type E_6:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	62	$A_2 \times A_1 \times A_1$	$\{1,2,4,6\}$	$A_4 + A_1$
7	66	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_6(a_3)$
11	70	A_1	$\{4\}$	$E_6(a_1)$
≥ 13	72	$\emptyset$	$\emptyset$	E_6

Type E_7:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	106	$A_3 \times A_2 \times A_1$	$\{1,2,3,5,6,7\}$	$A_4 + A_2$
7	114	$A_2 \times A_1 \times A_1 \times A_1$	$\{1,2,3,5,7\}$	A_6
11	120	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_7(a_3)$
13	122	$A_1 \times A_1$	$\{4,6\}$	$E_7(a_2)$
17	124	A_1	$\{4\}$	$E_7(a_1)$
≥ 19	126	$\emptyset$	$\emptyset$	E_7

Type E_8:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
7	212	$A_4 \times A_2 \times A_1$	$\{1,2,3,5,6,7,8\}$	$A_6 + A_1$
11	224	$A_2 \times A_2 \times A_1 \times A_1$	$\{1,2,3,5,6,8\}$	$E_8(a_6)$
13	228	$A_2 \times A_1 \times A_1 \times A_1$	$\{2,3,5,6,8\}$	$E_8(a_5)$
17	232	$A_1 \times A_1 \times A_1 \times A_1$	$\{2,3,5,7\}$	$E_8(a_4)$
19	234	$A_1 \times A_1 \times A_1$	$\{2,3,5\}$	$E_8(a_3)$
23	236	$A_1 \times A_1$	$\{4,6\}$	$E_8(a_2)$
29	238	A_1	$\{4\}$	$E_8(a_1)$
≥ 31	240	$\emptyset$	$\emptyset$	E_8

Type F_4:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	40	$A_2 \times A_1$	$\{1,3,4\}$	$F_4(a_3)$
7	44	$A_1 \times A_1$	$\{1,3\}$	$F_4(a_2)$
11	46	A_1	$\{3\}$	$F_4(a_1)$
≥ 13	48	$\emptyset$	$\emptyset$	F_4

Type G_2:

p	$\dim \mathcal{N}_1(\mathfrak{g})$	Φ_0	J	orbit
5	10	A_1	$\{2\}$	$G_2(a_1)$
≥ 7	12	$\emptyset$	$\emptyset$	G_2

3. Generalized Restricted Nullcones

3.1. Let G be a simple algebraic group over an arbitrary algebraically closed field k. The computation of the restricted nullcone motivates us to formulate a more general problem. Let $\widehat{\rho} : G \to GL(W)$ be a finite-dimensional representation of G. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ be its differential. We will assume that $\ker \rho \cap \mathcal{N}(\mathfrak{g}) = \{0\}$. Note that the representation ρ over a field of characteristic p is always a restricted representation.

Define

$$\mathcal{N}_{r,\rho}(\mathfrak{g}) = \{x \in \mathfrak{g} : \ \rho(x)^r = 0\}.$$

Then $\mathcal{N}_{r,\rho}(\mathfrak{g})$ is an algebraic variety contained in $\mathcal{N}(\mathfrak{g})$ which is invariant under the action of G. Since there are only finitely many G-orbits on $\mathcal{N}(\mathfrak{g})$, it is reasonable to try to describe $\mathcal{N}_{r,\rho}(\mathfrak{g})$ as the union of orbit closures.

The University of Georgia VIGRE Algebra Group considered this problem. In [UGA1], we computed $\mathcal{N}_{r,\rho}(\mathfrak{g})$ for $k = \mathbb{C}$ or when p is good under the assumptions that ρ is a minimal dimensional irreducible representation or the adjoint representation. Severy key ideas involved using and extending the work of Nakano and Tanisaki [NT] on the realization of orbit closures as support varieties.

As a corollary of the computations in [UGA1], when the characteristic of k is p, one can set $r = p$. Then

$$\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}_{p,\rho}(\mathfrak{g}).$$

From this we get a new method to determine the restricted nullcone without using the Jantzen conjecture or [NPV].

