Reducible Volterra retarded equations with infinite delay

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Abstract. Our objective here is to survey some of the results obtained for the reducible case of the Volterra retarded equations with infinite delay. In this special case one is able to address global questions for the nonlinear system generated by the Volterra equations using the results obtained for the reduced system. Qualitative questions like the existence of Liapunov functions and global attractors and, in general, questions regarding the behavior of the global solutions are considered.

1. **Introduction**

A system of retarded equations is said to be *reducible* if its global bounded solutions are solutions of a system of ordinary differential equations, the *reduced* system. The study of the global solutions of such systems is greatly facilitated by this feature since the study of the corresponding reduced

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equations is often a much more amenable problem. Among other issues, the reduced system is quite adequate for the study of invariant sets for the dynamical systems generated by the retarded equations and the consideration of their geometric properties.

In the following we survey some of the results obtained in [8] for the Volterra retarded equations with infinite delay of the form

$$\dot{x}_j(t) = x_j(t)\left[\varepsilon_j + \sum_{k=1}^{m} a_{jk}x_k(t) + \sum_{k=1}^{m} p_{jk}\int_{-\infty}^{0} e^{\theta}x_k(t+\theta)d\theta\right] \ , \qquad (1)$$

where ε_j, a_{jk} and p_{jk} are constants and $x_j = x_j(t) > 0$, for $j, k = 1, \ldots, m$. Due to the special form of the infinite delay terms these Volterra equations are reducible. In fact, their global bounded solutions (globally defined on $\mathbb{R}$) are solutions of a finite dimensional system of ordinary differential equations. Moreover, we can show that this relation between the solutions of the retarded equations and the reduced system extends to the global solutions (bounded or unbounded) which exhibit a certain backwards exponential behavior. This is the contents of our main result in Theorem 1.

These results are adequate for the study of the invariant sets for the dynamical system generated by the Volterra equations allowing us to address some global questions. For example, we can consider the existence and the qualitative properties of attractors using some results related to the existence of Liapunov functions for the infinite delay retarded system. Moreover, under special conditions on the parameters, we are able to discuss the occurrence of hamiltonian dynamics for the Volterra system.

These types of problems are not new in the scientific literature. We mention the monograph of MacDonald [7] that deals with systems of retarded equations appearing in biological models and is mainly devoted to the analysis of reducible equations. For functional differential equations with infinite delay already Fargue [2] gave conditions under which such systems can be reduced to ordinary differential equations. Later on Wörz-Busekros [12] used this result to discuss the stability of the equilibria of the Volterra system (1).

Equations of the form (1) are functional differential equations of retarded type. In standard notation these equations have the form $\dot{x}(t) = f(x_t)$, where $x_t(\theta) = x(t+\theta)$ for $\theta \in (-\infty, 0]$ and, in appropriate phase spaces, they generate nonlinear dynamical systems, [3, 5]. An adequate phase space for these equations is the separable Banach space $C_\gamma, \gamma > 0$, defined in the following way:

$$C_\gamma = \{\varphi \in C((-\infty, 0], \mathbb{R}^m) : \lim_{\theta\to-\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists } \} \ , \qquad (2)$$

(see Hino, Murakami and Naito [6]). The phase space we use for (1) is the set C_γ^+, which is open in C_γ, and whose elements are continuous paths $\varphi \in C((-\infty,0],\mathbb{R}_+^m)$. In fact, all the classical results on existence, uniqueness and continuous dependence of solutions for the equations with finite delay also hold in the case of infinite delay with these phase spaces, [6].

2. The Volterra equations

Motivated by the study of equations for biological models involving heredity Volterra considered in [11] the following system of retarded equations with infinite delay on $\mathbb{R}_+^m$

$$\dot{x}_j(t) = x_j(t)\left[\varepsilon_j + \sum_{k=1}^m a_{jk}x_k(t) + \sum_{k=1}^m \int_{-\infty}^0 F_{jk}(\theta)x_k(t+\theta)d\theta\right] , \qquad j = 1,\dots,m \ , \tag{3}$$

where the functions $F_{jk} = F_{jk}(\theta) \in C((-\infty,0])$ appearing in the delay terms, i.e. the *memory* functions, satisfy the integrability condition

$$\int_{-\infty}^0 F_{jk}(\theta)d\theta < +\infty \ . \tag{4}$$

If these functions have the form $F_{jk}(\theta) = p_{jk}e^\theta$ we obtain the Volterra system (1) of retarded equations with infinite delay that we are considering on this survey.

