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An introduction to the Thermodynamics of Conformal Repellers

Edson de Faria

Department of Mathematics, University of São Paulo, São Paulo

E-mail address: edson@ime.usp.br

Abstract. In this expository paper we discuss Bowen's thermodynamic formalism for conformal repellers, and possible connections with the theory of asymptotic Teichmüller spaces and uniformly asymptotically conformal dynamical systems.

Introduction

Our purpose in this expository article is to illustrate how a physical theory can be used to *prove* a mathematical result, thereby reversing the usual implication arrow Math $\longrightarrow$ Physics. This will be consistent with D. Ruelle's well-known assertion that our mathematics is "natural" [23].

Physical motivation. The physical theory at the base of the (reversed) arrow is statistical mechanics. The macroscopic laws of thermodynamics were established by Carnot, Clausius and Kelvin from early to mid-nineteenth century. The branch of Physics known today as statistical mechanics has the reductionist goal of deriving such macroscopic laws from purely microscopic principles, starting from the usual laws of classical mechanics. The basic paradigm is to replace the Newtonian study of the motion of individual particles, which is highly impractical for large (macroscopic) systems to say the least, by measurable *statistical averages*.

The physical foundations of statistical mechanics were laid down primarily by L. Boltzmann and J. W. Gibbs. Mathematicians from Poincaré

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onwards worried about providing proper mathematical foundations for statistical mechanics. The result of their efforts was ergodic theory, today a very fruitful branch of Dynamical Systems. Various key concepts such as ergodicity (Birkhoff, von Neumann, Khinchin), measure-theoretic entropy (Kolmogorov, Sinai, Bowen), Markov partitions, symbolic coding, topological entropy (Adler, Konheim, McAndrew, Bowen), Gibbs measures and equilibrium states (Bowen, Ruelle, Sinai) – these and more were either inspired by or direct generalizations of the physical notions of entropy, Gibbs ensembles, pressure, etc.

A mathematical problem. The story we wish to tell, like many other stories in modern Mathematics, starts with H. Poincaré. In his *Mémoires sur les groupes Kleinéens* (1883), Poincaré observed that if the (finitely many, say) generators of a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ containing no parabolic elements are slightly perturbed, the resulting Kleinian group G will be *quasi-Fuchsian*, i.e. it will possess an invariant region $\Delta \subset \widehat{\mathbb{C}}$ conformally equivalent to a disk. In a deep insight, all the more remarkable given the absence of computational evidence, Poincaré conjectured that whenever $\partial\Delta$ is not a circle, it must be a non-rectifiable Jordan curve. This conjecture was verified by Fricke and Klein at the end of the nineteenth century.

Despite great progress in the hyperbolic geometry and the theory of Fuchsian and Kleinian groups in the early part of the twentieth century, the study of the microscopic structure of limit sets of quasi-Fuchsian groups remained dormant for several decades.

In 1948, G. Mostow proved the following result. Since the boundary $\partial\Delta$ of the invariant region of a Fuchsian group as above is a Jordan curve, one can consider the Riemann maps of the inside and outside of $\partial\Delta$, and both maps extend homeomorphically to the curve itself. Hence, taking the composition of one of them with the inverse of the other, we get a self-homeomorphism h of the unit circle. Mostow's theorem can be stated as follows.

Theorem 1 (Mostow)**.** *If h is absolutely continuous, then it must be a Möbius transformation.*

This is an example of what is known as *geometric rigidity*. See figure 1, where $h = (h^-|_{\partial\Delta})^{-1} \circ h^+|_{\partial\mathbb{D}}$.

Then, in 1978, R. Bowen went much further than Poincaré, Fricke and Klein or Mostow, proving that $\partial\Delta$, if not a circle, always has Hausdorff dimension greater than 1. The way Bowen proved his result is really the highlight of our story. He used several of the above mentioned key ideas

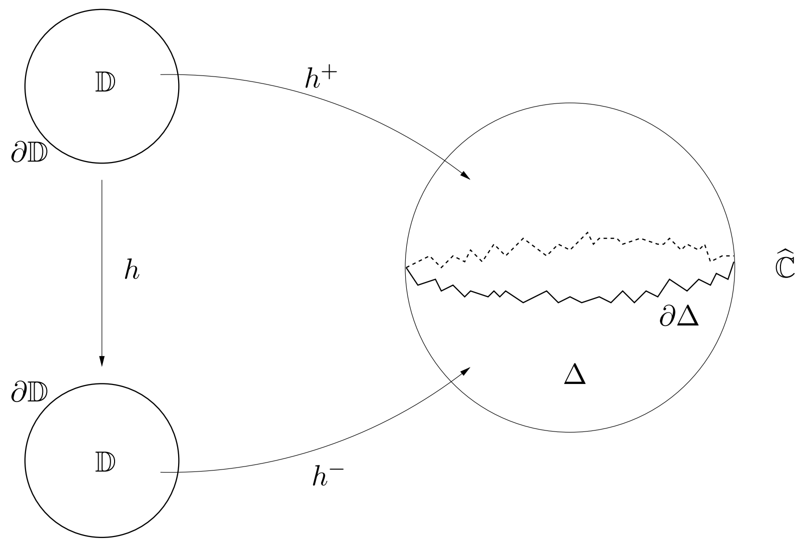

FIGURE 1. Mostow's theorem

of ergodic theory that were born out of thermodynamics and statistical mechanics.

Although circumscribed to a situation where one has conformal expansion, this story is very emblematic, and Bowen's work gave rise to a well developed subject. Today the relationship between ergodic ideas such as equilibrium states and various properties of a dynamical system such as the Hausdorff dimension of the non-wandering set, or its conformal dimension when the system is conformal, is reasonably well understood in a variety of contexts, from conformal dynamics to non-uniformly hyperbolic dynamics. See [2] for the latter, and [26] for the former, and the references therein.

We end this introduction with the following remark. In the context of unimodal maps, the problem of calculating the Hausdorff dimension of attractors such as the Feigenbaum Cantor set is sometimes solvable if one manages to replace the original map, which has critical points, by another map which is expanding and has the same invariant set (this can be done explicity for the Feigenbaum case). To the best of our knowledge, this has not been done explicitly for infinitely renormalizable unimodal maps with other combinatorics besides Feigenbaum. Coarser estimates on the Hausdorff dimension in these cases can be achieved by other means, as shown in [8, pp. 760-62].

1. **Conformal repellers**

The ideas and methods used by Bowen in [4] are considerably more general than the case of quasi-Fuchsian groups. They can be used to study a wider class of dynamical systems, the so-called *conformal repellers*. Here is a formal definition, suitable for our purposes.

Definition 1. *A conformal repeller consists of an open set $U \subset \widehat{\mathbb{C}}$, a compact subset $K \subset U$ and a pseudo-semigroup G of conformal transformations $g : U_g \to \widehat{\mathbb{C}}$, where each $U_g \subset U$ is open, such that*

(a) *the set K is G-invariant: $GK \subset K$;*
(b) *for each $g \in G$, the restriction $g|_{K \cap U_g}$ is expanding, i.e. $|g'(x)| \geq \lambda > 1$ for all $x \in K \cap U_g$.*

The constant λ is uniform (that is to say, independent of g).

We will also assume, in order to facilitate our exposition, that the action of G on K is *transitive* (i.e. there exists $x_0 \in K$ whose orbit Gx_0 is dense in K). Let us present a few examples. A common feature of all the examples bellow is that they can be given suitable *finite Markov partitions*, with which their dynamics can be nicely encoded. Such encoding will only be discussed for one of the examples, that of Cantor repellers.

