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On simple Lie algebras of dimension seven over fields of characteristic 2

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1. Introduction

The problem of classification of the simple Lie algebras over a field of characteristic $p > 7$ was solved in the middle of the 90's by H. Strade, R. Block and R. L. Wilson (see [B], [BW1], [BW2], [SW], [S89.1], [S92], [S92.1], [Wi]). In the beginning of the 2000's, A. Premet and H. Strade proved the classification results for $p = 5$ and 7 in a series of papers [PS1], [PS2], [PS3], but for $p = 2$ and $p = 3$ the problem is still open. Throughout this paper all algebras are defined over a fixed algebraically closed field k of characteristic 2 containing the prime field $\mathbb{F}_2$. We start with some basic definitions and known facts.

Definition 1.1. *A Lie algebra L over k is a Lie* **2-algebra** *if there exists a map $L \to L$, $x \longmapsto x^{[2]}$, called* **2-map**, *such that*

$$(x + \lambda y)^{[2]} = x^{[2]} + \lambda^2 y^{[2]} + \lambda[x, y], \text{ for all } x, y \in L, \lambda \in k.$$

It is well known fact that for every algebra A over a field k of characteristic 2 the corresponding Lie algebra $Der_k A$ of $k-$derivations of A has the natural structure of $2-$Lie algebra such that $d^{[2]}(a) = d^2(a) = d(d((a))$.

Definition 1.2. *Let L be a Lie algebra such that $Z(L) = 0$, which is also called a* **centerless** *Lie algebra. The* **2-closure of L in** $Der_k(L)$,

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denoted by L_2, is the smallest subalgebra of $Der_k(L)$ containing L and closed under the 2-map.

According to H. Strade [S89], the *toral rank* of L is the maximal dimension $T(L)$ of the toral subalgebras of L. By definition, a toral subalgebra is an abelian subalgebra with a basis $\{t_1, \ldots, t_n\}$ such that $t_i^{[2]} = t_i, i = 1, \ldots, n$. The *absolute toral rank* $TR(L)$ of a centerless Lie algebra L is $T(L_2)$ — toral rank of $2-$closure of L defined above.

The first results for the classification problem in characteristic 2 are as follows.

Theorem 1.1 (S. Skryabin, [Sk])**.** *Let L be a simple finite dimensional Lie k-algebra over an algebraically closed field k of characteristic 2. Then L has absolute toral rank greater or equal to 2.*

In the case of absolute toral rank 2, A. Grichkov and A. Premet announced the following result:

Theorem 1.2 (A. Premet, A. Grichkov [GP])**.** *Let L be a simple Lie k-algebra of finite dimension with k an algebraically closed field of characteristic 2. If the absolute toral rank of L is 2, then L is classical of dimension 3, 8, 14 or 26.*

The toral rank 3 is a much more difficult case and it is still open. In this work we begin the study of the simple Lie algebras of dimension seven and absolute toral rank 3 over an algebraically closed field k of characteristic 2.

In the literature up to this date there appeared only three types of the simple Lie 2-algebras of dimension 7 and absolute toral rank 3: the Witt-Zassenhaus algebra $\overline{W(1;3)}$ [Ju], the Hamiltonian algebra H_2 [SF](p. 144) (this algebra corresponds to a non-standard 2-form) and a family $L(\varepsilon)$, called the Kostrikin-Dzhumadil'daev algebras, that depends on one parameter $\varepsilon \in k$ [K]. Here we calculate some features of these algebras such as their group of 2-automorphisms and their varieties of idempotent and nilpotent elements. We also present some Cartan decompositions for these algebras. The study of the algebras W and H_2 is motivated by the following conjecture.

Conjecture 1.1. *Let L be a simple finite dimensional Lie algebra over an algebraically closed field of characteristic 2. If $dimL > 3$ then L contains a subalgebra W or H_2.*

In this paper we prove that all simple Kostrikin-Dszumadil'daev 7-dimensional Lie algebras are isomorphic to the Hamiltonian algebra H_2.

This is a reason why we sometimes use in this paper the notation K instead of H_2 for this algebra.

In a second paper we will prove that, for dimension 7 and absolute toral rank 3, a simple Lie 2-algebra is either isomorphic to a Witt-Zassenhaus or to a Hamiltonian algebra.

Definition 1.3. *Let L be a Lie 2-algebra. A k-linear map $\varphi : L \to L$ is a* 2**-automorphism** *of L provided that $\varphi(x^{[2]}) = (\varphi(x))^{[2]}$ for all $x \in L$. Denote by $Aut_{k,2}(L)$ the group of all 2-automorphisms of L.*

Note that by definition of Lie $2-$algebras, every $2-$automorphism of a Lie $2-$algebra is an automorphism of L, but inverse is not true.

Throughout this paper we denote by $\bar{a}$ the element $a+1$, for $a \in k$, and $\langle M \rangle$ is the k-vector space spanned by the set M.

2. The Witt-Zassenhaus algebra

The simple Witt-Zassenhaus Lie algebra, denoted here by $W = \overline{W(1;3)}$, can be constructed using different approaches as one can see in [Ju], [SF] or [K]. Here we consider a basis $\{y_i : -1 \leq i \leq 5\}$ for W and denote its 2-closure in $Der_k(W)$ by $W_2 = \langle \eta, \kappa, \kappa^{[2]}, y_i : -1 \leq i \leq 5 \rangle$. The Lie multiplication in W_2 is given by the table below. Note that the diagonal of this table exhibits the elements $x^{[2]}$, for each $x \in W_2$.