The flow generated by these equations crucially depends on the parameters ε_j, a_{jk} and p_{jk}, with $j,k = 1,\dots,m$. For further reference we introduce the matrices $A = (a_{jk})$ and $P = (p_{jk})$, respectively the *interaction* and the *perturbation* matrix. In order to describe the flow dependence on the parameters some classical results are easily adapted to these equations. For example, one immediately observes that there exist equilibrium solutions $\mathbf{q} = (q_1,\dots,q_m) \in \mathbb{R}_+^m$ of (1) if, and only if,

$$\varepsilon_j = -\sum_{k=1}^m (a_{jk}+p_{jk})q_k \ , \ j = 1,\dots,m \ . \tag{5}$$

In the following, we let $0 < \gamma < 1$. Next, for $x_j = x_j(t) \in C_\gamma$, we define the functions

$$y_j(t) := \int_{-\infty}^0 e^\theta x_j(t+\theta)\, d\theta \quad , \quad j = 1,\dots,m \ .$$

Then, an integration by parts yields

$$\dot{y}_j = \int_{-\infty}^{0} e^{\theta}\dot{x}_k(t+\theta)d\theta = x_k(t) - \int_{-\infty}^{0} e^{\theta}x_k(t+\theta)d\theta = x_j - y_j \ .$$

Therefore, the functions $(x_j, y_j) = (x_j(t), y_j(t))$, $j = 1, \ldots, m$, satisfy the system of ordinary differential equations

$$\left\{ \begin{array}{l} \dot{x}_j = \varepsilon_j x_j + \sum_{k=1}^{m} a_{jk} x_j x_k + \sum_{k=1}^{m} p_{jk} x_j y_k \\ \dot{y}_j = x_j - y_j \ . \end{array} \right. \tag{6}$$

Moreover, from the definition of $y_j(t)$ it follows that

$$e^{\theta} y_j(\theta) = \int_{-\infty}^{0} e^{\theta+\tau} x_j(\theta+\tau)d\tau = \int_{-\infty}^{\theta} e^{t} x_j(t)dt \ ,$$

and, since $\gamma < 1$, we have

$$\lim_{\theta\to-\infty} e^{\theta} y_j(\theta) = 0 \ , \ j = 1, \ldots, m \ . \tag{7}$$

This yields the first part of the next result:

Theorem 1: *Any global solution $x_j = x_j(t)$ of the Volterra system (1) defines a global solution $(x_j, y_j) = (x_j(t), y_j(t))$ of the ODE system (6). Moreover, this solution satisfies the condition (7).*

Conversely, if $(x_j, y_j) = (x_j(t), y_j(t))$ is a global solution of the ODE system (6) such that condition (7) is satisfied, then $x_j = x_j(t)$ is a global solution of (1).

To obtain the converse part of this theorem we observe that, if $(x_j, y_j) = (x_j(t), y_j(t))$ is a global solution of the ODE system (6), then from the integration of the equation $\dot{y}_j = x_j - y_j$ we obtain

$$e^{t} y_j(t) = e^{t_0} y_j(t_0) + \int_{t_0}^{t} e^{\tau} x_j(\tau)d\tau \ ..$$

In this case, taking the limit $t_0 \to -\infty$ and using (7) we have

$$y_j(t) = e^{-t} \int_{-\infty}^{t} e^{\tau} x_j(\tau)d\tau = \int_{-\infty}^{0} e^{\theta} x_j(t+\theta) \ d\theta \ ,$$

which implies that, in fact, $x_j = x_j(t)$ is a global solution of (1).

Therefore, all the information on the global solutions of (6) is useful for the description of the flow of (1). One first observation is the following immediate consequence of the analysis of (6):

Proposition 2: *The set $\mathbb{R}^m_+ \times \mathbb{R}^m_+$ is invariant under the flow of the ODE system (6).*

As it turns out, the dynamics of (6), for adequate perturbation matrices, is related to the dynamics of the Lotka–Volterra equations

$$\dot{x}_j = \varepsilon_j x_j + \sum_{k=1}^{m} a_{jk} x_j x_k \quad , \quad j = 1, \dots, m \ . \tag{8}$$

3. The relation with the Lotka–Volterra system

Duarte, Fernandes and Oliva in [1] analysed extensively these equations discussing the existence of invariant sets of (8) where the generated flow has a hamiltonian structure. Such systems have been considered by many authors since they were introduced by Volterra in [11]. In particular, we mention the results of Redhefer and his coauthors [10, 9] that, in a series of papers, introduced a reduced graph to describe the dynamics of this system, (see [1] for further references). In the following we will use the approach of [1] to analyse the dynamics of (6) in some particular interesting cases for the perturbation matrix.