1.1. **Co-compact Fuchsian groups.**

The first example is, not surprisingly, the situation studied by Poincaré. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ be a *co-compact* Fuchsian group with invariant disk equal to the unit disk $\mathbb{D}$. By co-compact we mean that the quotient $\mathbb{D}/\Gamma$ is compact. This implies in particular that Γ contains no parabolic elements and is finitely generated. We can find a finite list $g_1, g_2, \ldots, g_n \in \Gamma$ of hyperbolic generators and a fundamental domain $\Delta \Subset \mathbb{D}$ for the action of Γ on $\mathbb{D}$ which is a $2n$-gon with geodesic sides. These sides are identified in pairs by the g_i's. The images of Δ by the various elements of Γ produce a *tiling* of $\mathbb{D}$, and the limit set of Γ is equal to $\partial\mathbb{D}$. Using these facts, one can prove that, given any two distinct points $a, b \in \partial\mathbb{D}$ and $\varepsilon > 0$, there exist $p, q \in \partial\mathbb{D}$ with $|p - a| < \varepsilon$ and $|q - b| < \varepsilon$, and a hyperbolic element $h \in \Gamma$ such that p is the expanding fixed point and q is the attracting fixed point of h. From this, it is not difficult to find finitely many hyperbolic elements $h_1, h_2, \ldots, h_m \in \Gamma$, and disks $D_1, D_2, \ldots, D_m$, each centered at the corresponding expanding fixed point of h_i such that

$$\bigcup_{j=1}^{m} D_j \supset \partial\mathbb{D} ,$$

and such that the pseudo semigroup generated by $\{h_1|_{D_1}, h_2|_{D_2}, \ldots, h_m|_{D_m}\}$ satifies the definition of conformal repeller, with $K = \partial\mathbb{D}$. The details are left as an exercise to the reader. For background on Fuchsian groups, see [3] or [17].

1.2. **Schottky groups.** A *Schottky group* yields another example of conformal repeller. Let $D_1, D_2, \ldots, D_{2m} \subset \widehat{\mathbb{C}}$ be $2m$ pairwise disjoint closed disks in the sphere, and for each $1 \leq i \leq m$ let $g_i \in PSL(2, \mathbb{C})$ be a linear fractional transformation mapping the interior of D_i onto the exterior of D_{i+n}. Then the g_i's freely generate a subgroup $\Gamma \subset PSL(2, \mathbb{C})$, which is called a Schottky group. The limit set of this group is easily seen to be a Cantor set. Just as in the previous example, one can find inside such a Schottky group a semigroup whose action will be expanding when restricted to the limit set.

1.3. **Jordan repellers.** Let $U, V \subset \widehat{\mathbb{C}}$ be two nested annuli, with $\overline{U} \subset V$, with none of the components of $\widehat{\mathbb{C}} \setminus \overline{U}$ contained in V. Let $f : U \to V$ be a holomorphic, degree d covering map. We look at the invariant, compact set $J_f = \cap_{n \geq 0} f^{-n}(V) \subset U$. Replacing f by a suitable power if necessary, we can always assume that $f|_{J_f}$ is expanding, so that once again we have a conformal repeller. In this case it is not difficult to see that J_f is a Jordan curve. This type of repeller is called a *Jordan* repeller. Here is a family of explicit examples. We can start with the polynomial map $f(z) = z^2$, take an annular neighborhood V of the unit circle, and then take $U = f^{-1}(V)$. Then we perturb the map slightly to $f(z) = z^2 + c$, with c small. The corresponding J_f is nothing but the Julia set of f, and it is a Jordan curve. Bowen's results imply that this Jordan curve always has Hausdorff dimension > 1 if $c \neq 0$.

1.4. **Cantor repellers.** Another example is provided by *Cantor repellers*, those in which, as the name suggests, the invariant set is a Cantor set. Here is the formal definition.

Definition 2. *A Cantor repeller consists of two open sets $U, V \subseteq \mathbb{C}$ and a holomorphic map $f : U \to V$ satisfying the following conditions:*

1. *The domain U is the union of Jordan domains $U_1, U_2, \ldots, U_m$ (for some $m \geq 2$) having pairwise disjoint closures;*
2. *The co-domain V is the union of Jordan domains $V_1, V_2, \ldots, V_M$ (for some $M \geq 1$) having pairwise disjoint closures;*
3. *For each $i \in \{1, 2, \ldots, m\}$ there exists $j(i) \in \{1, 2, \ldots, M\}$ such that $f|_{U_i}$ maps U_i conformally onto $V_{j(i)}$;*
4. *We have $\overline{U} \subset V$;*
5. *The limit set $J_f = \cap_{n \geq 0} f^{-n}(V)$ has the locally eventually onto property.*

Remark 1. *Note that the limit set J_f of a Cantor repeller is a compact, perfect and totally disconnected set, i.e. a Cantor set, hence the name.*

Proposition 1. *Every Cantor repeller is a conformal repeller.*

Proof. All we have to do is to verify that the expansion property in the definition of conformal repeller holds true for a Cantor repeller. We exploit hyperbolic contraction (Schwarz's lemma). If $\Omega \subset \mathbb{C}$ is simply connected domain, we denote by d_Ω its hyperbolic metric.

Each U_i is compactly contained in $V_{k(i)}$ for some $k(i) \in \{1, 2, \ldots, M\}$. The inclusion $U_i \to V_{k(i)}$ is a contraction of the corresponding hyperbolic metrics. In other words, there exists $\lambda_i > 1$ such that for all $z, w \in U_i$ we have

$$d_{U_i}(z, w) \;\geq\; \lambda_i d_{V_{k(i)}}(z, w) \;.$$

Let $\lambda = \min\{\lambda_i \,:\, 1 \leq i \leq m\} > 1$. If $z, w \in U_i$ and $f(z), f(w) \in V_j$ then, since f maps U_i conformally onto V_j, we have

$$d_{V_j}(f(z), f(w)) \;=\; d_{U_i}(z, w) \;\geq\; \lambda d_{V_{k(i)}}(z, w) \;.$$

Using this inequality and an easy inductive argument (exercise), we see that if $z, w \in U_i$ are in the same connected component of $f^{-n}(V_j)$ for some j, then

$$d_{V_j}(f^n(z), f^n(w)) \;\geq\; \lambda^n d_{V_{k(i)}}(z, w) \;. \tag{1}$$

Now suppose that z, w are points in $K_1 = J_f \cap V_{k(i)}$. Then $f^n(z), f^n(w) \in K_2 = J_f \cap V_j$. Since K_1 and K_2 are compact, the hyperbolic metrics $d_{V_{k(i)}}$ and d_{V_j} over K_1 and K_2 respectively are both comparable to the Euclidean metric. Hence there exists $C > 0$ depending on $V_{k(i)}, V_j, K_1$ and K_2 such that the inequality (1) translates into

$$|f^n(z) - f^n(w)| \;\geq\; C\lambda^n |z - w| \;. \tag{2}$$

Dividing both sides of (2) by $|z - w|$, fixing z and letting $w \to z$, we get $|(f^n)'(z)| \geq C\lambda^n$, as we wanted.

□

The topological dynamics of a Cantor repeller is fairly easy to describe. First, we give a symbolic code for f in the limit set. We define the transition matrix of (f, J_f) as follows. Let $J_i = J_f \cap U_i$ for $i = 1, 2, \ldots, m$, and let the square matrix $A = (a_{ij})_{m\times m}$ be such that $a_{ij} = 1$ if $f(J_i) \supseteq J_j$ and $a_{ij} = 0$ otherwise. Now let $\Sigma_A \subseteq \{1, 2, \ldots, m\}^{\mathbb{N}}$ be the subspace of the space of infinite one-sided sequences in the m symbols $1, 2, \ldots, m$ defined by the condition that $x = (x_n)_{n\in\mathbb{N}} \in \Sigma_A$ if an only if $a_{x_n x_{n+1}} = 1$ for all n. We endow the set $\{1, 2, \ldots, m\}$ with the discrete topology and the cartesian product space $\{1, 2, \ldots, m\}^{\mathbb{N}}$ with the product topology, which makes it a compact space. Hence Σ_A is also a compact (Hausdorff, hence metrizable) space. It is *invariant* under the shift map $\sigma : \{1,2,\dots,m\}^{\mathbb{N}} \hookleftarrow$, given by $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. The dynamical system (Σ_A,σ) is called the *subshift of finite type* associated to the transition matrix A.