The 2-closure W_2 of the Witt-Zassenhaus algebra W

	η	κ	$\kappa^{[2]}$	y_{-1}	y_0	y_1	y_2	y_3	y_4	y_5
η	0	y_4	y_2	y_5	0	0	0	0	0	0
κ	y_4	$\kappa^{[2]}$	0	0	0	y_{-1}	y_0	y_1	y_2	y_3
$\kappa^{[2]}$	y_2	0	0	0	0	0	0	y_{-1}	y_0	y_1
y_{-1}	y_5	0	0	κ	y_{-1}	y_0	y_1	y_2	y_3	y_4
y_0	0	0	0	y_{-1}	y_0	y_1	0	y_3	0	y_5
y_1	0	y_{-1}	0	y_0	y_1	y_2	0	y_4	y_5	0
y_2	0	y_0	0	y_1	0	0	0	y_5	0	0
y_3	0	y_1	y_{-1}	y_2	y_3	y_4	y_5	η	0	0
y_4	0	y_2	y_0	y_3	0	y_5	0	0	0	0
y_5	0	y_3	y_1	y_4	y_5	0	0	0	0	0

2.1. The group of 2-automorphisms $G_1 = Aut_{k,2}(W_2)$.

Proposition 2.1. *The group G_1 of 2-automorphisms of W_2 is defined on the basis elements of W_2, for $\varphi = \varphi(\alpha_{-1}, \alpha_1, \alpha_3, \alpha_4, \alpha_5) \in G_1$ and*

$\alpha_{-1} \neq 0$, *by:*

$$\begin{array}{rcl}
\varphi: \; y_{-1} & \longmapsto & \alpha_{-1}\, y_{-1} \,+\, \alpha_1\, y_1 \,+\, \alpha_3\, y_3 \,+\, \alpha_4\, y_4 \,+\, \alpha_5\, y_5 \\
y_0 & \longmapsto & y_0 \,+\, \alpha_4\, \alpha_{-1}^{-1}\, y_5 \\
y_1 & \longmapsto & \alpha_{-1}^{-1}\, y_1 \,+\, \alpha_3\, \alpha_{-1}^{-2}\, y_5 \\
y_2 & \longmapsto & \alpha_{-1}^{-2}\, y_2 \\
y_3 & \longmapsto & \alpha_{-1}^{-3}\, y_3 \,+\, \alpha_1\, \alpha_{-1}^{-4}\, y_5 \\
y_4 & \longmapsto & \alpha_{-1}^{-4}\, y_4 \\
y_5 & \longmapsto & \alpha_{-1}^{-5}\, y_5 \\
\eta & \longmapsto & \alpha_{-1}^{-6}\, \eta \\
\kappa & \longmapsto & \alpha_{-1}^{2}\, \kappa \,+\, \alpha_3^2\, \eta \,+\, \alpha_{-1}\, \alpha_1\, y_0 \,+\, (\alpha_1^2 \,+\, \alpha_{-1}\, \alpha_3)\, y_2 \,+ \\
 & & \alpha_{-1}\, \alpha_4\, y_3 \,+ (\alpha_1\, \alpha_3 \,+\, \alpha_{-1}\, \alpha_5)\, y_4 \,+\, \alpha_1\, \alpha_4\, y_5 \\
\kappa^{[2]} & \longmapsto & \alpha_{-1}^{4}\, \kappa^{[2]} \,+\, \alpha_{-1}^{2}\, \alpha_4^2\, \eta \,+\, \alpha_{-1}^{3}\, \alpha_3\, y_0 \,+\, \alpha_{-1}^{3}\, \alpha_4\, y_1 \,+ \\
 & & \alpha_{-1}^{2}(\alpha_1\, \alpha_3 \,+\, \alpha_{-1}\, \alpha_5)\, y_2 \,+\, \alpha_{-1}^{2}\, \alpha_3^2\, y_4 \,+\, \alpha_{-1}^{2}\, \alpha_3\, \alpha_4\, y_5.
\end{array}$$

Note that $\dim_k G_1 = 5$ *for every field* k *of characteristic* 2.

Proof. It is not difficult to prove that, for all $0 \neq \alpha_{-1}, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in k$, a map ϕ defined as in the proposition is a 2-automorphism of W_2. In order to prove that every 2-automorphism of W_2 is defined exactly like this, we first construct some G_1-invariant subspaces and subsets of W_2. Construct some G_1-invariant subspaces and subsets of W_2.

It is clear that all subsets defined below are G_1-invariant subsets. Note that $W = [W_2, W_2]$.

1. $V_1 = \{x \in W : x^{[2]} = 0\} = Span_k\{y_2, y_4, y_5\}$,
2. $V_2 = \{x \in W : [x, V_1] \subseteq V_1\} = Span_k\{y_0, y_1, y_2, y_3, y_4, y_5\}$,
3. $V_3 = [V_2, V_2] = Span_k\{y_1, y_3, y_4, y_5\}$,
4. $V_4 = [V_3, V_3] = Span_k\{y_4, y_5\}$,
5. $V_5 = \{x \in V_3 : [x, V_3] = 0\} = ky_5$,
6. $V_6 = \{x \in V_1 : dim[x, W_2] = 3\} = ky_2$.

Let ψ be an arbitrary 2-automorphism of W_2. Since V_5 is G_1-invariant, we may suppose that $y_5^{\psi} = y_5$, $y_{-1}^{\psi} = \sum_{i=-1}^{5} r_i y_i$. By $[y_{-1}, y_i] = y_{i-1}$, $i = 0, \ldots, 5$, we have

$$y_4^{\psi} = r_{-1}y_4, y_3^{\psi} = r_{-1}^2 y_3 + r_{-1}r_1y_5, y_2^{\psi} = r_{-1}^3 y_2 + r_{-1}^2 r_0 y_3 + r_{-1}(r_0r_1 + r_2r_{-1})y_5.$$

Since $r_{-1} \neq 0$ and V_6 is G_1-invariant, $r_0 = r_2 = 0$. Using some 2-automorphism $\phi(\alpha_{-1}, \alpha_1, \alpha_3, \alpha_4, \alpha_5)$ we may suppose that $r_0 = r_1 = r_2 =$

$r_3 = r_4 = r_5 = 0$. Hence,

$$y_{-1}^{\psi} = r_{-1}y_{-1}, \quad y_4^{\psi} = r_{-1}y_4, \quad y_3^{\psi} = r_{-1}^2 y_3,$$
$$y_2^{\psi} = r_{-1}^3 y_2, \quad y_1^{\psi} = r_{-1}^4 y_1, \quad y_0^{\psi} = r_{-1}^5 y_0.$$

By $[y_0, y_5] = y_5$, we get $r_{-1}^5 = 1$. Then $\psi = \phi(r_{-1}, 0, 0, 0, 0)$.