An essential assumption for the discussion of the global bounded orbits of (6) is the existence of equilibrium points. In fact, repeating the argument in [1] we can relate the asymptotic behavior of the orbits in $\mathbb{R}^m_+ \times \mathbb{R}^m_+$ with the existence of an equilibrium point of (6), that is a point $(\mathbf{q}, \mathbf{q}) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ such that $\mathbf{q} = (q_1, \dots, q_m)$ satisfies

$$\varepsilon_j + \sum_{k=1}^{m} (a_{jk} + p_{jk}) q_k = 0 \quad , \quad j = 1, \dots, m \ . \tag{9}$$

Proposition 3: *There exists a fixed point $(\mathbf{q}, \mathbf{q}) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ of system (6) if and only if $\mathbb{R}^m_+ \times \mathbb{R}^m_+$ contains some α- or ω-limit point.*

Proof: Consider the affine operator $L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ given by

$$\begin{aligned} L(x_1, \dots, x_m, y_1, \dots, y_m)_j &= \varepsilon_j + \sum_{k=1}^{m} (a_{jk} x_k + p_{jk} y_k) \\ L(x_1, \dots, x_m, y_1, \dots, y_m)_{m+j} &= x_j - y_j \end{aligned} \tag{10}$$

for $j = 1, \dots, m$. If there is no fixed point of (6), then $0 \notin K = L(\mathbb{R}^m_+ \times \mathbb{R}^m_+)$. Hence, there exists a hyperplane H through the origin of $\mathbb{R}^m \times \mathbb{R}^m$ disjoint from the convex set K, and we can choose $\mathbf{c} = (c_1, \dots, c_{2m}) \in H^\perp$ such that

$\mathbf{c}\cdot\mathbf{z} > 0$ for all $\mathbf{z} \in K$. Then we consider the function $U : \mathbb{R}^m_+ \times \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$U(x_1, \dots, x_m, y_1, \dots, y_m) = \sum_{j=1}^{m} (c_j \log x_j + c_{m+j} y_j)$$

and, if $\mathbf{w}(t) = (x_j(t), y_j(t))$ is a solution of (6), we obtain

$$\frac{d}{dt} U(\mathbf{w}(t)) = \sum_{j=1}^{m} (c_j L(\mathbf{w}(t))_j + c_{m+j} L(\mathbf{w}(t))_{m+j}) = \mathbf{c} \cdot L(\mathbf{w}(t)) > 0 \ .$$

This excludes the existence of any α- or ω-limit point, for which one would have $\dot{U} = 0$, and concludes the proof. □

For further reference we recall here some previous notation and results related to the Lotka–Volterra system. For a fixed diagonal matrix $D = \mathrm{diag}(d_j)$, with $d_j \neq 0$, $j = 1, \dots, m$, the transformation $X_j = x_j/d_j, Y_j = y_j/d_j$ is a gauge symmetry taking the system (6) into

$$\begin{cases} \dot{X}_j = \varepsilon_j X_j + \displaystyle\sum_{k=1}^{m} a_{jk} d_k X_j X_k + \sum_{k=1}^{m} p_{jk} d_k X_j Y_k \\ \dot{Y}_j = X_j - Y_j \end{cases} \tag{11}$$

which is again a system of the form (6) with interaction matrix AD and perturbation matrix PD. A choice of $A = (a_{jk})$ and $P = (p_{jk})$ in the equivalence class under the above gauge transformation is called a *choice of gauge* and, attending to the invariance of $\mathbb{R}^m_+ \times \mathbb{R}^m_+$ given by Proposition 2, we will only use matrices $D = \mathrm{diag}(d_j)$ with $d_j > 0$, $j = 1, \dots, m$, in order to preserve the phase space.

Following [1], an interaction matrix $A = (a_{jk})$ will be called *dissipative* if there exists a diagonal matrix $D > 0$ such that $AD \leq 0$, and will be called *conservative* if there exists a diagonal matrix $D > 0$ such that AD is skew-symmetric.

We also need the following notion of stably dissipative interaction matrix, which is essential in the study of these systems. A perturbation of an interaction matrix $A = (a_{jk})$ is a matrix $B = (b_{jk})$ such that $b_{jk} = 0$ if and only if $a_{jk} = 0$. Then, the interaction matrix $A = (a_{jk})$ is called *stably dissipative* if every perturbation of A sufficiently close to it is also dissipative, that is, there is a $\delta > 0$ such that all the perturbations B of A with $\max_{jk} |a_{jk} - b_{jk}| < \delta$ are dissipative.

For completion we recall here a result that will be used in our analysis, (for reference see [10, 1]):

Lemma 4: *If the interaction matrix A is stably dissipative then there exists a gauge transformation with $D > 0$ such that $AD \leq 0$ and the following condition is satisfied: if $\sum_{j,k=1}^{m} a_{jk} d_k w_k w_j = 0$ then $a_{jj} w_j = 0$ for all $j = 1, \dots, m$.*

We remark that if the resulting matrix $\tilde{A} = AD$ satisfies $\tilde{A} \leq 0$ and $\tilde{a}_{jj} = 0$ for all $j = 1, \dots, m$ then $\tilde{A}$ is skew-symmetric. This observation will be useful in the discussion of the dynamics generated by (6).