As an example, figure 2 shows a Cantor repeller with transition matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Theorem 2. *The map f restricted to its limit set J_f is topologically conjugate to the subshift (Σ_A,σ) with transition matrix A.*

Proof. Given $x \in J_f$, we know that for each $n \geq 0$ there exists a unique $i_n \in \{1,2,\dots,m\}$ such that $f^n(x) \in J_{i_n}$. We define the *itinerary* of x to be the sequence $\theta_x = i_0 i_1 \cdots i_n \cdots \in \{1,2,\dots,m\}^{\mathbb{N}}$. Note that $a_{i_n i_{n+1}} = 1$ for all $n \geq 0$, so that in fact $\theta_x \in \Sigma_A$. Hence we have a well-defined map $h : J_f \to \Sigma_A$ given by $h(x) = \theta_x$. We leave to the reader the task of proving that h is the desired conjugacy. $\square$

When filling in the blanks in the above proof, the reader will not fail to notice the following. Let us agree to call a finite sequence $i_0 i_1 \cdots i_n \in \{1,2,\dots,m\}^{n+1}$ *admissible* if $a_{i_k i_{k+1}} = 1$ for $k = 0,1,\dots,n-1$. The connected components of $f^{-(n+1)}(V)$ can be labeled inductively by admissible sequences. Indeed, assuming that the components of $f^{-n}(V)$ have already been labeled, and given an admissible sequence $i_0 i_1 \cdots i_n$, let $U_{i_0 i_1 \cdots i_n}$ be the unique connected component of $f^{-(n+1)}(V)$ contained in $U_{i_0 i_1 \cdots i_{n-1}}$ with

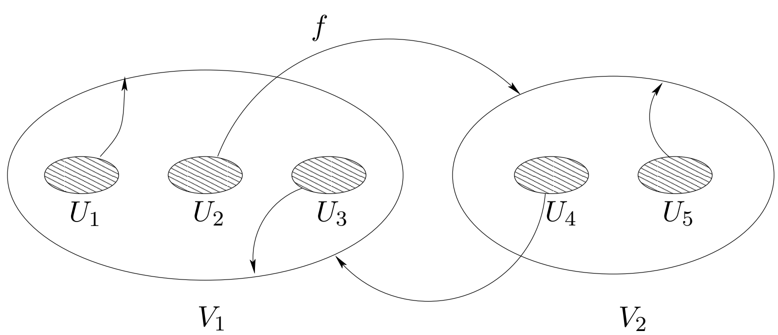

FIGURE 2. A Cantor repeller

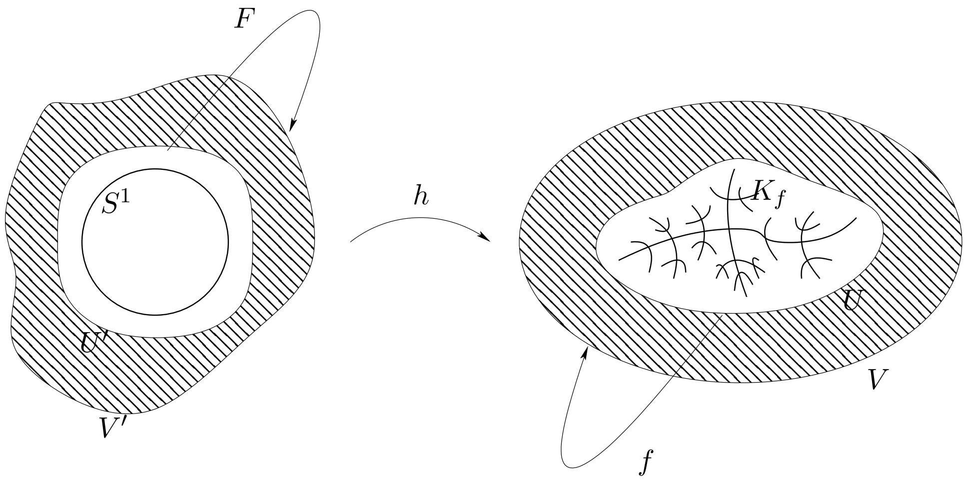

FIGURE 3. The Ghys-Sullivan construction.

the property that $f(U_{i_0i_1\cdots i_n}) = U_{i_1i_2\cdots i_n}$. Each open set $U_{i_0i_1\cdots i_n}$ is a topological disk, because the open sets U_i at the base of the induction already are. The reader will see that if $x \in J_f$ has itinerary $\theta_x = i_0i_1\cdots i_n\cdots$, then in fact

$$\{x\} = \bigcap_{n=0}^{\infty} U_{i_0i_1\cdots i_n} .$$

This happens because the diameters of the topological disks $U_{i_0i_1\cdots i_n}$ shrink to zero as $n \to \infty$.

For each admissible sequence $i_0i_1\cdots i_n$, we define $J_{i_0i_1\cdots i_n} = J_f \cap U_{i_0i_1\cdots i_n}$ and call it the *cylinder* with prefix $i_0i_1\cdots i_n$. This clearly agrees with the image of the set of all sequences in Σ_A with prefix $i_0i_1\cdots i_n$ under h^{-1}, where h is the conjugacy built in theorem 2 above. The set of all such cylinders (with prefix given by a sequence of length $n+1$) will be denoted by $\mathscr{A}_n$.

1.5. **Quadratic-like maps.** Conformal repellers can be useful even in situations where there is no obvious expansion. It may happen, for instance, that the non-wandering set of our dynamical system contains critical points. This is the case with *quadratic-like maps* having connected filled-in Julia sets.

Definition 3. *Let $U, V \subset \widehat{\mathbb{C}}$ be topological disks with $\overline{U} \subset V$, and let $f : U \to V$ be a holomorphic branched covering map onto V with a unique quadratic critical point $c \in U$ ($f'(c) = 0$, $f''(c) \neq 0$). We say that f is a* quadratic-like map. *Its filled-in Julia set is*

$$K_f \;=\; \bigcap_{n\geq 0} f^{-n}(V)\ ,$$

which is easily seen to be a compact and f-invariant set.

The theory of quadratic-like maps was started by Douady and Hubbard and is a very well developed subject, see for instance [11]. When $c \in K_f$, it is easy to see that K_f is connected: indeed, in this case $f^{-n}(V)$ is a topological disk for all $n \geq 0$, and since such disks are nested, their intersection K_f is connected. The dynamics in the filled-in Julia set is certainly not expanding, but we can still extract from it an expanding dynamical system – a conformal repeller – by means of the following construction due to Ghys and Sullivan (see figure 3). Since K_f is connected, its complement $\widehat{\mathbb{C}} \setminus K_f$ is simply connected. Consider the Riemann map $h : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus K_f$, normalized so that, say, $h(\infty) = \infty$ and $h'(\infty) = 1$. Consider the annuli $V' = h^{-1}(V)$ and $U' = h^{-1}(U)$ and let $F : U' \to V'$ be the map given by $F = h^{-1} \circ f \circ h$. This map is holomorphic, it has no critical points, and it is a covering map of degree 2. Moreover, F extends continuously to the unit circle $S^1 = \partial U' \cap \partial V' = \partial\mathbb{D}$.

Now we can consider the reflections of both U' and V' across $\partial\mathbb{D}$, that is to say, their images U'' and V'' under geometric inversion about the unit circle, and extend F to a map from the open annulus $U_* = U' \cup S^1 \cup U''$ to the open annulus $V_* = V' \cup S^1 \cup V''$ using the Schwarz reflection principle. This map which we still denote by F is a holomorphic double covering of U_* onto V_*, and its restriction to S^1 is a degree two real-analytic endomorphism of the circle. Thus we see that $F : U_* \to V_*$ defines a Jordan repeller with limit set $K = S^1$.

1.6. **Holomorphic pairs.** In [6], the author developed a theory of *holomorphic commuting pairs*, conformal dynamical systems akin to quadratic-like maps suitable for the study of universality properties of critical circle maps. An analogue of the Ghys-Sullivan construction is performed in that paper, and the resulting repeller turns out to be a Cantor repeller of a very specific topological type; see [6] for details.

2. Thermodynamics and statistical mechanics

Let us digress a bit into showing how, in a very simple context, thermodynamics is explained by statistical mechanics. We avoid the full reductionist step of explaining these statistical considerations from the laws of classical mechanics, as this would take us too far afield.