At last, $\eta^{\psi} = (y_3^{\psi})^{[2]}$, $\kappa^{\psi} = (y_{-1}^{\psi})^{[2]}$, since ψ is an 2-automorphism. □

2.2. **Idempotent and Nilpotent Elements of W_2.** The sets of nilpotent and idempotent elements of a Lie algebra are quite important features of the algebra structure as they allow us to construct different subalgebras and study the relations among them. In fact a method based on a study of the orbits of toral elements with respect to the automorphism group of the algebra and on an investigation of the centralizer of a toral element was already used in several papers describing the structure of tori and Cartan subalgebras of a Lie p-algebra, for a prime p, see [S92], [BW2] [R], [W].

Proposition 2.2. *For the Lie $2-$algebra W_2, the* **variety of idempotent elements** *is given by* $I(W) = \bigcup_{\delta=1}^{3} I_W^{\delta}$, *where*

$$I_W^1 = \{a^4\kappa^{[2]} + a^2\kappa + b^2\eta + ay_{-1} + c\,y_0 + (\bar{c} + b)\,y_1 + (\bar{c}^2 + b + d)\,y_2 + b\,y_3 + d\,y_4 + (\bar{c}b + d)\,y_5 : a \in k^*, b, c, d \in k\},$$
$$I_W^2 = \{a^2\eta + y_0 + b\,y_1 + b^2 y_2 + a\,y_3 + ab\,y_4 + c\,y_5 : a \in k^*, b, c \in k\},$$
$$I_W^3 = \{y_0 + a\,y_1 + a^2 y_2 + b\,y_5 : a, b \in k\}.$$

Moreover, $I_W^1 = \{\kappa^{[2]} + \kappa + y_{-1} + y_1 + y_2\}^{G_1}$; *that is, all elements of* I_W^1 *belong to the same orbit under the* G_1*-action.*

$I_W^2 = \cup_{b\in k/\mathbf{Z}_3}\{\eta + y_0 + by_1 + b^2y_2 + y_3 + by_4\}^{G_1}$, *where* $\mathbf{Z}_3 = \{1, \delta, \delta^2 = 1 + \delta\}$.

$I_W^3 = y_0^{G_1} \cup \{y_0 + y_1 + y_2\}^{G_1}$.

Proof. Let $t^{[2]} = t = b_1\kappa^{[2]} + b_2\kappa + b_3\eta + ay_{-1} + a_0y_0 + a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5$. Comparing the coefficients at $k^{[2]}, \ldots, y_5$, by Table I we get:

$$b_1 = b_2^2,\ b_2 = a^2,\ b_3 = a_3^2, \tag{1}$$
$$a = a^4a_3 + b_2a_1 + aa_0, \tag{2}$$
$$a_0 = a_0^2 + b_1a_4 + a_2b_2 + aa_1, \tag{3}$$
$$a_1 = b_1a_5 + b_2a_3 + aa_2 + a_0a_1, \tag{4}$$
$$a_2 = b_1b_3 + b_2a_4 + aa_3 + a_1^2, \tag{5}$$
$$a_3 = b_2a_5 + aa_4 + a_0a_3, \tag{6}$$

$$a_4 = b_2b_3 + aa_5 + a_3a_1, \quad (7)$$
$$a_5 = ab_3 + a_1a_4 + a_2a_3 + a_0a_5, \quad (8)$$

Note that $0 \neq t$ is an idempotent if and only if we have all equalities (1)–(8). By (1), we have $b_1 = a^4$. Suppose that $a \neq 0$. Using (2) we get

$$a_0 = 1 + aa_1 + a^3a_3. \quad (9)$$

By (5) we get

$$a_2 = a^4a_3^2 + a^2a_4 + a_1^2 + aa_3. \quad (10)$$

By (7) we have

$$a_4 = aa_5 + a_3a_1 + a^2a_3^2,\ a_2 = a^3a_5 + a^2a_3a_1 + a_1^2 + aa_3; \quad (11)$$

then $t = a^4\kappa^{[2]} + a^2 + \kappa + a_3^2\eta + ay_{-1} + (1 + aa_1 + a^3a_3)y_0 + a_1y_1 + (a^3a_5 + a^2a_3a_1 + a_1^2 + aa_3)y_2 + a_3y_3 + (aa_5 + a_3a_1 + a^2a_3^2)y_4 + a_5y_5$ is an idempotent.

In the case $a = 0$ the calculations are analogous but more easy.

All statements about the conjugation of idempotents are easy to prove. For example, consider the set I_W^2. If $b = 0$ then $t = a^2\eta + y_0 + a\,y_3 + c\,y_5 = (\eta + y_0 + y_3)^\phi$, where $\phi = \phi(x, y, 0, 0, 0)$, $x^3 = 1/a$, $y = xc/a$. Suppose that $b \neq 0$. In this case $t = a^2\,\eta + y_0 + b\,y_1 + b^2\,y_2 + a\,y_3 + ab\,y_4 + c\,y_5$ is conjugated with $t(b_1) = \eta + y_0 + b_1\,y_1 + b_1^2\,y_2 + y_3 + b_1\,y_4$. Suppose that $t(b_1)$ is conjugated with $t(b_2) = \eta + y_0 + b_2\,y_1 + b_2^2\,y_2 + y_3 + b_2\,y_4$, then $t(b_1)^\phi = t(b_2)$, $\phi = \phi(x, y, z, p, q)$. Hence, $x^3 = 1$ and $b_1x = b_2$. □

Proposition 2.3. *The variety $N(W)$ of 2-nilpotent elements is given by $N(W) = \{x \in W_2 : x^{[2]} = 0\} = \bigcup_{i=1}^3 N_W^i$, where*
$N_W^1 = \{a\eta + b\,y_2 + c\,y_4 + d\,y_5 : a \in k^*, b, c, d \in k\}$
$N_W^2 = \{a\kappa^{[2]} + \frac{b^2}{a}\,\eta + c\,y_0 + b\,y_1 + d\,y_2 + \frac{c^2}{a}\,y_4 + \frac{bc}{a}\,y_5 : a \in k^*, b, c, d \in k\}$
$N_W^3 = \{a\,y_2 + b\,y_4 + c\,y_5 : a, b, c \in k\} \subseteq W$.