3.2. The computation of the restricted nullcone: In [UGA2], the UGA VIGRE Algebra Group computed generalized restricted nullcones and restricted nullcones for fields of bad characteristic. We record here the computation of the restricted nullcone. For the classical Lie algebras the labelling of orbits follows the conventions given in [He].

Theorem (A). *Let G be a simple classical connected algebraic group over k where* $\operatorname{char} k = 2$. *For each type $\Phi = B_l$, C_l and D_l the following holds.*

(i) If Φ is of type B_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_2^l, 1_1)}$.

(ii) If Φ is of type C_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_1^l)}$.

(iii) If Φ is of type D_l then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(2_2^l)}$.

Theorem (B). *Let G be an exceptional algebraic group with p a bad prime. Then* *(i) If Φ is of type E_6 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ $(p = 3)$, $3A_1$ $(p = 2)$.*

(ii) If Φ is of type E_7 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = 2A_2 + A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.

(iii) If Φ is of type E_8 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = A_4 + A_3$ $(p = 5)$, $2A_2 + 2A_1$ $(p = 3)$, $4A_1$ $(p = 2)$.

(iv) If Φ is of type F_4 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = A_1 + \widetilde{A_2}$ $(p = 3)$, $A_1 + \widetilde{A_1}$ $(p = 2)$.

(v) If Φ is of type G_2 then $\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(X)}$ where $X = G_2(a_1)$ $(p = 3)$, $\widetilde{A_1}$ $(p = 2)$.

Observe that the restricted nullcone need not be the closure of a Richardson orbit when the field has bad characterstic. For example when $p = 2$, $\Phi = G_2$,

$$\mathcal{N}_1(\mathfrak{g}) = \overline{\mathcal{O}(\widetilde{A_1})}$$

and $\mathcal{O}(\widetilde{A_1})$ is not Richardson.

3.3. These computations settle the question posed by Friedlander and Parshall for Lie algebras of reductive groups over fields of arbitrary prime characteristic.

Corollary. *Let G be a reductive connected algebraic group and p be any prime. Then $\mathcal{N}_1(\mathfrak{g})$ is irreducible.*

3.4. Let $\mathcal{U}(G)$ be the variety of unipotent elements of G. It is well-known that when $k = \mathbb{C}$, the exponential map gives a G-equivariant isomorphism between $\mathcal{N}(\mathfrak{g})$ and $\mathcal{U}(G)$. When k is a field of good characteristic, there is also G-equivariant isomorphism between these two varieties given by a Springer isomorphism. For fields of bad characterstic these varieties are in general not isomorphic because the number of conjugacy classes in $\mathcal{U}(G)$ and nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ can be different. Set

$$\mathcal{U}_1(G) = \{u \in \mathcal{U} : \ u^p = 1\}.$$

In [UGA2] an amazing fact about the conjugacy classes in $\mathcal{U}_1(G)$ and nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ was discovered which holds for any prime.

Theorem. *Let G be a connected reductive algebraic group over an algebraically closed field of characteristic $p > 0$. There exists a one-to-one correspondence between unipotent orbits in $\mathcal{U}_1(G)$ and nilpotent orbits in $\mathcal{N}_1(\mathfrak{g})$ which respects the closure ordering of orbits. This correspondence also respects dimensions of unipotent classes and nilpotent orbits.*

The preceding result indicates that there might be a natural G-equivariant isomorphism between $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ which works for all primes. Several experts that we have consulted agree that constructing such a map would be a worthwhile endeavor.

On the following page we present the closure ordering relations (Hasse diagram) for $\mathcal{U}_1(G)$ and $\mathcal{N}_1(\mathfrak{g})$ when $\Phi = E_8$ ($p = 2, 3, 5$).