4. Nonsingular symmetric perturbation matrix

Under the hypothesis of existence of an equilibrium point, (6) has a Liapunov function V if the interaction and perturbation matrices satisfy certain sufficient conditions (see [12]). When the interaction matrix A and the perturbation matrix P are both negative semidefinite, $A \leq 0$, $P \leq 0$, and P is symmetric, we set

$$V = \sum_{j=1}^{m} (x_j - q_j \log x_j) - \frac{1}{2} \sum_{j,k=1}^{m} p_{jk} (y_j - q_j)(y_k - q_k) \ . \tag{12}$$

Computing the derivative of V along the solutions of (6) and using the symmetry of P we obtain

$$\dot{V} = \sum_{j=1}^{m} (x_j - q_j)(\varepsilon_j + \sum_{k=1}^{m} (a_{jk} x_k + p_{jk} y_k)) - \sum_{j,k=1}^{m} p_{jk} (x_j - y_j)(y_k - q_k) \ , \tag{13}$$

and from (9) we have

$$\dot{V} = \sum_{j=1}^{m} (x_j - q_j)(\sum_{k=1}^{m} a_{jk} (x_k - q_k) + p_{jk} (y_k - q_k)) - \sum_{j,k=1}^{m} p_{jk} (x_j - y_j)(y_k - q_k) \ . \tag{14}$$

Therefore, we conclude

$$\dot{V} = \sum_{j,k=1}^{m} a_{jk} (x_j - q_j)(x_k - q_k) + \sum_{j,k=1}^{m} p_{jk} (y_j - q_j)(y_k - q_k) \leq 0 \ . \tag{15}$$

To study the asymptotic behavior of $(x(t), y(t))$ we need to consider the subset of the phase space where $\dot{V} = 0$, which by LaSalle's principle contains the ω-limit set of the orbit $(x(t), y(t))$. If we consider first the case $P < 0$ then, from (15), on the set $\dot{V} = 0$ we have

$$y_j = q_j \quad , \quad j = 1, \dots, m \ , \tag{16}$$

and, from the second equation in (6) it follows that also

$$x_j = q_j \quad , \quad j = 1, \ldots, m \ . \tag{17}$$

Since $A \leq 0$ then, from $P < 0$ we have that $A + P < 0$, therefore $A + P$ is nonsingular and the solution $\mathbf{q}$ of (9) is unique. We then conclude that the system (6) is dissipative and possesses a global attractor which is a singleton, $\mathcal{A} = \{(\mathbf{q}, \mathbf{q})\}$. Since the solutions $x(t) \in C_\gamma^+$, $t \geq 0$, of the Volterra equations (1) determine solutions $(x_j(t), y_j(t)) \in \mathbb{R}_+^m \times \mathbb{R}_+^m$, $t \geq 0$, we conclude the following

Proposition 5: *If (9) has a solution $\mathbf{q} \in \mathbb{R}_+^m$, the interaction matrix A is negative semidefinite, $A \leq 0$, and the perturbation matrix P is symmetric negative definite $P < 0$, then the Volterra system (1) is dissipative with a global attractor $\mathcal{A}_0 = \{(x_j(t)) = \mathbf{q}\} \subset C_\gamma^+$ corresponding to the unique solution $\mathbf{q}$ of (9).*

If the perturbation matrix $P = \operatorname{diag}(p_{jj}) < 0$ is diagonal then, for any choice of gauge, the perturbation matrix $PD < 0$ is also diagonal. Therefore, the previous Proposition also holds when the interaction matrix A is stably dissipative and the perturbation matrix $P < 0$ is diagonal.

5. Singular diagonal perturbation matrix

Next we consider a diagonal singular perturbation matrix P. Here the diagonal case is considered only for simplicity since the analysis is easily extended to nondiagonal cases.

As an introduction, we first consider $P = 0$. Then, the equations for x_j decouple from the rest of the system and we obtain

$$\dot{x}_j = \varepsilon_j x_j + \sum_{k=1}^{m} a_{jk} x_j x_k \tag{18}$$

$$\dot{y}_j = x_j - y_j \tag{19}$$

for $j = 1, \ldots, m$. In this case, (18) is just the Lotka–Volterra system (8). Furthermore, (19) can be integrated to obtain the general solution

$$y_j(t) = Ce^{-t} + \int_{-\infty}^{0} e^{\theta} x_j(t + \theta) \, d\theta \ . \tag{20}$$