As we stated in the introduction, the macroscopic laws of thermodynamics were discovered by Carnot, Clausius and Kelvin (among others). We want to explain the two most important of these laws from statistical principles. Our arguments here will be for the most part heuristic, but the reader can rest assured that they can be made rigorous. For rigorous treatments, see [21] or [19].

Consider a very dilute (or *ideal*) gas, i.e. a gas in which the interactions between its constituent molecules are so weak that they can be safely ignored. We shall ignore external forces as well. Following tradition, we denote by V its volume, by P its pressure, by U its internal (or total) energy, and by T its temperature. These are all macroscopic properties of the gas. The idea of *thermal equilibrium* is hereto taken for granted, and it leads to the so-called *zeroth law* which will not be discussed.

If an amount δQ of heat is furnished to the gas, part of this energy adds to its internal energy, and part is converted into work as the gas expands. This is the contents of the *first law*, and it is simply the principle of conservation energy. In infinitesimal form, the first law can be written

$$dU \;=\; \delta Q - P\,dV \;.$$

There is no reason why δQ should be the exact differential of a function. However, the *second law* postulates the existence of a function $S = S(V, P, \cdots)$ called *entropy* whose differential dS is equal to δQ divided by the temperature, in other words

$$dS \;=\; \frac{\delta Q}{T} \;.$$

These two laws, despite their simplicity, are quite powerful. We note here that they can be combined into one formula

$$dS \;=\; \frac{dU + P\,dV}{T} \;.$$

Simple consequences of these formula are, of course (thinking of S as a function of U and V),

$$\frac{\partial S}{\partial U} \;=\; \frac{1}{T} \quad ; \quad \frac{\partial S}{\partial V} \;=\; \frac{P}{V} \;.$$

The stroke of genius by Boltzmann was to explain S in microscopic terms, taking it as a primitive variable and then using the above relations to *define* temperature and pressure.

Here is how entropy is explained by Boltzmann, via an elementary probabilistic argument. Let us imagine that our ideal gas is composed of N molecules and that each of these can occupy finitely many "states"

$1, 2, \ldots, k$ with energy levels $u_1, u_2, \ldots, u_k$. Let us be a bit more precise. The set of all possible configurations – or states – of our gas is $\Omega = \{1, 2, \ldots, k\}^N$. Given $\omega = (\omega_1, \omega_2, \ldots, \omega_k) \in \Omega$, let

$$n_i = n_i(\omega) = \#\{\, j : \omega_j = i \,\} \quad \text{for } i = 1, 2, \ldots, k \ .$$

Then $N = n_1 + n_2 + \cdots + n_k$ and the total (internal) energy of the gas in the state ω is

$$U \ = \ n_1 u_1 + n_2 u_2 + \cdots + n_k u_k \ .$$

Now we suppose that the (indistinguishable) molecules in the gas have each an individual probability p_i of being in the state i. Here $p_i \geq 0$ and of course $\sum p_i = 1$. This induces a probability distribution $P(\omega)$ – the product distribution – in Ω. If a configuration $\omega \in \Omega$ is selected at random according to such distribution, the expected number of molecules occupying the state i is therefore Np_i.

Definition 4. *An equilibrium state for the gas is a state $\omega \in \Omega$ for which $n_i = Np_i$, for all $i = 1, 2, \ldots, k$.*

We are now ready for Boltzmann's definition of entropy.

Definition 5. *The entropy of the gas (relative to a given probability distribution of energy states) is given by $S = k_B \log (\#$ equilibrium states$)$, where $k_B > 0$ is a universal constant.*

The constant k_B is called *Boltzmann's constant* (more usually denoted simply by the letter k). That there should be such a constant in the formula defining entropy is unavoidable. The second law of thermodynamics dictates that S is not a pure number, but has physical dimensions (it is given in, say, Joules per Kelvin).

It is easy to estimate S for our ideal gas with the help of Stirling's formula

$$n! \ \simeq \ \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n} \ ,$$

which we write in logarithmic form as

$$\log (n!) \ \simeq \ \left(n + \frac{1}{2}\right) \log n - n + c \ , \tag{3}$$

where $c = \frac{1}{2} \log (2\pi)$. The number of equilibrium states is $N!/(N_1! \cdots N_k!)$, where $N_i = Np_i$ for each i (of course N_i is not necessarily an integer, but that will not affect our estimate). Thus, applying (3) several times, we get

$$\begin{aligned} S \ &= \ k_B \log \left(\frac{N!}{N_1! N_2! \cdots N_k!} \right) \\ &\simeq \ \left(N + \frac{1}{2}\right) \log N - N + c - \sum_{i=1}^{k} \left[\left(N_i + \frac{1}{2}\right) \log N_i - N_i + c \right] \ , \end{aligned}$$

and after after a few simple computations we deduce that

$$S \simeq -k_B N \sum_{i=1}^{k} p_i \log p_i \ . \tag{4}$$

This is an approximate formula, but it is meant to became an exact formula in the limit as $N \to \infty$. Regarding a system with a very large number as infinite corresponds to what physicists call *taking the thermodynamic limit*. The expression in (4) can be recast in terms of the probability distribution $\{P(\omega) : \omega \in \Omega\}$ over the product space Ω. Since we have

$$P(\omega) = \frac{N!}{N_1!N_2!\cdots N_k!}\, p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \ ,$$

it follows from (4) and some straightforward computations that the right-hand side of (4) is

$$S = -k_B \sum_{\omega\in\Omega} P(\omega) \log P(\omega) \ . \tag{5}$$

This we now take as the *definition* of entropy of such a probability distribution.

The first basic foundational principle of statistical mechanics laid down by Boltzmann and Gibbs states that, when the total energy U of the gas is fixed, nature chooses the probabilities p_i in such a way as to *maximize* entropy. What is this special distribution? Mathematically, the problem is to maximize

$$S = -k_B N \sum_{i=1}^{k} p_i \log p_i \ .$$

subject to the constraints $\sum p_i = 1$ and $\sum p_i u_i = U/N$. This is a simple problem with Lagrange multipliers whose solution is

$$p_i = \frac{e^{-\beta u_i}}{\sum_{j=1}^{k} e^{-\beta u_j}} \quad ; \quad i = 1, 2, \ldots, k \ ,$$

where β is the unique root of the equation

$$\sum_{j=1}^{k} u_j e^{-\beta u_j} = \frac{U}{N} \sum_{j=1}^{k} e^{-\beta u_j} \ .$$

The probability distribution obtained in this fashion is called a *Gibbs distribution*. The denominator appearing in these expressions, namely $Z_\beta = \sum_{j=1}^{k} e^{-\beta u_j}$, is called the *partition function* of the gas.

Another very important notion in thermodynamics is that of *free energy*. It is defined as $F = U - TS$, and physically it is the energy in the gas that is available for work. If we write it infinitesimally, we have $dF = dU - TdS - SdT$, and if we combine this relation with the first and second laws, we see that

$$dF = -P\,dV - S\,dT .$$

This tells us in particular that

$$P = -\frac{\partial F}{\partial V} . \tag{6}$$

This can be taken as the statistical mechanical *definition* of pressure, as long as we manage to write the free energy in purely microscopic terms. And this we can certainly do for our ideal gas. The second foundational principle of Boltzmann and Gibbs tells us that in a isothermal process (i.e., at fixed temperature), nature chooses the probabilities so as to *minimize the free energy*. Using this, one arrives after a similar computation as before to the same Gibbs distribution probabilities, this time with $\beta = 1/k_B T$, and one sees also that

$$F = -\frac{N}{\beta} \log Z_\beta . \tag{7}$$

3. Ergodic ideas: entropy, pressure and Gibbs states

Given the above motivation from statistical mechanics, the concepts from ergodic theory and topological dynamics to be introduced below will, we hope, seem more natural.

3.1. **Entropy in dynamics.** The concept of entropy was introduced in ergodic theory by Kolmogorov and Sinai in 1959. Their definition was undoubtedly inspired not only by the Boltzmann-Gibbs entropy, especially in the form (5), but also by Shannon's notion of *information entropy*, or *uncertainty*. See [16] for a good account.