Moreover,

i) $N_W^1 = \{a\eta + y_2 + cy_4 + dy_5 : 0 \neq a, d, c \in k\}^{G_1} \cup \{a\eta + y_4 + dy_5 : 0 \neq a, d \in k,\}^{G_1} \cup \{\eta + dy_5 : d \in k/\mathbf{Z}_3\}^{G_1}$, *here $k/\mathbf{Z}_3$ is the set of orbits of the following $\mathbf{Z}_3-$action on $k : x \to \delta x$, $\delta^3 = 1$.*

ii) $N_W^2 = \{\kappa^{[2]}\}^{G_1}$ *forms one orbit under the G_1-action.*

iii) $N_W^3 = \{y_2 + by_4 + cy_5 : b, c \in k\}^{G_1} \cup \{y_4 + cy_5 : c \in k\}^{G_1} \cup y_5^{G_1}$.

We note also that the G_1-stabilizers of the elements in N_W^3 have dimension 4, but they may be defined over different fields.

Proof. The set $N(W)$ we can describe as the set $I(W)$ but more easy. Consider the set of G_1-orbits of the natural G_1-action on $N(W)$. It is easy to see that $(N_W^1)^{G_1} = N_W^1$. Let $n = a\eta + b\,y_2 + c\,y_4 + d\,y_5 \in N_W^1$

and $b \neq 0$. Then we can find a diagonal automorphism $\phi = \phi(\alpha, 0, 0, 0, 0)$ such that $n^\phi = a_1\eta + y_2 + c_1y_4 + d_1y_5$. Note that for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in k$ we have $n^\phi = n^{\phi(\alpha,\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$. If $n^\phi = (a_2\eta + y_2 + c_2y_4 + d_2y_5)^{\phi(\beta,0,0,0,0)}$, then $\beta^2 = 1$ and $\phi(\beta, 0, 0, 0, 0) = 1$. It means that $a_1\eta + y_2 + c_1y_4 + d_1y_5$ is the unique representative of its G_1−orbit.

Analogously we proceed in the case $b = 0$, $c \neq 0$. Suppose that $b = c = 0$. As above we can find a diagonal automorphism ϕ such that $(a\eta + dy_5)^\phi = \eta + d_1y_5$. Let $\psi = \phi(\beta, 0, 0, 0, 0)$ and $(\eta + d_1y_5)^\phi = \eta + d_2y_5$. Therefore, $\beta^6 = 1$ and $\beta^{-5}d_1 = d_2$. Then $\beta = \delta \in k$, $\delta^3 = 1$, $\beta^{-5} = \delta$, and d_1, d_2 are contained in the same $\mathbf{Z}_3$−orbit.

The other cases may be considered analogously. □

3. The Kostrikin-Dzhumadil'daev algebras

The Kostrikin-Dzhumadil'daev Lie algebras $L(\varepsilon)$ (or KD-algebras, for brevity) of dimension 7 form a family depending on one parameter $\varepsilon \in k$ (see Example 7.2 of [K]). The multiplication table of basis elements in $L(\varepsilon)$ is as follows:

A KD-algebra $L(\varepsilon)$

	$L(\varepsilon)_{-1}$		$L(\varepsilon)_0$		$L(\varepsilon)_1$		$L(\varepsilon)_2$
	u_0	u_1	e_0	e_1	f_0	f_1	g
u_0	$\cdot$	0	$\varepsilon\, u_0$	$\bar\varepsilon\, u_1$	e_0	e_1	f_1
u_1	0	$\cdot$	$\bar\varepsilon\, u_1$	$\varepsilon\, u_0$	e_1	e_0	f_0
e_0	$\varepsilon\, u_0$	$\bar\varepsilon\, u_1$	$\cdot$	e_1	$\varepsilon\, f_0$	$\bar\varepsilon\, f_1$	g
e_1	$\bar\varepsilon\, u_1$	$\varepsilon\, u_0$	e_1	$\cdot$	$\varepsilon\, f_1$	$\bar\varepsilon\, f_0$	0
f_0	e_0	e_1	$\varepsilon\, f_0$	$\varepsilon\, f_1$	$\cdot$	g	0
f_1	e_1	e_0	$\bar\varepsilon\, f_1$	$\bar\varepsilon\, f_0$	g	$\cdot$	0
g	f_1	f_0	g	0	0	0	$\cdot$

Firstly note that for $\varepsilon = 0$ or $\varepsilon = 1$ the algebra $L(\varepsilon)$ is semi-simple but not simple. It is an easy exercise to prove that L_0 and L_1 are isomorphic. For $\varepsilon \notin \{0, 1\}$, the following theorem holds.

Theorem 3.1. *Given* $\varepsilon \notin \{0, 1\}$*, the corresponding simple* KD*-algebra* $L(\varepsilon)$ *is isomorphic to the Hamiltonian algebra* $H_2 = H((2, 1), \omega)$*.*

Proof. For $\varepsilon \in k \setminus \{0, 1\}$, consider the Lie algebra $L(\varepsilon)$ as given above and apply the following changing of basis: $V_0 = \sqrt{\varepsilon\bar\varepsilon}(u_0 + u_1)$, $V_1 = \varepsilon\, u_0 + \bar\varepsilon\, u_1$, $F_0 = f_0 + f_1$, $F_0 = \dfrac{1}{\sqrt{\varepsilon\bar\varepsilon}}(\bar\varepsilon f_0 + \varepsilon f_1)$, $E_1 = \dfrac{e_1}{\sqrt{\varepsilon\bar\varepsilon}}$, $E_0 =$

$e_0 + e_1$, $G = \frac{g}{\sqrt{\varepsilon\bar{\varepsilon}}}$. Hence, $L(\varepsilon)$ is isomorphic to the Lie algebra $K = \langle V_0, V_1, E_0, E_1, F_0, F_1, G\rangle$ given by the Lie multiplication table below. It is easy to see that a basis of the 2-closure K_2 may be chosen as follows: $\{t, m, n, V_0, V_1, E_0, E_1, F_0, F_1, G\}$ and the multiplication table in K_2 is the following:

The 2-closure K_2 of the KD-algebra K

	t	m	n	V_0	V_1	E_1	E_0	F_1	F_0	G
t	t	0	0	V_0	V_1	0	0	F_1	F_0	0
m	0	0	E_0	0	0	0	0	V_1	V_0	E_1
n	0	E_0	0	0	F_1	G	0	0	0	0
V_0	V_0	0	0	0	0	V_1	0	0	E_0	F_1
V_1	V_1	0	F_1	0	m	V_0	V_1	E_0	E_1	F_0
E_1	0	0	G	V_1	V_0	t	E_1	F_0	F_1	0
E_0	0	0	0	0	V_1	E_1	E_0	F_1	0	G
F_1	F_1	V_1	0	0	E_0	F_0	F_1	n	G	0
F_0	F_0	V_0	0	E_0	E_1	F_1	0	G	n	0
G	0	E_1	0	F_1	F_0	0	G	0	0	0