Since this solution satisfies condition (7) of Theorem 1 if and only if $C = 0$, we obtain the solutions $y_j(t)$ as functions of $x_j(t)$ in the form

$$y_j(t) = \int_{-\infty}^{0} e^{\theta} x_j(t + \theta) \, d\theta \ . \tag{21}$$

We remark that, if x_j is a bounded solution, then y_j is bounded as well. Also, if x_j is periodic, then y_j is periodic too (but in general out of phase). Moreover, one can show that the average behaviors of x_j and y_j are the same in the sense that, if the following limits exist, they are identical

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T x_j(t)\ dt = \lim_{T\to+\infty} \frac{1}{T}\int_0^T y_j(t)\ dt\ . \tag{22}$$

In fact, this average behavior is related to the existence of equilibrium points as it follows from [1, Proposition 2.3]. If (18) has a unique fixed point $\mathbf{q} \in \mathbb{R}^n_+$ then the averages (22) turn out to be q_j. Moreover, in this case $x_j = q_j$ then implies $y_j = q_j$.

For completeness we recall here the results of [1] for the Lotka–Volterra system. Using (9), the equations (18) become

$$\dot{x}_j = x_j \sum_{k=1}^{m} a_{jk}(x_k - q_k) \quad , \quad j = 1, \ldots, m\ . \tag{23}$$

The diagonal entries of the interaction matrix $A \leq 0$ satisfy $a_{jj} \leq 0$. If A is stably dissipative, using the choice of interaction matrix given by Lemma 4, on the set where $\dot{V} = 0$ we obtain

$$a_{jj}(x_j - q_j) = 0 \quad , \quad j = 1, \ldots, m\ . \tag{24}$$

If we have $a_{jj} < 0$ for all $j = 1, \ldots, m$, from (24) and (21) it follows that

$$x_j = y_j = q_j \quad , \quad j = 1, \ldots, m\ . \tag{25}$$

Moreover, A is nonsingular. In fact, if A is a singular matrix there is a nonzero vector $(w_1, \ldots, w_m)$ such that $\sum_{k=1}^m a_{jk}w_k = 0$ for $j = 1, \ldots, m$. Then, the previous Lemma 4 implies that $a_{jj}w_j = 0$ for all $j = 1, \ldots, m$ from which it follows that $a_{jj} = 0$ for some j, a contradiction. Again we conclude that the solution $\mathbf{q}$ of (9) is unique. Therefore, also in this case if $\mathbf{q} \in \mathbb{R}^m_+$ the system (6) is dissipative and possesses a global attractor which is a singleton, $\mathcal{A} = \{(\mathbf{q}, \mathbf{q})\}$.

If, on the other hand, we have $a_{jj} = 0$ for some j, we need to use the original equation (6) to obtain more information regarding the dynamics on the set where $\dot{V} = 0$. In this case we may still have $x_j = q_j$ for some values of j. We split the variables x_j into two groups that, upon reordering, correspond to $j \in \{1, \ldots, n\}$ and $j \in \{n+1, \ldots, m\}$ with $1 \leq n < m$. In the second group we include all the variables for which we have $x_j = q_j$ (either because $a_{jj} < 0$ or due to a reduced graph argument involving the interaction matrix, see [1] for details). In the first group we include all the other variables x_j. Then, the solutions of (6) on the set $\dot{V} = 0$ satisfy new equations of the form

$$\dot{x}_j = x_j \sum_{k=1}^{n} \tilde{a}_{jk}(x_k - q_k) \quad , \quad j = 1, \ldots, n \tag{26}$$

and

$$x_j = q_j \ , \ j = n+1, \ldots, m \tag{27}$$

where the reduced interaction matrix $\tilde{A} = (\tilde{a}_{jk})$ satisfies

$$\tilde{a}_{jj} = 0 \quad , \quad j = 1, \ldots, n \tag{28}$$

and is negative semidefinite, $\tilde{A} \leq 0$. From our previous remark, these two conditions together imply that $\tilde{A}$ is skew–symmetric and the reduced interaction matrix is conservative. It follows from [1] that the reduced system (26) has a hamiltonian formulation. This is contained in the next result that essentially corresponds to [1, Theorem 4.5].

Theorem 6: *Consider the system of Lotka–Volterra equations (18) coupled with the linear equations (19) and assume that: (i) the Lotka–Volterra system (18) has a singular point* $\mathbf{q} \in \mathbb{R}^m_+$*; and, (ii) the interaction matrix* A *is stably dissipative. Then the dynamics of this system on the set* $\dot{V} = 0$ *can be described by a Lotka–Volterra system (26) of dimension* $n \leq m$ *together with (27), and* $y_j(t) = \int_{-\infty}^{0} e^{\theta} x_j(t+\theta)\, d\theta$ *for* $j = 1, \ldots, m$*. Moreover, the dynamics of (26) is hamiltonian.*

The hamiltonian formulation of (26) leads to a very rich and complex behavior for the solutions. To support this observation we mention that [1] presents an example for which the hamiltonian system for (26) is non–integrable and the flow contains families of strongly hyperbolic periodic orbits.