Let $(X, \mathscr{B})$ be a measurable space, let μ be a (probability) measure on X, and let $T : X \to X$ be a measure-preserving transformation (meaning $T^{-1}B \in \mathscr{B}$ for all $B \in \mathscr{B}$). We define the entropy of T with respect to μ as follows. First we associate to each partition $\mathscr{A} = \{A_1, A_2, \ldots, A_m\}$ of X into measurable sets $A_j \in \mathscr{B}$ (pairwise disjoint modulo null-sets) its own entropy, namely the number

$$h(\mathscr{A}, \mu) = -\sum_{A \in \mathscr{A}} \mu(A) \log \mu(A) .$$

Note the similarity with (5). This is always a non-negative number (we assume that $0 \cdot \log 0 = 0$, encouraged by the continuity of $x \mapsto x \log x$). Next, given two partitions $\mathscr{A}$ and $\mathscr{B}$ of X, we define their *join* as the partition $\mathscr{A} \vee \mathscr{B} = \{A \cap B : A \in \mathscr{A} \text{ and } B \in \mathscr{B}\}$. This can of course be extended to the join of any number of partitions. In particular, given a partition $\mathscr{A}$, we define for each $n \geq 0$ the partition $\mathscr{A}_n = \mathscr{A} \vee T^{-1}\mathscr{A} \vee \cdots \vee T^{-n+1}\mathscr{A}$, where $T^{-j}\mathscr{A} = \{T^{-j}(A) : A \in \mathscr{A}\}$ (which is obviously a partition of X for each j). It is easy to see that $h(\mathscr{A}_n, \mu)$ is a sub-additive sequence, i.e.

$$h(\mathscr{A}_{m+n}, \mu) \;\leq\; h(\mathscr{A}_m, \mu) + h(\mathscr{A}_n, \mu) \ .$$

(this follows at once from the fact that $h(T^{-j}\mathscr{A}_n, \mu) = h(\mathscr{A}_n, \mu)$ for all $j, n \geq 0$, for T is measure-preserving). Then a well-known lemma implies that the following limit exists:

$$h_\mu(T, \mathscr{A}) \;=\; \lim_{n\to\infty} \frac{1}{n} h(\mathscr{A}_n, \mu) \ .$$

Finally, we let

$$h_\mu(T) \;=\; \sup_{\mathscr{A}} h_\mu(T, \mathscr{A}) \ ,$$

and call it the *entropy* of T with respect to μ. This is an invariant under measurable conjugacies. It is also an additive invariant under composition, so that $h_\mu(T^n) = nh_\mu(T)$. The entropy invariant is very useful in telling ergodic systems apart (it turns out to be even a complete invariant for Bernoulli shifts).

When X is a compact metric space and T is continuous, and we let μ run through the set $M(X, T)$ of T-invariant Borel probability measures on X, we get the following remarkable fact, known as the *variational principle*, relating the topological entropy of T (see [27]) with the measure-theoretic entropies just defined:

$$h_{\text{top}}(T) \;=\; \sup_{\mu \in M(X,T)} h_\mu(T) \tag{8}$$

As it turns out, for (irreducible, aperiodic) subshifts of finite type, the topological entropy $h_{\text{top}}(T)$ is always strictly positive. It is in fact equal to $\log \lambda$, where λ is the largest eigenvalue of the shift's transition matrix (again, see [27]).

3.2. **Topological pressure.** Let us move to a concept that properly speaking belongs not to ergodic theory, but to *topological dynamics*. Just as with measure-theoretic entropy, the concept of topological pressure was inspired by statistical mechanics, this time on the expression of pressure as a certain derivative of the free energy, see (6)and (7). Rather than giving the most general definition, we introduce the concept of topological pressure in the specific context of repellers. For a more general treatment, see [27] or [18].

Let $\varphi : J_f \to \mathbb{R}$ be a continuous function, and let $B = J_{i_0 i_1 \cdots i_n} \in \mathscr{A}_n$ be a cylinder of J_f. We write

$$\varphi_B \;=\; \sup_{x\in B} \varphi(x) \;.$$

Let us also consider the Birkhoff sums of φ, namely

$$S_n\varphi \;=\; \sum_{j=0}^{n-1} \varphi \circ f^j \;.$$

Each of these sums is, on its own right, a continuous function on the limit set J_f.

Theorem 3. *For every continuous function $\varphi : J_f \to \mathbb{R}$, the limit*

$$P(\varphi) \;=\; \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{B\in\mathscr{A}_n} e^{(S_n\varphi)_B} \right) \tag{9}$$

exists.

Proof. Let (p_n) be the sequence given by

$$p_n \;=\; \sum_{B\in\mathscr{A}_n} e^{(S_n\varphi)_B} \;.$$

We claim that (p_n) is sub-multiplicative, in the sense that $p_{m+n} \leq p_m \cdot p_n$ for all $m, n \geq 0$. To see why, note that for all $x \in J_f$ we have

$$S_{m+n}\varphi(x) \;=\; S_m\varphi(x) + S_n\varphi(f^m(x)) \;. \tag{10}$$

Given any cylinder $B \in \mathscr{A}_{m+n}$, we know that there exist cylinders $B' \in \mathscr{A}_m$ and $B'' \in \mathscr{A}_n$ such that $B = B' \cap f^{-m}(B'')$. Taking the supremum in (10) over all $x \in B$, we get

$$(S_{m+n}\varphi)_B \;\leq\; (S_m\varphi)_{B'} + (S_n\varphi)_{B''} \;.$$

Therefore we have

$$\begin{aligned}
p_{m+m} \;&=\; \sum_{B\in\mathscr{A}_{m+n}} e^{(S_{m+n}\varphi)_B} \\
&\leq\; \sum_{B'\in\mathscr{A}_m} \sum_{B''\in\mathscr{A}_n} e^{(S_m\varphi)_{B'} + (S_n\varphi)_{B''}} \\
&=\; \left(\sum_{B'\in\mathscr{A}_m} e^{(S_m\varphi)_{B'}} \right) \left(\sum_{B''\in\mathscr{A}_n} e^{(S_n\varphi)_{B''}} \right) \;=\; p_m \cdot p_n \;.
\end{aligned}$$

This proves our claim. Hence $(\log p_n)$ is a *sub-additive sequence*. The theorem follows, then, from a well-known lemma concerning sub-additive sequences. □

The limit (9) whose existence has been established by this theorem is called the *topological pressure* of φ. When $\varphi = 0$, such limit agrees in fact with the *topological entropy* of $f|_{J_f}$, see [27].

The topological pressure satisfies a variational principle that is a generalization of (8). Indeed, one can prove (see [27]) that

$$P(\varphi) \;=\; \sup_{\mu\in M(X,T)} \{h_\mu(T) + \int_X \varphi\, d\mu\} \;.$$

In particular, $P(0) = h_{\mathrm{top}}(T) \geq 0$.

3.3. **Equilibrium states.** Another notion from ergodic theory that will be needed below is that of a Gibbs or equilibrium measure. This concept once again has its origins in statistical mechanics, see §2. Its use in dynamical systems was pioneered by Sinai, Bowen and Ruelle (see [28] for more details). Again, we restrict our discussion to the specific context at hand. Let us consider a (continuous) function $\varphi : J_f \to \mathbb{R}$.

Definition 6. *An equilibrium (or Gibbs) measure for φ is a Borel measure μ supported on J_f for which there exist constants $K \geq 1$ and $C > 0$ such that, for all cylinders $B \in \mathscr{A}_n$ and all $x \in B$ we have*

$$\frac{1}{K} \;\leq\; \frac{\mu(B)}{e^{S_n\varphi(x)+Cn}} \;\leq\; K \;. \tag{11}$$

Note that there exists *at most one* value of C for which (11) holds. The existence and uniqueness of an equilibrium measure for a given φ are not always guaranteed. A sufficient condition is to require that φ be Hölder continuous. Let us define the n-th variation of φ to be

$$\mathrm{Var}_n(\varphi) \;=\; \max\{|\varphi(x) - \varphi(y)| \,:\, x, y \in B,\; B \in \mathscr{A}_n\} \;.$$

We say that φ is Hölder continuous if the n-th variation of φ decreases exponentially with n, *i.e.* if there exist constants $c > 0$ and $0 < \alpha < 1$ such that $\mathrm{Var}_n(\varphi) \leq c\alpha^n$ for all n. See the next section for more.