Note that K has a Cartan subalgebra $C = k\{E_0, F_0, V_0\}$ of toral rank one (but the absolute toral rank of C is equal to two!) Recall that Skryabin's Theorem 6.2 [Sk] asserts (in particular) that every finite dimensional simple Lie algebra L over a field of characteristic 2 with a Cartan subalgebra C of toral rank one is isomorphic to a Hamiltonian algebra if $dim L/L_0 = 2$, where L_0 is a maximal subalgebra that contains C. In our case $K_0 = Span_k\{E_0, F_0, V_0, G, F_1\}$ and $dim K/K_0 = 2$. Hence K is a Hamiltonian algebra by Skryabin's Theorem. On the other hand there exists a unique 7-dimensional Hamiltonian algebra $H_2 = H((2,1), \omega)$, where $\omega = (1 + x_1^{(3)} x_2) dx_1 \wedge dx_2$ is a non-standard 2-form. □

From now on we will denote a KD-algebra $L(\varepsilon)$, for $\varepsilon \notin \{0,1\}$, simply by K and its 2-closure by K_2, as in the theorem above.

3.1. **The group of 2-automorphisms $G_2 = Aut_{k,2}(K_2)$.**

Proposition 3.1. *The group of* 2*-automorphisms* G_2 *of the Lie* 2*-algebra* K_2 *is defined on its basis elements, for* $\varphi = \varphi(a, b, c) \in G_2$ *and* $a \neq 0$ *, by:*

$$\begin{array}{rrcl} \varphi: & E_0 & \longmapsto & E_0 + a^{-2} b^2 G \\ & G & \longmapsto & a^2 G \\ & F_0 & \longmapsto & a F_0 \\ & F_1 & \longmapsto & a F_1 + b G \\ & E_1 & \longmapsto & E_1 + a^{-1} b F_1 + c G \\ & V_0 & \longmapsto & a^{-1} V_0 + a^{-2} b E_0 + a^{-3} b^2 F_1 + a^{-3} b^2 F_0 + a^{-4} b^3 G \\ & V_1 & \longmapsto & a^{-1} V_1 + a^{-2} b E_1 + a^{-1} c F_1 + a^{-3} b^2 F_0 + (a^{-2} b c + \\ & & & a^{-4} b^3) G \\ & n & \longmapsto & a^2 n \\ & t & \longmapsto & t + a^{-2} b^2 n + a^{-1} b F_0 \\ & m & \longmapsto & a^{-2} m + a^{-4} b^2 t + (a^{-2} c^2 + a^{-6} b^4) n + a^{-3} b V_0 + \\ & & & a^{-4} b^2 E_1 + a^{-2} c E_0 + a^{-5} b^3 F_1 + a^{-5} b^3 F_0 + \\ & & & a^{-4} b^2 c G. \end{array}$$

Note that $\dim_k G_2 = 3$ *for every field* k *of characteristic* 2*.*

Proof. Let ϕ be an automorphism of K_2. Then $\{x \in K : x^{[2]} = x\}^{\phi} = \{x \in K : x^{[2]} = x\} = \{E_0 + aG : a \in k\}$; in particular, $E_0^{\phi} = E_0 + aG$.

For all $a_1, a_2 \in k$, the map $E_0 + a_2 G \to E_0 + a_1 G$ may be extended to an automorphism $\psi = \psi_{a_1,a_2}$. Hence, $E_0^{\phi\psi_{0,a}} = E_0$ and we may assume that $E_0^{\phi} = E_0$. Let $S = Ann_K E_0 = Span_k\{V_0, E_0, F_0\}$. Then $S^{\phi} = S$ and $V_0^{\phi} = aV_0$, $0 \neq a \in k$, since $kV_0 = \{x \in S : x^{[2]} = 0\}$. It is easy to see that the map $\tau : E_0 \to E_0, V_0 \to a^{-1}V_0$, $V_1 \to a^{-1}V_1$, $F_1 \to aF_1$, $F_0 \to aF_0$, $G \to a^2 G$ is an automorphism. Therefore, $V_0^{\phi\tau} = V_0$ and we may suppose that $E_0^{\phi} = E_0$, $V_0^{\phi} = V_0$. Since $\{x \in S : x^{[4]} = 0\}^{\phi} = \{x \in S : x^{[4]} = 0\} = kV_0 \cup kF_0$, we have $F_0^{\phi} = F_0$. Analogously, if $T = \{x \in K : [x, E_0] = x\}$ then $Ann_T F_0 = kG$ and $G^{\phi} = G$. We have $E_1^{\phi} = E_1 + aF_1 + bG$, then

$$[E_1^{\phi}, F_0^{\phi}] = [E_1, F_0]^{\phi} = F_1^{\phi} = F_1 = [E_1 + aF_1 + bG, F_0] = F_1 + aG,$$

and $a = 0$. Furthermore,

$$V_1^{\phi} = [E_1, V_0]^{\phi} = [E_1^{\phi}, V_0^{\phi}] = [E_1 + bG, V_0] = V_1 + bF_1.$$

It is easy to see that ϕ is an automorphism. Hence, $dim G_2 = 3$. □

3.2. **Idempotent and Nilpotent Elements of** K_2**.**

Proposition 3.2. *For the* 2*-closure* K_2 *of the* KD*-algebra, the* **variety of idempotent elements** $I(K) = \{x \in k_2 : 0 \neq x^{[2]} = x\}$ *is given by*