We now consider the crucial case where $p_{jj} = 0$ for $j = 1, \ldots, l$ with $1 \leq l < m$, and $p_{jj} < 0$ for $j = l+1, \ldots, m$. In this case, (15) and (6) imply that on the set where $\dot{V} = 0$ we have

$$y_j = x_j = q_j \ , \ j = l+1, \ldots, m \ . \tag{29}$$

Using (9), the equations (6) then become

$$\begin{aligned} \dot{x}_j &= x_j \sum_{k=1}^{l} a_{jk}(x_k - q_k) \\ \dot{y}_j &= x_j - y_j \end{aligned} \quad , \quad j = 1, \ldots, l \tag{30}$$

together with (29). Again the equations for x_j, $j = 1, \dots, l$, are decoupled and correspond to a Lotka–Volterra system. In fact, these equations can be written in the form

$$\dot{x}_j = x_j \tilde{\varepsilon}_j + \sum_{k=1}^{l} a_{jk} x_j x_k \quad , \quad j = 1, \dots, l \tag{31}$$

with

$$\tilde{\varepsilon}_j = \varepsilon_j + \sum_{k=l+1}^{m} a_{jk} q_k \quad , \quad j = 1, \dots, l \ . \tag{32}$$

Moreover, from (9) and this last equation we obtain

$$\tilde{\varepsilon}_j + \sum_{k=1}^{l} a_{jk} q_k = 0 \quad , \quad j = 1, \dots, l \ , \tag{33}$$

which ensures that $\tilde{\mathbf{q}} = (q_1, \dots, q_l)$ is an equilibrium of (31).

If the reduced matrix is stably dissipative, with the proper choice of interaction matrix given by the previous Lemma 4, we obtain

$$a_{jj}(x_j - q_j) = 0 \quad , \quad j = 1, \dots, l \ . \tag{34}$$

Therefore, by the previous argument, (31) can lead to a reduced system with a hamiltonian structure.

However, (29) also implies the following equations

$$\sum_{k=1}^{l} a_{jk}(x_k - q_k) = 0 \quad , \quad j = l+1, \dots, m \tag{35}$$

which, then constitute constraints that must be satisfied by the solutions of (6) on the set $\dot{V} = 0$. Solving these equations, we determine r variables x_j in terms of the remaining $s = l - r$ variables, where r denotes the rank of the reduced matrix $R = (a_{jk})_{l+1 \le j \le m, 1 \le k \le l}$. With an eventual reordering of the variables we can assume that the first s variables are free, and write

$$x_j = q_j + \sum_{k=1}^{s} \beta_{jk}(x_k - q_k) \quad , \quad j = s+1, \dots, l \ . \tag{36}$$

where the coefficients $(\beta_{jk})_{s+1 \le j \le l, 1 \le k \le s}$ depend on the entries of the matrix R. Then, we can write the first s equations of (31) again in the form

$$\dot{x}_j = x_j \varepsilon_j^* + \sum_{k=1}^{s} a_{jk}^* x_j x_k \quad , \quad j = 1, \dots, s \ , \tag{37}$$

with

$$\varepsilon_j^* = \tilde{\varepsilon}_j + \sum_{k=s+1}^{l} a_{jk}(q_k - \sum_{i=1}^{s} \beta_{ki} q_i) \quad , \quad j = 1, \ldots, s \; , \tag{38}$$

and

$$a_{jk}^* = a_{jk} + \sum_{i=s+1}^{l} a_{ji}\beta_{ik} \quad , \quad j = 1, \ldots, s, \; k = 1, \ldots, s \; . \tag{39}$$

The remaining equations (31), for $j = s+1, \ldots, l$, can also be written in the form (37), that is

$$\dot{x}_j = x_j \varepsilon_j^* + \sum_{k=1}^{s} a_{jk}^* x_j x_k \quad , \quad j = s+1, \ldots, l \; , \tag{40}$$

with

$$\varepsilon_j^* = \tilde{\varepsilon}_j + \sum_{k=s+1}^{l} a_{jk}(q_k - \sum_{i=1}^{s} \beta_{ki} q_i) \quad , \quad j = s+1, \ldots, l \; ,$$

and

$$a_{jk}^* = a_{jk} + \sum_{i=s+1}^{l} a_{ji}\beta_{ik} \quad , \quad j = s+1, \ldots, l, \; k = 1, \ldots, s \; .$$