3.4. **The Ruelle operator.** We work in the context of subshifts of finite type, but the discussion to follow is considerably more general. See for instance [27]. In particular, everything we say here could be written directly for our Cantor repeller (f, J_f).

Let $X \subseteq \{1, 2, \ldots, m\}^{\mathbb{N}}$ be such a subshift, and let $T : X \to X$ be the shift map. We can assume that X is endowed with a metric that attributes diameter 2^{-n} for each n-cylinder in $\mathscr{A}_n$. Suppose we are given a continuous function $\varphi : X \to \mathbb{R}$. For each $n \geq 1$, let

$$\mathrm{Var}_n(\varphi) \;=\; \sup_{A\in\mathscr{A}_n}\;\sup_{x,y\in A}|\varphi(x)-\varphi(y)|\;,$$

where, as before, $\mathscr{A}_n$ is the set of all n-cylinders of X. We write

$$\|\varphi\|_{BV} \;=\; \|\varphi\| + \sum_{n=1}^{\infty}\mathrm{Var}_n(\varphi)\;.$$

Here and throughout, $\|\cdot\|$ denotes the sup norm of $C(X)$. When $\|\varphi\|_{BV} < \infty$ we say that φ has bounded variation. The space of all such φ's having bounded variation is a Banach space denoted $BV(X)$, with norm given by the above expression. Note that if $\varphi \in BV(X)$ then φ is necessarily continuous: in fact, $\varphi \in C(X)$ if and only if $\mathrm{Var}_n(\varphi) \to 0$ as $n \to \infty$. The inclusion $BV(X) \hookrightarrow C(X)$ is a linear contraction.

Given $\varphi \in BV(X)$, we define a linear operator $\mathscr{L}_\varphi : C(X) \to C(X)$ by

$$\mathscr{L}_\varphi f(x) \;=\; \sum_{y\in T^{-1}(x)} e^{\varphi(y)} f(y)\;. \tag{12}$$

Since the shift map T is at most m-to-one on X, the sum in the right-hand side of (12) is finite and therefore the operator $\mathscr{L}_\varphi$ is well-defined. It is called the *Ruelle operator*, or *transfer operator*, associated with φ.

Here is a simple remark concerning the Ruelle operator which explains its connection with topological pressure. Note that if we take $f \in C(X)$ and compute its n-th iterated image under $\mathscr{L}_\varphi$, we get

$$\mathscr{L}_\varphi^n f(x) \;=\; \sum_{y\in T^{-n}(x)} e^{S_n\varphi(y)} f(y)\;,$$

where as before $S_n\varphi = \varphi + \varphi\circ T + \cdots + \varphi\circ T^{n-1}$ is the n-th Birkhoff sum of φ. Hence, if $f \neq 0$, we have

$$\left\|\frac{\mathscr{L}_\varphi^n f}{\|f\|}\right\|^{1/n} \;\leq\; \left(\sum_{A\in\mathscr{A}_n} e^{(S_n\varphi)_A}\right)^{1/n} \;\longrightarrow\; e^{P(\varphi)} \tag{13}$$

as $n \to \infty$. It follows at once from (13) that the *spectral radius* of $\mathscr{L}_\varphi$ is $\leq e^{P(\varphi)}$. This much is apparent, but the following more subtle result yields much more information about the spectrum of the Ruelle operator.

Theorem 4 (Ruelle-Perron-Frobenius)**.** *For each $\varphi \in BV(X)$, the Ruelle operator $\mathscr{L}_\varphi$ has the following properties.*

(a) *It has a simple eigenvalue $\beta > 0$ with associated eigenfunction $f_\beta > 0$;*

(b) *We have* $P(\varphi) = \beta$*;*

(c) *There exists a unique Borel probability measure* μ *on* X *such that* $\mathscr{L}_\varphi^*(\mu) = \beta\mu$*, and* $\beta^{-n}\mathscr{L}_\varphi^n(f)$ *converges in* $C(X)$ *to the function*

$$\frac{\int_X f\,d\mu}{\int_X f_\beta\,d\mu}\,f_\beta\ ,$$

for all $f \in C(X)$*.*

For the (non-trivial) proof, which involves the use of Schauder's fixed-point theorem several times, we refer the reader to [28, ch. 4].

How does the topological pressure $P(\varphi)$ vary as a function of φ? Given that $P(\varphi)$ is detected from the spectral radius of $\mathscr{L}_\varphi$, the answer lies in the following theorem from functional analysis. See [20, p. 166] for a proof.

Theorem 5 (Kato-Rellich)**.** *Let* E *be a complex Banach space, and let* $L(E)$ *be the space of bounded linear operators* $E \to E$ *with the operator norm topology. If* $T_0 \in L(E)$ *has a simple eigenvalue* λ_0 *which is isolated in the spectrum of* T_0*, then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the following property. If* $T \in L(E)$ *is such that* $\|T - T_0\| < \delta$*, then* T *has an isolated eigenvalue* $\lambda(T)$ *in its spectrum, and this eigenvalue is close to* λ_0 *in the sense that* $|\lambda(T) - \lambda_0| < \varepsilon$*; in other words,* $\mathrm{sp}(T) \cap D(\lambda_0, \varepsilon) = \{\lambda(T)\}$*. Moreover, the function* $T \to \lambda(T)$ *is holomorphic, and for each* T *there is an eigenvector* $v_T \in E$ *with eigenvalue* λ_T *such that* $T \to v_T$ *is also holomorphic.*

Note however that theorem 4 in the form given *does not* guarantee that β is an isolated eigenvalue. This will be true if we impose the additional condition that the function φ be Hölder continuous, in the sense that $\mathrm{Var}_n \leq C2^{-\alpha n}$ for some $C > 0$ and $\alpha > 0$. In this case, we can consider $V \subset C(X)$ consisting of all those functions f such that

$$\|f\|_V \ = \ \|f\| + \sup\{2^{n\alpha}\mathrm{Var}_n(f)\ :\ n \geq 1\}\ <\ \infty\ .$$

Then V is a Banach space under this norm, and one verifies easily that $\mathscr{L}_\varphi(V) \subset V$, and also that the eigenfunction $f_\beta \in V$. Hence we can consider the Ruelle operator acting only on V. To be able to apply the Kato-Rellich theorem to this situation, one simply considers the complex Banach space of functions of the form $f + ig$ with $f, g \in V$. After some work, one arrives at the following result.

Corollary 1. *If* φ *is Hölder continuous, there exists a unique* T*-invariant probability measure* μ *such that* $\mathscr{L}_\varphi^*(\mu) = \mu$*, and this measure is an equilibrium measure. Moreover, the function* $t \mapsto P(t\varphi)$ *is real-analytic.*

See the excellent survey [28] for a proof. Given the identification of our repeller (f, J_f) with a subshift as considered here, the above corollary could also be stated as follows.

Corollary 2. *Let $\varphi : J_f \to \mathbb{R}$ be Hölder continuous. The there exists a unique probability equilibrium measure for (f, J_f, φ). Moreover, the pressure of $t\varphi$ varies real-analytically with t.*

4. Hausdorff dimension

We digress a bit to introduce the notion of Hausdorff dimension. For more details and complete proofs of the assertions made here, see the standard reference [14]. The exposition here is taken from [7].

First we define the so-called *Hausdorff outer measures* on $\mathbb{R}^n$. Given real numbers $s > 0$ and $\varepsilon > 0$ and any (Borel) set $E \subseteq \mathbb{R}^n$, let

$$\mu_s^\varepsilon(E) \;=\; \inf_{\mathscr{B}} \sum_{B \in \mathscr{B}} |B|^s \;,$$

the infimum being taken over *all* coverings $\mathscr{B}$ of the set E by balls $B \in \mathscr{B}$ with diameter $|B| \leq \varepsilon$. Note that $\mu_s^\varepsilon(E)$ is, for fixed s and E, a non-increasing function of ε. Hence we can define $\mu_s(E) = \lim_{\varepsilon\to 0} \mu_s^\varepsilon(E)$. It is straightforward to prove that μ_s is an outer measure, for all $s > 0$. It is also an easy exercise to check that $\mu_s(E) = 0$ for all $E \subseteq \mathbb{R}^n$ when $s > n$.