$I^1_K = \bigcup_{i=1}^{6} I^i_K$, *where*
$I^1_K = \{\alpha^2 t + \xi^{-2} m + \xi^2(b+\bar\alpha\bar a)^2 n + a\xi^{-1} V_0 + \xi^{-1} V_1 + \alpha\bar\alpha E_1 + b E_0 + \xi(b+\bar\alpha(\alpha a+\bar\alpha))F_1 + \xi\bar\alpha(\alpha a + a + \alpha) F_0 + \xi^2\bar\alpha(b\alpha + \alpha a + a) G : \alpha, a, b \in k, \xi \in k^*\}$
$I^2_K = \{t + \xi^2(b^2+b+c)^2 n + \xi^{-1} V_0 + b E_0 + c\xi F_1 + \xi(b^2+b) F_0 + \xi^2 b c G : \xi, b, c \in k\}$
$I^3_K = \{ t + \xi^{-1}c^2 n + E_0 + c\xi F_0 + \xi^2 d G : \xi, c, d \in k\}$
$I^4_K = \{ t + \xi^2(c_0+c_1)^2 n + \xi c_1 F_1 + c_0 \xi F_0 + \xi^2 c_0 c_1 G : \xi, c_0, c_1 \in k\}$
$I^5_K = \{ \delta t + \delta a^2 n + a(\delta F_0 + F_1) + E_1 + E_0 + d G : \delta^2+\delta+1 = 0,\ a, d \in k\}$
$I^6_K = \{ E_0 + d G : d \in k\}$.

Proposition 3.3. *The* **variety of nilpotent elements** $N(K) = \{x \in K_2 : x^{[2]} = 0\}$ *is described as follows:* $N(K) = \bigcup_{i=1}^{6} N^i_K$, *where*
$N^1_K = \{t + \beta m + (c^2+\beta d^2) n + \beta c V_0 + E_1 + \beta d E_0 + c(F_1 + F_0) + d G : \beta, d, c \in k\}$
$N^2_K = \{ t + c^2 n + E_1 + c(F_0 + F_1) + d G : d, c \in k\}$
$N^3_K = \{ n + d G : d \in k\}$, $\quad N^4_K = \{ n + a V_0 : a \in k\}$
$N^5_K = \{ n + b^3 V_0 + d b^2 E_0 + b d^2(F_0 + F_1) + d^3 G : d, b \in k\}$
$N^6_K = \{ \alpha^3 V_0 + \alpha^2\gamma E_0 + \alpha\gamma^2 (F_0 + F_1) + \gamma^3 G : \alpha, \gamma \in k\}$.

Proofs of Propositions 3.2 and 3.3 are analogous to the proof of Proposition 2.2. □

Proposition 3.4. *The G_2-orbits of the variety* $I(K) = \bigcup_{i=1}^{7} OI^i_K$ *are*
$I^1_K = OI^1_K = \cup_{\lambda\in k} OI^1_{K,\lambda}$, $OI^1_{K,\lambda} = \{ t + m + \lambda V_0 + V_1 \}^{G_2}$
$I^2_K = OI^2_K = \cup_{b\in k} OI^2_{K,b}$, $OI^2_{K,b} = \{ t + V_0 + b E_0 + b\bar b (F_1 + F_0) + b^2\bar b G \}^{G_2}$
$I^3_K = OI^3_K = \cup_{d\in k} OI^3_{K,d}$, $OI^3_{K,d} = \{ t + E_0 + d G \}^{G_2}$
$I^4_K = OI^4_K \cup OI^5_K$, $OI^4_K = \{ t \}^{G_2}$ $OI^5_K = \{ t + F_1 + F_0 + G \}^{G_2}$
$I^5_K = OI^6_K = \{ \delta t + E_1 + E_0 : \delta^2 + \delta + 1 = 0\}^{G_2}$
$I^6_K = OI^7_K = \{ E_0 \}^{G_2}$.

Proof. Show that $I^1_K = \cup_{\lambda\in k} OI^1_{K,\lambda}$. Denote by $\phi(a, b, c)$ an automorphism from Proposition 3.1. Let $a_1 = \xi$, $b_1 = \xi^2(1+\alpha)$, $c_1 = \xi(\xi^{-3}b^2 + \xi(b + \bar a\bar\alpha))$, $\lambda = a_1(a\xi^{-1} + a_1^{-3} b_1)$. Then by direct calculation we get

$$(t + m + \lambda V_0 + V_1)^{\phi(a_1,b_1,c_1)} = \alpha^2 t + \xi^{-2} m + \xi^2(b+\bar\alpha\bar a)^2 n + a\xi^{-1} V_0 +$$

$$\xi^{-1} V_1 + \alpha\bar\alpha E_1 + b E_0 + \xi(b+\bar\alpha(\alpha a+\bar\alpha))F_1 + \xi\bar\alpha(\alpha a+a+\alpha) F_0 + \xi^2\bar\alpha(b\alpha +$$

$\alpha a + a) G \in I^1_{K,\lambda}$.

The other cases may be considered analogously. For example,

$$I^6_K = \{ \delta t + E_1 + E_0 : \delta^2 + \delta + 1 = 0, \}^{G_2},$$

since $(\delta t + E_1 + E_0)^{\phi(1,a,d+a^2)} = \delta t + \delta a^2 n + a(\delta F_0 + F_1) + E_1 + E_0 + d G$. □

Note that $N^5_K \subset K$. We have the following result on the varieties of nilpotent and idempotent elements.

Theorem 3.2. *The varieties $I(A)$ and $N(A)$, for $A \in \{W, K\}$, are irreducible.*

Proof. We write a detailed proof for the variety $I(K)$ and leave the other cases to the reader. It suffices to prove that the first orbit includes in its closure (in the Zariski topology) all the other orbits. Observe that a generic element of the orbit $orb(1)$, in projective coordinates, is written as: $f(\lambda, \xi, \alpha, b, a) = \lambda^4\xi^2\alpha^2 t + \lambda^8 m + \xi^4(b^2\lambda^2 + (\lambda+\alpha)^2(\lambda+a)^2) n + \lambda^6 a\xi V_0 + \lambda^7\xi V_1 + \lambda^4\xi^2\alpha(\lambda+\alpha) E_1 + \lambda^5\xi^2 b E_0 + \lambda^2\xi^3(b\lambda^2 + (\lambda+\alpha)(\alpha a + \lambda^2 + \alpha\lambda)) F_1 + \lambda^2\xi^3(\lambda+\alpha)(\lambda\alpha + a\alpha + a\lambda) F_0 + \lambda\xi^4(\lambda+\alpha)(b\alpha + a\alpha + a\lambda) G$.

