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A counterexample to the existence of a Poisson structure on a twisted group algebra

Eliana Zoque[1]

Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637

E-mail address: `elizoque@math.uchicago.edu`

Abstract. Crawley-Boevey [1] introduced the definition of a noncommutative Poisson structure on an associative algebra A that extends the notion of the usual Poisson bracket. Let (V, ω) be a symplectic manifold and G be a finite group of symplectimorphisms of V. Consider the twisted group algebra $A = \mathbb{C}[V]\#G$. We produce a counterexample to prove that it is not always possible to define a noncommutative poisson structure on $\mathbb{C}[V]\#G$ that extends the Poisson bracket on $\mathbb{C}[V]^G$.

1. **Introduction**

Crawley-Boevey [1] defined a noncommutative Poisson structure on an associative algebra A over a ring K as a Lie bracket $\langle -, - \rangle$ on $A/[A, A]$ such that for each $a \in A$ the map $\langle \overline{a}, - \rangle : A/[A, A] \to A/[A, A]$ is induced by a derivation $d_a : A \to A$; i.e. $\langle \overline{a}, \overline{b} \rangle = \overline{d_a(b)}$ where the map $a \mapsto \overline{a}$ is the projection $A \to A/[A, A]$. When A is commutative a noncommutative Poisson structure is the same as a Poisson bracket.

Let (V, ω) be a symplectic manifold, with the usual Poisson bracket $\{-, -\}$ on $\mathbb{C}[V]$. Let G be a finite group of symplectimorphisms of V. Consider the twisted group algebra $A = \mathbb{C}[V]\#G$. The algebra of G-invariant polymonials $\mathbb{C}[V]^G$ is contained in $A/[A, A]$. We produce a counterexample to prove that it is not always possible to define a noncommutative poisson structure on $\mathbb{C}[V]\#G$ that extends the Poisson bracket on $\mathbb{C}[V]^G$.

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2. **Twisted group algebra and derivations**

From now on, let $A = \mathbb{C}[V]\#G$.($\mathbb{C}$ can be replaced by any field of characteristic 0.) We use the symbol ${}^g\psi$ to denote the left action of $g \in G$ on $\psi \in \mathbb{C}[V]$. For every $g \in G$ we denote $(-)_g$ the projection $A \to \mathbb{C}[V]$ into the g-part, i.e, $(\psi h)_g = \psi\delta_{g,h}$ if $\psi \in \mathbb{C}[V], h \in G$. Let $G = C_0 \cup C_1 \cup \cdots$ be the conjugacy classes of G, with $C_0 = \{1\}$.

It is proved in [4] that

$$\frac{A}{[A,A]} = HH_0(A) = \left(HH_0\left(\mathbb{C}[V], \mathbb{C}[V]\#G\right)\right)^G,$$

therefore

$$\begin{aligned}
\frac{A}{[A,A]} &= \left(\bigoplus_{g\in G} HH_0\left(\mathbb{C}[V], \mathbb{C}[V]g\right)\right)^G \\
&= \left(\bigoplus_{g\in G} \frac{\mathbb{C}[V]}{\langle \varphi - {}^g\varphi \,:\, \varphi \in \mathbb{C}[V]\rangle} g\right)^G \\
&= \bigoplus_i \left(\frac{\mathbb{C}[V]}{\langle \varphi - {}^{g_i}\varphi \,:\, \varphi \in \mathbb{C}[V]\rangle}\right)^{G_{g_i}} g_i
\end{aligned}$$

where g_i is an arbitrary element of C_i and $G_g = \{h \in G | gh = hg\}$. The first summand is precisely $\mathbb{C}[V]^G$. Let P_i be the projection

$$A \to \left(\frac{\mathbb{C}[V]}{\langle \varphi - {}^{g_i}\varphi \,:\, \varphi \in \mathbb{C}[V]\rangle}\right)^{G_{g_i}} g_i.$$

The Poisson bracket gives us a family of derivations $d_\psi : \mathbb{C}[V]^G \to \mathbb{C}[V]^G, \phi \mapsto \{\psi, \phi\}$ for $\psi \in \mathbb{C}[V]^G$; and we want to extend it to a larger family. The following Lemma restricts the possibilities.