However, in general the compatibility with (36) impose further restrictions that need to be satisfied by the solutions on the set $\dot{V} = 0$. Therefore, we may obtain $x_j = q_j$ also for some $j = s, s-1, \ldots$, in which case we repeat the previous procedure. By iteration one eventually ends up with a reduced Lotka–Volterra system

$$\dot{x}_j = x_j \varepsilon_j^* + \sum_{k=1}^{s_0} a_{jk}^* x_j x_k \quad , \quad j = 1, \ldots, s_0 \; , \tag{41}$$

coupled with equations

$$x_j = q_j + \sum_{k=1}^{s_0} \tilde{\beta}_{jk}(x_k - q_k) \quad , \quad j = s_0 + 1, \ldots, l \tag{42}$$

for the variables $x_j, j = 1, \ldots, l$, together with

$$\dot{y}_j = x_j - y_j \quad , \quad j = 1, \ldots, l \tag{43}$$

for the variables $y_j, j = 1, \ldots, l$, and

$$x_j = y_j = q_j \quad , \quad j = l, \ldots, m \tag{44}$$

for all the remaining variables. Summarizing these results we have the following

Theorem 7: *Consider the Volterra system (1) and assume that:*

(i) *There exists an equilibrium solution with* $\mathbf{q} \in \mathbb{R}_+^m$ *given by (9); and*
(ii) *The perturbation matrix* P *is diagonal and singular, with*

$$p_{jj} = 0, j = 1, \ldots, l \quad , \quad p_{jj} < 0, j = l+1, \ldots, m \ .$$

Then the dynamics of (1) can be described by a Lotka–Volterra system (41) of dimension $s_0 \leq l$ *together with equations (42).*

Moreover, $y_j(t) = \int_{-\infty}^{0} e^{\theta} x_j(t+\theta)\, d\theta$ *for* $j = 1, \ldots, m$.

Then, the results of [1] apply again to the Lotka–Volterra system (41), eventually further reducing the dimension s_0, and we obtain the following

Corollary 8: *If in addition to the hypothesis (i) and (ii) of Theorem 7, we assume that the reduced interaction matrix* $A^* = (a^*_{jk})_{1 \leq j \leq s_0, 1 \leq k \leq s_0}$ *is stably dissipative, then the dynamics of (1) can be described by a reduced Lotka–Volterra system (41) with dimension* $s_0 \leq l$ *together with equations (42), and the dynamics of (41) is hamiltonian.*

6. An illustrative example

To illustrate these results we consider the following reducible Volterra retarded equation with infinite delay on $\mathbb{R}_+^6$

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_1 x_2 \\
\dot{x}_2 &= x_2 - x_2(x_1 - \delta x_3) \\
\dot{x}_3 &= -x_3 - x_3(\delta x_2 - x_4 + \alpha x_6) \\
\dot{x}_4 &= x_4 - x_4 x_3 \\
\dot{x}_5 &= -x_5 + x_5 x_6 \\
\dot{x}_6 &= 2x_6 + x_6(\alpha x_3 - x_5) - x_6 \int_{-\infty}^{0} e^{\theta} x_6(t+\theta) d\theta \ .
\end{aligned} \tag{45}$$

The interaction matrix A and perturbation matrix P of the corresponding system of ODEs (6) are given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & \delta & 0 & 0 & 0 \\ 0 & -\delta & 0 & 1 & 0 & -\alpha \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 & -1 & 0 \end{bmatrix} \tag{46}$$

and

$$p_{66} = -1 \quad , \quad p_{jk} = 0 \ \text{ otherwise,} \tag{47}$$

and the vector of growth coefficients is given by $\varepsilon = (-1, 1, -1, 1, -1, 2)$.

Solving explicitly the system (5), we obtain

$$\mathbf{q} = (1+\delta, 1, 1, 1+\delta+\alpha, 1+\alpha, 1) \ , \tag{48}$$

and conclude that (45) has a fixed point $\mathbf{q} \in \mathbb{R}^6_+$ when $\delta > -1$ and $\alpha > \max\{-1, -1-\delta\}$.

One can easily verify that the matrix A is stably dissipative. This can be checked either directly or by noticing that the reduced graph of A, as introduced by [10], is a tree (see again [1]). Moreover, A is skew–symmetric.