Definition 7. *The Hausdorff dimension of a (Borel) set $E \subseteq \mathbb{R}^n$ is*

$$\dim_H(E) \;=\; \inf\,\{s > 0 \,:\, \mu_s(E) = 0\} \;.$$

In particular, the Hausdorff dimension of any $E \subseteq \mathbb{R}^n$ is always $\leq n$. One can show that, if $d = \dim_H(E)$, then $\mu_s(E) = \infty$ if $s < d$ and $\mu_s(E) = 0$ if $s > d$. Hausdorff dimension is a *diffeomorphism invariant*: if $E \subseteq \mathbb{R}^n$ is a Borel set and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism, then $\dim_H(g(E)) = \dim_H(E)$.

Calculating the exact value of the Hausdorff dimension of a set can be rather tricky. As a rule, good *upper* bounds are usually easier to get than good *lower* bounds. An extremely useful tool for good lower bounds is the following result.

Lemma 1. *Let $E \subseteq \mathbb{R}^n$ be a Borel set, and let μ be a (Borel) measure with support in E. Suppose there exist $s > 0$, $\varepsilon > 0$ and $C > 0$ such that $\mu(A) \leq C|A|^s$ for all measurable sets $A \subseteq E$ with $|A| \leq \varepsilon$. Then $\dim_H(E) \geq s$.*

Proof. Let $\mathscr{B}$ be any (countable) covering of E by balls of diameter $\leq \delta < \varepsilon$. Then for all $B \in \mathscr{B}$ we have

$$\mu(B) = \mu(B \cap E) \leq C|B \cap E|^s \leq C|B|^s \ .$$

Therefore

$$\begin{aligned} \sum_{B\in\mathscr{B}} |B|^s \ &\geq \ \frac{1}{C} \sum_{B\in\mathscr{B}} \mu(B) \ \geq \ \frac{1}{C}\, \mu\left(\bigcup_{B\in\mathscr{B}} B\right) \\ &= \ \frac{1}{C}\, \mu(E) \ > \ 0 \end{aligned}$$

Taking the infimum over all such coverings, we get $\mu_s^\delta(E) \geq C^{-1}\mu(E) > 0$. Letting $\delta \to 0$, we deduce that $\mu_s(E) > 0$, and this means of course that $\dim_H(E) \geq s$. □

The above lemma is known in the literature as *mass distribution principle* (see for instance [14, p. 60]). In some places it is called *Frostmann's lemma*, but this last name should be reserved to the more difficult result proved by Frostmann, namely the *converse* to the above lemma (which however will not be needed here). The mass distribution principle is extremely useful. As an exercise, the reader may use it to calculate the Hausdorff dimension of the standard triadic Cantor set.

There are many other useful dimensions in dynamics. For their definitions and properties, and to understand how they relate to each other in the context of conformal dynamics, we recommend [26].

5. Bowen's theorem and beyond

Let us now return to the problem of computing the Hausdorff dimension of our Cantor repeller J_f. Let us consider the function $\psi = -\log|f'|$ restricted to J_f. Note that the expansion property (a) defining a conformal repeller implies that, for all sufficiently large n, the Birkhoff sums $S_n\psi$ are *negative* everywhere in J_f. This fact will be used below. But first we need the following lemma, which estimates the sizes of cylinders of J_f in terms of the values of these Birkhoff sums on points of the cylinders.

Lemma 2. *There exists a constant $C > 0$ such that, for all cylinders $B \in \mathscr{A}_n$ and all points $x \in B$, we have*

$$C^{-1}e^{S_n\psi(x)} \ \leq \ |B| \leq C\, e^{S_n\psi(x)} \ .$$

The proof uses some Koebe conformal distortion estimates combined with the contraction of hyperbolic metrics given by Schwarz's lemma. See [7] for details.

This lemma tells us in particular that the sizes of cylinders in $\mathscr{A}_n$ decrease at an exponential rate as $n \to \infty$. In particular, $\varphi = -\log|f'|$ is Hölder continuous in J_f. Hence by corollary 2, there exists an equilibrium measure for φ. This fact will be used in the following theorem due to Bowen, the culmination of our efforts in this section.

Theorem 6 (Bowen)**.** *Let (f, J_f) be a Cantor repeller. Then the Hausdorff dimension of its limit set J_f is the unique real number t such that $P(-t\log|f'|) = 0$.*

Proof. Let us first prove that a value of t with the stated property exists. We will write $\varphi = -\log|f'|$ as before. Recall that

$$P(t\varphi) \;=\; \lim_{n\to\infty} \frac{1}{n} \log p_n(t) \;,$$

where

$$p_n(t) \;=\; \sum_{B\in\mathscr{A}_n} e^{t(S_n\varphi)_B} \;. \tag{14}$$

As observed before, $P(0)$ is equal to the topological entropy of (f, J_f), and this turns out to be a positive number, so $P(0) > 0$. On the other hand, because f is expanding in J_f, say $|f'| \geq \lambda > 1$ there, we have

$$\varphi \;=\; -\log|f'| \;\leq\; -\log\lambda \;<\; 0 \;.$$

Since $\mathscr{A}_n$ contains at most N^{n+1} cylinders (where N is the number of components U_i in the domain of f), we see from (14) that $p_n(t) \leq N^{n+1}e^{-\lambda tn}$. Hence

$$\frac{1}{n}\log p_n(t) \;\leq\; \left(1 + \frac{1}{n}\right)\log N - \lambda t \;,$$

and this tells us that

$$P(t\varphi) \;\leq\; \log N - \lambda t \;.$$

This shows that $P(t\varphi) \to -\infty$ as $t \to \infty$. A similar sort of argument also shows that P is a decreasing function of t. Therefore there exists a unique value of t for which $P(t\varphi) = 0$. Let us denote this special value of t by δ. Note that $\delta > 0$.

Now we need to show that δ is the Hausdorff dimension of J_f. First we claim that $\dim_H(J_f) \leq \delta$. Let us use the coverings of J_f given by the $\mathscr{A}_n$'s themselves. Given any $t > \delta$ and $\varepsilon > 0$, choose n so large that every $B \in \mathscr{A}_n$ has diameter less than ε. Applying lemma 2, we get

$$\begin{aligned} \mu_t^{\varepsilon}(J_f) \;&\leq\; \sum_{B\in\mathscr{A}_n} |B|^t \\ &\leq\; C^t \sum_{B\in\mathscr{A}_n} e^{t(S_n\varphi)_B} \;=\; C^t p_n(t) \;, \end{aligned}$$

where C is the constant in that lemma. From this, it follows that

$$\mu_t^{\varepsilon}(J_f) \;\leq\; C^t e^{n[P(t\varphi)-\epsilon_n]} \;, \tag{15}$$

where $\epsilon_n = P(t\varphi) - (\log p_n(t)/n)$ tends to zero as $n \to \infty$. Since $P(t\varphi) < 0$, the right-hand side of (15) also goes to zero as $n \to \infty$, and thus $\mu_t^{\varepsilon}(J_f) = 0$ for all $\varepsilon > 0$. Hence $\mu_t(J_f) = 0$, for all $t > \delta$, and this shows that $\dim_H(J_f) \leq \delta$ as claimed.