1) Now we make the following substitutions: $b = \frac{1}{\lambda}$, $\xi = \frac{\lambda^3}{(1+\lambda)^3}$, $a = \dfrac{1}{\lambda(\lambda+1)}$, $\alpha = 1$ and $\bar\lambda = \lambda + 1$. Hence,

$$\begin{aligned} f(\lambda, \frac{\lambda^3}{\bar\lambda^3}, 1, \lambda^{-1}, \frac{1}{\lambda\bar\lambda}) &= \frac{\lambda^{10}}{\bar\lambda^6}(t + n + E_0 + F_0) + \lambda^8 m + \frac{\lambda^8}{\bar\lambda^4} V_0 + \frac{\lambda^{10}}{\bar\lambda^3} V_1 + \\ &\quad \frac{\lambda^{10}}{\bar\lambda^5} E_1 + \frac{\lambda^{10}}{\bar\lambda^6} F_1. \end{aligned}$$

Let χ be the closure (in the Zariski topology) of the orbit OI^1_K. Then we have $\lambda^{10}(t + n + E_0 + F_0) + \bar\lambda(\bar\lambda^5\lambda^8 m + \bar\lambda\lambda^8 V_0 + \bar\lambda^2\lambda^{10} V_1 + \bar\lambda\lambda^{10} E_1) + \lambda^{10} F_1 \in \chi$. Hence, for $\lambda = 1$, one gets $u = t + n + E_0 + F_0 \in \chi$. Applying the automorphism $\varphi(a,b,c)$ with $a^2 = b, c = 0$ to u we obtain $u^\varphi = t + E_0 + a^2 G \in \chi$. Therefore, OI^3_K is contained in χ.

2) Putting $\xi = a$, $\lambda = \alpha$, $\alpha_1 = \frac{\alpha}{a}$, $b_1 = \frac{b}{\alpha}$, we have

$$\begin{aligned} f = f(\alpha, a, \alpha, b, a) &= \alpha^6 a^2 t + \alpha^8 m + \alpha^2 b^2 a^4 n + \alpha^6 a^2 V_0 + \alpha^7 a V_1 + \\ &\quad \alpha^5 a^2 b E_0 + \alpha^4 a^3 b F_1. \end{aligned}$$

Hence, $\frac{f}{\alpha^6 a^2} = (t + V_0) + \alpha_1^2 m + \alpha_1 V_1 + \left(\frac{b_1}{\alpha_1}\right)^2 n + b_1 E_0 + \frac{b_1}{\alpha_1} F_1 = \bar f(\alpha_1, b_1)$. Therefore, $\bar f(\alpha_1, \tau\alpha_1) = (t + V_0 + \tau^2 n + \tau F_1) + \alpha_1^2 m +$

$\alpha_1 V_1 + \alpha_1 E_0$. Thus, for $\alpha = 0$, one gets $g = t + V_0 + \tau^2 n + \tau F_1 \in \chi$. Applying the automorphism $\varphi = \varphi(1, \tau, 0)$ to g, we obtain $g^\varphi = t + V_0 + \tau E_0 + \tau\bar{\tau}(F_0 + F_1) + \tau^2\bar{\tau} G \in \chi$. Therefore, OI_K^2 is also contained in χ.

3) Now put $b = 0$ and $\lambda = a$ in f. Then
$f = a^4\xi^2\alpha^2 t + a^8 m + a^7\xi(V_0 + V_1) + a^4\xi^2\alpha(a+\alpha)E_1 + a^4\xi^3(a+\alpha)^2(F_0 + F_1) + a^2\xi^4(a+\alpha)^2 G$.

Substituting $a_1 = \frac{a}{\xi}$, $a_2 = \frac{a}{\alpha}$ we have:

$$g = \frac{f}{a^4\xi^2\alpha^2} = t + a_1^2 a_2^2 m + (1 + a_2)E_1 + a_1 a_2^2(V_0 + V_1) + \frac{(1 + a_2)a_2}{a_1}(F_0 + F_1) + \frac{(1 + a_2)^2}{a_1^2} G.$$

For $a_1 = a_2 + 1$ one gets $g = t + \bar{a_2}^2 a_2^2 m + \bar{a_2} E_1 + \bar{a_2} a_2^2(V_0 + V_1) + a_2(F_0 + F_1) + G$. Hence, if $a_2 = 1$, then $g = t + F_0 + F_1 + G \in \chi$, that is, OI_K^4 is contained in χ.

4) Let $\lambda = \tau\alpha = b$, $a = \tau^2\alpha$ and so, as $\tau^2 + \tau = 1$, we have $\alpha + \lambda = \tau^2\alpha$. Hence,

$f(\tau\alpha, \xi, \tau\alpha, \tau^2\alpha) = \tau\alpha^6\xi^2 t + \alpha^6\xi^2(E_0 + E_1) + \tau^2\alpha^8 m + \tau^2\alpha^7\xi V_0 + \alpha^7\xi V_1$.

By substituting $\rho = \dfrac{\alpha}{\xi}$, one gets $\dfrac{f}{\alpha^6\xi^2} = (\tau t + E_0 + E_1) + \tau^2\rho^2 m + \tau^2\rho V_0 + \rho V_1$. For $\rho = 0$ we have $\tau t + E_0 + E_1 \in \chi$. Therefore, $OI_K^6 \subset \chi$.

5) Applying the automorphism $\varphi = \varphi(a, 0, 0)$ to $g = t + F_0 + F_1 + G$, we get $g^\varphi = t + a(F_0 + F_1) + a^2 G$. Hence, for $a = 0$, the orbit of t is also contained in χ.

6) Finally, to prove that $OI_K^7 \subset \chi$, consider $\frac{1}{b}(t + V_0 + b E_0 + b\bar{b}(F_1 + F_0) + b^2\bar{b} G) = at + aV_0 + E_0 + \bar{b}(F_1 + F_0) + b\bar{b} G$, with $a \in k$. In this way, for $a = 0, b = 1$, in the Zariski topology, E_0 lies in the closure of OI_K^2, which is contained in χ. □

3.3. **Cartan decompositions.** An interesting and important problem for a Lie 2-algebra is the classification of its Cartan subalgebras up to automorphisms. Here we give some examples of Cartan subalgebras of K_2 and W_2 such that the corresponding Cartan decomposition is defined over a field $\mathbf{F}_4$ for W_2 and over $\mathbf{F}_2$ for the algebra K_2.