Using the Liapunov function V given by (12) we have that

$$\dot{V} = -(y_6 - q_6)^2 \leq 0 \ . \tag{49}$$

Therefore, (49) and (19) imply that, on the set where $\dot{V} = 0$, we have

$$x_6 = y_6 = q_6 \ . \tag{50}$$

The remaining variables $x_j, 1 \leq j \leq 5$, satisfy the Lotka–Volterra system

$$\dot{x}_j = \tilde{\varepsilon}_j x_j + \sum_{j=1}^{5} \tilde{a}_{j,k} x_j x_k \ , \ 1 \leq j \leq 5 \ , \tag{51}$$

where $\tilde{\varepsilon}_j = \varepsilon_j + a_{j6} q_6$, $1 \leq j \leq 5$, yielding $\tilde{\varepsilon} = (-1, 1, -1-\alpha, 1, 0)$, and the corresponding interaction matrix is

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & \delta & 0 & 0 \\ 0 & -\delta & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \ . \tag{52}$$

Again we have that $\tilde{A}$ is stably dissipative and skew–symmetric. In addition, we obtain one constraint (35) from the remaining equation, which in this case has the form

$$\alpha(x_3 - q_3) - (x_5 - q_5) = 0 \ . \tag{53}$$

Then, we incorporate this information in the Lotka–Volterra system (51). We compute x_5 from (53) and take $x_1, \ldots, x_4$ as free variables, reducing the system to

$$\dot{x}_j = \varepsilon^*_j x_j + \sum_{j=1}^{4} a^*_{j,k} x_j x_k \ , \ 1 \leq j \leq 4 \ . \tag{54}$$

Since $\tilde{a}_{j5} = 0$, here we have $\varepsilon^* = (-1, 1, -1 - \alpha, 1)$ and

$$A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} . \tag{55}$$

Once more, A^* is stably dissipative and skew–symmetric. Consequently, the dynamics of (54) is hamiltonian. However, we also need to take into account the possible constraint arising from the remaining equation

$$\dot{x}_5 = 0 . \tag{56}$$

If $\alpha = 0$, then (53) implies that $x_5 = q_5$ and (56) does not constitute a new constraint. This is the case in which the equations (45) decouple. The variables x_5, x_6, are fixed, while the variables $x_1, \dots, x_4$, satisfy the Hamiltonian system (54). This system corresponds to a toy model considered in some length in [1] to illustrate the complexity of the dynamics that can occur in a 4-dimensional Lotka–Volterra system. It turns out that, for adequate values of the internal coupling $\delta \neq 0$, the system (54) is non–integrable and its phase portrait contains strongly hyperbolic periodic orbits (see [1] for details).

If, however, $\alpha \neq 0$, then from (56) and (53) we have that

$$\dot{x}_3 = 0 , \tag{57}$$

which is a new constraint. From (54) and (55), this constraint has the form

$$\delta(x_2 - q_2) - (x_4 - q_4) = 0 . \tag{58}$$

and we can incorporate it again in the Lotka–Volterra system (54). After computing x_4 from (58), the remaining variables satisfy the reduced system

$$\dot{x}_j = \hat{\varepsilon}_j x_j + \sum_{j=1}^{3} \hat{a}_{j,k} x_j x_k \ , \ 1 \leq j \leq 3 , \tag{59}$$

where $\hat{\varepsilon} = (-1, 1, 0)$ and

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \delta \\ 0 & 0 & 0 \end{bmatrix} . \tag{60}$$

As before, we have a possible constraint due to the remaining equation

$$\dot{x}_4 = -x_4(x_3 - q_3) . \tag{61}$$

Now, if $\delta = 0$, then (58) implies that $x_4 = q_4$, hence $\dot{x}_4 = 0$, and from (61) we obtain that $x_3 = q_3$. Again, the equations (45) decouple and there are no further constraints. In this case, x_1, x_2 satisfy

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_2 - x_2 x_1 \end{aligned} , \tag{62}$$

which is the two dimensional Lotka–Volterra system with Hamiltonian function

$$H(x_1, x_2) = x_1 + x_2 - \log(x_1 x_2) \ . \tag{63}$$

This provides the complete phase portrait of the global bounded solutions for this case. We have the periodic solutions corresponding to the level curves of $H(x_1, x_2)$, together with $x_j = q_j$, $3 \leq j \leq 6$, and the y_j obtained from

$$y_j = \int_{-\infty}^{0} e^{\theta} x_j(t+\theta)\ d\theta \ ,\ 1 \leq j \leq 6 \ . \tag{64}$$

If $\delta \neq 0$, then from (61) and (57) we conclude that the only bounded solution corresponds to $x_3 = q_3$ and $x_4 = q_4$. Then, we have $\dot{x}_4 = 0$ which, from (58), implies $\dot{x}_2 = 0$. From the equations (59), (60), we then conclude that also $x_2 = q_2$ and $x_1 = q_1$. Therefore, in the coupled case the only bounded solution of the Volterra system (45) corresponds to the unique equilibrium given by (48) and the global attractor reduces to the singleton $\mathcal{A}_0 = \{(x_j(t))_{1 \leq j \leq 6} = \mathbf{q}\}$.

This example shows up the strong dissipative effect of the negative retarded perturbation term on the global dynamics of the Volterra retarded system.

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