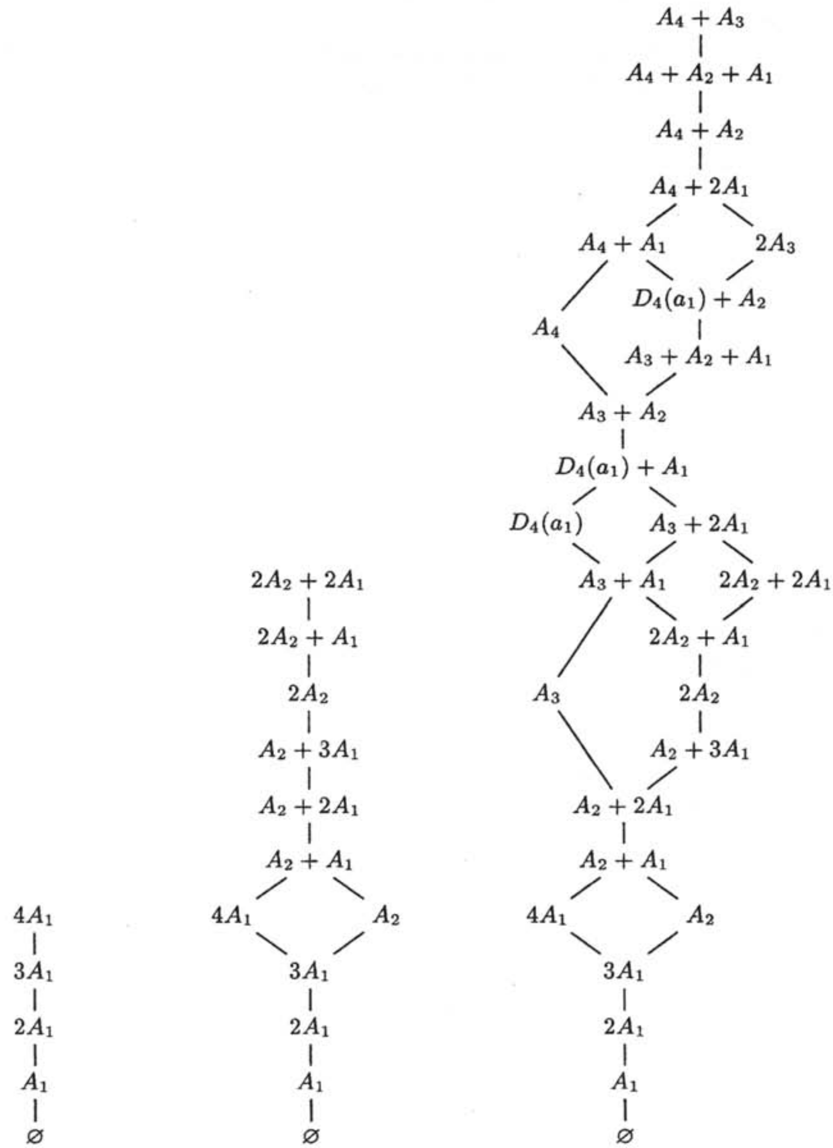

Figure 1: E_8 for $p = 2$ (left), $p = 3$, (center), and $p = 5$ (right)

4. Support varieties for induced modules: bad primes Recall that for good primes, by Theorem 2.1,

$$\dim V_{\mathfrak{g}}(H^0(\lambda)) = \dim G \cdot \mathfrak{u}_J = |\Phi| - |\Phi_\lambda|.$$

In [UGA3], it was shown that this formula also holds for all primes.

Theorem. *Let G be a reductive algebraic group. Then for any prime p, one has*

$$\dim V_{\mathfrak{g}}(H^0(\lambda)) = |\Phi| - |\Phi_\lambda|.$$

This formula is verified by determining $V_{\mathfrak{g}}(H^0(\lambda))$. This is accomplished by using and modifying results established in [NPV]. As with the restricted nullcone, for fields of bad characteristic, $V_{\mathfrak{g}}(H^0(\lambda))$ is always irreducible (i.e. the closure of an orbit). Furthermore, the orbits realizing $V_{\mathfrak{g}}(H^0(\lambda))$ in this case need not be Richardson.

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