Conjecture 3.1. *A toral subalgebra of A_2 of dimension 3 always has an idempotent from I_A^1, $A \in \{W, K\}$. Let T be a toral subalgebra of W_2 of dimension 3. Suppose that T is defined over a field $\mathbf{F}$, then $\mathbf{F}_4 \subseteq \mathbf{F}$.*

A particular example of a toral Cartan subalgebra T of W_2 is generated by $\{t_1, t_2, t_3\}$ where $t_1 = \eta + y_0 + y_3$, $t_2 = \kappa^{[2]} + \kappa + y_{-1} + y_1 + y_2$, $t_3 = \delta^2(\kappa + y_1) + \delta(\kappa^{[2]} + y_{-1} + y_2)$, with $\delta^2 + \delta + 1 = 0$, $\delta^3 = 1$, $\delta \in k^*$.

Let $\mathcal{G} = \langle \alpha, \beta, \gamma \rangle$ be an elementary abelian group of order 8. A Cartan decomposition of W_2 with respect to T is given by

$$W_2 = T \oplus \sum_{\xi \in \mathcal{G}} \oplus L_\xi,$$

where $L_\xi = \langle e_\xi \rangle$ and $e_\alpha = y_{-1} + y_2, e_\beta = \delta^2(y_0 + y_3) + (y_2 + y_5) + \delta\, y_2$, $e_\gamma = y_0 + y_2 + y_3 + y_4 + y_5$, $e_{\alpha+\beta} = y_{-1} + y_2 + y_5 + \delta(y_1 + y_4) + \delta^2 y_3$, $e_{\alpha+\gamma} = y_{-1} + y_1 + y_2 + y_3 + y_4 + y_5$, $e_{\beta+\gamma} = \delta(y_0 + y_3) + (y_2 + y_5) + \delta^2 y_4$ and $e_{\alpha+\beta+\gamma} = y_{-1} + y_2 + y_5 + \delta\, y_3 + \delta^2\,(y_1 + y_4)$.

In the diagonal of the table below, we present the elements $e_\xi^{[2]}$, $\xi \in \mathcal{G}$ and $\tilde{t} = t_3 + \delta(t_1 + t_2)$, $\check{t} = \delta^2 t_1 + \delta t_2 + t_3$. Note that this Cartan decomposition occurs over a field k with four elements.

	e_α	e_β	e_γ	$e_{\alpha+\beta}$	$e_{\alpha+\gamma}$	$e_{\beta+\gamma}$	$e_{\alpha+\beta+\gamma}$
e_α	$t_3 + \delta t_2$	$\delta^2 e_{\alpha+\beta}$	$e_{\alpha+\gamma}$	$\delta^2 e_\beta$	e_γ	$\delta e_{\alpha+\beta+\gamma}$	$\delta e_{\beta+\gamma}$
e_β	$\delta^2 e_{\alpha+\beta}$	δt_1	0	$\delta^2 e_\alpha$	$\delta^2 e_{\alpha+\beta+\gamma}$	0	$e_{\alpha+\gamma}$
e_γ	$e_{\alpha+\gamma}$	0	t_1	$e_{\alpha+\beta+\gamma}$	e_α	0	$e_{\alpha+\beta}$
$e_{\alpha+\beta}$	$\delta^2 e_\beta$	$\delta^2 e_\alpha$	$e_{\alpha+\beta+\gamma}$	$\tilde{t}$	$\delta e_{\beta+\gamma}$	$\delta e_{\alpha+\gamma}$	e_γ
$e_{\alpha+\gamma}$	e_γ	$\delta^2 e_{\alpha+\beta+\gamma}$	e_α	$\delta e_{\beta+\gamma}$	$t_3 + t_1 + \delta t_2$	$\delta e_{\alpha+\beta}$	$\delta^2 e_\beta$
$e_{\beta+\gamma}$	$\delta e_{\alpha+\beta+\gamma}$	0	0	$\delta e_{\alpha+\gamma}$	$\delta e_{\alpha+\beta}$	$\delta^2 t_1$	δe_α
$e_{\alpha+\beta+\gamma}$	$\delta e_{\beta+\gamma}$	$\delta^2 e_{\alpha+\gamma}$	$e_{\alpha+\beta}$	e_γ	$\delta^2 e_\beta$	δe_α	$\check{t}$

Consider the following elements of K_2:

$t_1 = m + E_0 + V_1$	$a_1 = E_1 + F_0 + G$	$b_1 = V_0 + F_0 + G$
$t_2 = t + n + F_1$	$a_2 = E_0 + V_1$	$b_2 = E_0 + F_1$
$t_3 = t + m + V_1$	$a_3 = E_1 + F_0$	$b_3 = V_0$
		$b = V_1 + E_1 + F_1$

Let $T = \langle t_i : i = 1, 2, 3 \rangle$ with $t_i^{[2]} = t_i$. It is easy to verify that $[a_i, t_j] = \delta_{ij}\, a_i$, $I(K) = \{t \in K : t^{[2]} = t\} = \{\alpha a_1 + a_2 + \alpha a_3 + b_2 + b : \alpha \in k\}$. This gives a decomposition of K_2 on root spaces, and we have the following Lie multiplication table, where in the diagonal are written the elements $x^{[2]}$. Observe that this multiplication is defined over the prime field $\mathbb{F}_2$.

	t_1	t_2	t_3	a_1	a_2	b_1	a_3	b_2	b_3	b
t_1	t_1	0	0	a_1	0	b_1	0	b_2	0	b
t_2	0	t_2	0	0	a_2	b_1	0	0	b_3	b
t_3	0	0	t_3	0	0	0	a_3	b_2	b_3	b
a_1	a_1	0	0	t_2	b_1	a_2	0	a_3	b	b_3
a_2	0	a_2	0	b_1	t_1	a_1	b_3	b	0	b_2
b_1	b_1	b_1	0	a_2	a_1	$t_1 + t_2 + t_3$	b	0	b_2	a_3
a_3	0	0	a_3	0	b_3	b	t_2	a_1	a_2	b_1
b_2	b_2	0	b_2	a_3	b	0	a_1	$t_1 + t_2 + t_3$	0	a_2
b_3	0	b_3	b_3	b	0	b_2	a_2	0	0	0
b	b	b	b	b_3	b_2	a_3	b_1	a_2	0	$t_2 + t_3$

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