In order to reverse this inequality, we apply the mass distribution principle, using the equilibrium measure μ for the potential $\delta\varphi$, whose existence is guaranteed by corollary 2. We need to check that μ satisfies the hypothesis of that principle with $s = \delta$. In other words, we need to show that

$$\mu(D(x,r)) \;\leq\; Cr^{\delta} \tag{16}$$

for every disk of sufficiently small radius r centered at an arbitrary point $x \in J_f$. Given such a disk, let $n = n(r,x)$ be chosen so that

$$|(f^{n-1})'(x)| \;<\; r^{-1} \;\leq\; |(f^n)'(x)| \;.$$

Let $\mathscr{B} \subseteq \mathscr{A}_n$ be the set of all cylinders B in $\mathscr{A}_n$ such that $B \cap D(x,r) \neq \emptyset$. Then

$$\mu(D(x,r)) \;\leq\; \sum_{B\in\mathscr{B}} \mu(B) \;. \tag{17}$$

It is not difficult to see (exercise) that the number of elements of $\mathscr{B}$ is bounded by a constant independent of n. Moreover, since μ is an equilibrium measure for $\delta\varphi$ and $P(\delta\varphi) = 0$, we see from (11) that

$$\mu(B) \;\leq\; C_1 e^{\delta S_n\varphi(y)} \;, \tag{18}$$

for every cylinder $B \in \mathscr{A}_n$, where $y \in B$ is arbitrary. For cylinders in $\mathscr{B}$, we can in fact take $y \in B \cap D(x,r)$. Some standard estimates on conformal distortion (see [7, ch. 3]) applied to a suitable inverse branch of f^n yield

$$e^{S_n\varphi(y)} \;=\; |(f^n)'(y)|^{-1} \;\leq\; C_2|(f^n)'(x)|^{-1} \;.$$

Taking this information back to (18) yields

$$\mu(B) \;\leq\; C_3|(f^n)'(x)|^{-\delta} \;\leq\; C_3 r^{\delta} \;,$$

by our choice of n. Using this last inequality in (17) we get (16). This shows that μ indeed satisfies the hypothesis of lemma 1, and therefore $\dim_H(J_f) \geq \delta$. This completes the proof of Bowen's theorem. $\square$

5.1. **Further developments.** As we mentioned earlier, Bowen's theorem is only the beginning of a fascinating subject. Bowen's formula is still valid for other conformal repellers, such as Julia sets of *expanding* rational maps. In particular, it holds true in the case of a quadratic polynomial $f_c(z) = z^2 + c$ with $|c|$ small. In this case, the limit set – that is, the Julia set $J(f_c)$ – is a quasi-circle, *i.e.* the image of a round circle under a quasi-conformal map. For a proof of Bowen's formula covering such cases, see [28]. In [22], D. Ruelle proved an asymptotic formula for the Hausdorff dimension of $J(f_c)$ for $|c|$ near zero, namely

$$\dim_H(J(f_c)) \;=\; 1 + \frac{|c|^2}{4\log 2} + o(|c|^2)\ .$$

The proof involves the study of the first two derivatives of the pressure function. The Hausdorff dimension of $J(f_c)$ for c in the main cardioid of the Mandelbrot set is a real-analytic function of c, and it attains its minimum value 1 at $c = 0$. Once again, see [28, ch. 6].

In recent years, a great deal of work has been done in attempting to generalize both Bowen and Ruelle's formulas to situations where expansion fails, as in the case of parabolic rational maps. An excellent survey of this area is [26].

6. Asymptotically conformal repellers

The above result by Bowen would remain valid if we considered *uniformly asymptotically conformal* (u.a.c.) Cantor repellers, instead of conformal ones (see [10] for more on u.a.c. Cantor repellers). We assume that the reader is familiar with the basic facts about quasiconformal maps (see for instance [1]).

Let U be an open set, and let $\Lambda \subset U$ be a compact set. We say that a quasiconformal map $f : U \to \mathbb{C}$ is *asymptotically conformal* at Λ if for each $\varepsilon > 0$ there exists a neighborhood U_ε of Λ in U such that the qc-dilatation of f satisfies $K_f(z) \le 1 + \varepsilon$ for all $z \in U_\varepsilon$.

Now, a pseudo semigroup G of quasiconformal maps is said to be a uniformly asymptotically conformal near its non-wandering set Λ if for every $\varepsilon > 0$ there exists a neighborhood U_ε of Λ such for every word w in elements of G and every $z \in U_\varepsilon$ such that $w(z) \in U_\varepsilon$, we have $K_w(z) \le 1 + \varepsilon$. Using this notion, we can formulate the notion of a uniformly asymptotically conformal Cantor repeller, imitating definition 2. As we remarked, Bowen's theory can be completely adapted to this situation. In particular, one has a theory of Gibbs states for u.a.c. Cantor repellers.

In [10], we defined an appropriate notion of Teichmüller space for such u.a.c. dynamical systems. An interesting problem is to understand the relationship between such Teichmüller space and the space of Gibbs states on the u.a.c. system. For example, in the case when the repeller is topologically the full one-sided shift on two symbols, we showed in [10] that the Teichmüller space is acted upon discretely by a group called *Thompson's group*, a very interesting algebraic object. For the relevant definitions and properties of this object, see [10] and [5]. So we can end this section with a problem.

Problem 1. *Describe the action of Thompson's group on the Gibbs states of a uniformly asymptotically conformal Cantor repeller.*

The dynamical action of Thompson's group constructed in [10] involves the notion of *dual* of a Cantor repeller. For a binary Cantor set such as the one referred to above, this dual object is another Cantor repeller of the same type. For other more general subshifts of finite type, the construction is more involved, and depends on the theory of *branched Riemann surfaces*, as will be shown in the forthcoming article [9].

7. Riemann surface laminations

In this final section, we merely sketch possible connections with Sullivan's dynamical theory of *Riemann surface laminations*.

These objects are generalizations of foliations of a space whose leaves are Riemann surfaces. Let X be a (Hausdorff, second countable) topological space. A *Riemann surface lamination structure* (or RSL-structure for short) on X consists of an atlas whose charts (U_i, φ_i) are such that

(i) Each $U_i \subset X$ is open, and the union of all such open sets is a covering of X;

(ii) Each $\varphi : U_i \to D_i \times T_i$ is a homeomorphism, where $D_i \subset \mathbb{C}$ is an open disk and T_i is some (Hausdorff) topological space;

(iii) The chart transitions

$$\varphi_{ij} \ = \ \varphi_j \circ \varphi_i^{-1} \ : \ \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

have the special form $\varphi_{ij}(z,t) = (h_{ij}(z,t), \sigma_{ij}(t))$, where h_{ij} is continuous on both variables and holomorphic in z, and σ_{ij} is continuous.

Endowed with such structure, X is what we call a Riemann surface lamination. Just as in the case of foliations, one can define the leaves of a lamination: these turn out to be Riemann surfaces in a natural way. A Riemann surface lamination is said to be *hyperbolic* if all its leaves are hyperbolic Riemann surfaces.

Riemann surface laminations have been used by Sullivan in [25] to establish the contraction (without a rate) of the renormalization operator for quadratic-like maps. Sullivan's theory was adapted to the context of critical circle maps by the author in [6].

A special case of the following theorem was proved in [6]. The proof given there applies, *mutatis mutandis*, to the present more general setting.

Theorem 7. *To each uniformly asymptotically conformal Cantor repeller (U, f, V) one can associate a compact, hyperbolic Riemann surface lamination $\mathscr{L}(U, f, V)$ in such a way that*

(a) *If (U, f, V) and $(\tilde{U}, \tilde{f}, \tilde{V})$ represent the same germ, then the corresponding Rimann surface laminations $\mathscr{L}(U, f, V)$ and $\mathscr{L}(\tilde{U}, \tilde{f}, \tilde{V})$ are isomorphic.*

(b) *Every qc-conjugacy $(U_1, f_1, V_1) \sim (U_2, f_2, V_2)$ between two Cantor repellers induces a qc RSL-isomorphism*

$$\mathscr{L}(U_1, f_1, V_1) \sim \mathscr{L}(U_2, f_2, V_2)$$

between the corresponding laminations.

One can proceed in analogy with the case of Riemann surfaces and define the *Teichmüller space* of a Riemann surface lamination. See [6] for the definition. By the above theorem, the dynamics of the germ of a Cantor repeller up to conformal conjugacy is faithfully represented by the corresponding lamination up to suitable RSL-isomorphism, *i.e.*, by an element of such Teichmüller space. Hence we can formulate the following problem.

Problem 2. *Give a detailed description of the Gibbs states of an asymptotically conformal Cantor repeller in terms of the Teichmüller space of the associated Riemann surface lamination.*

Many other qustions can be asked. For example, in the case of the binary Cantor sets discussed in the previous section, an interesting problem is to study the action of Thompson's F group on the Teichmüller space of the corresponding lamination.

For an extremely elegant exposition of the general theory of Riemann surface laminations, the reader should consult [15].

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