Lemma 1. Let $d : A \to A$ be any derivation. If $x \in \mathbb{C}[V]^g \neq \mathbb{C}[V]$ then $(d(x))_g = 0$.

Proof. Let $y \notin \mathbb{C}[V]^g$. The equality $d(xy) = d(yx)$ implies $d(x)y + xd(y) = d(y)x + yd(x)$. The g-part of this equality is

$$(d(x))_g\, gy + x(d(y))_g g = (d(y))_g gx + y\,(d(x))_g\, g$$

or

$$(d(x))_g\, {}^gy + x\,(d(y))_g = (d(y))_g\, {}^gx + y\,(d(x))_g\,.$$

Since ${}^gx = x, {}^gy \neq y$ we conclude $(d(x))_g\,({}^gy - y) = 0$, so $(d(x))_g = 0$ □

Therefore if the action of G on V is faithful and $g \neq 1$, the g-part of the derivative an element of $\mathbb{C}[V]^g$ is zero. This implies that for every $\psi \in \mathbb{C}[V]^G$, $d(\psi) \in \mathbb{C}[V] \subset A$.

The condition $\langle \overline{\psi g}, \overline{\phi h} \rangle = -\langle \overline{\phi h}, \overline{\psi g} \rangle$ implies $\overline{d_{\psi g}(\phi h)} = -\overline{d_{\phi h}(\psi g)}$. Consider the case $\phi, \psi \in \mathbb{C}[V]^G$, $h = 1$ and $g \in C_i, i \neq 0$. Since $P_i\left(d_{\psi g}(\phi)\right) = 0$, we must have $0 = P_i\left(d_\phi(\psi g)\right) = P_i\left(d_\phi(\psi) g + \psi d_\phi(g)\right)$. The only terms that must be taken into account are $d_\phi(\psi) g + \psi \sum \left(d_\phi(g)\right)_{hgh^{-1}} hgh^{-1}$. Modulo $[A, A]$ this is equal to

$$\left(d_\phi(\psi) + \sum_h {}^{h^{-1}}\left(\psi \left(d_\phi(g)\right)_{hgh^{-1}}\right)\right) g = \left(d_\phi(\psi) + \psi \sigma_{\phi,g}\right) g$$

where $\sigma_{\phi,g} = \sum_h {}^{h^{-1}}\left(\left(d_\phi(g)\right)_{hgh^{-1}}\right)$ does not depend on ψ.

We want $0 = P_i\left(\left(d_\phi(\psi) + \psi \sigma_{\phi,g}\right) g\right) = P_i\left(\left(\{\phi, \psi\} + \psi \sigma_{\phi,g}\right) g\right)$ since we want a Poisson structure extending the usual Poisson bracket on $\mathbb{C}[V]^G$. Therefore a neccesary condition for the existance of the Poisson structure is the existance of $\sigma_{\phi,g} \in \mathbb{C}[V]$ so that

$$P_i\left(\left(\{\phi, \psi\} + \psi \sigma_{\phi,g}\right) g\right) = 0 \tag{1}$$

for every $\psi \in \mathbb{C}[V]$. We will see that this is not always possible.

3. The counterexample

Let $V = \mathbb{C}^4$ with linear coordinates $\{x_1, x_2, x_3, x_4\}$ and the symplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$, so $\mathbb{C}[V] = \mathbb{C}[x_1, x_2, x_3, x_4]$, and

$$\{\phi, \psi\} = \frac{\partial \phi}{\partial x_1}\frac{\partial \psi}{\partial x_2} - \frac{\partial \phi}{\partial x_2}\frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_3}\frac{\partial \psi}{\partial x_4} - \frac{\partial \phi}{\partial x_4}\frac{\partial \psi}{\partial x_3}.$$

Let $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. Let e, b, c be the generators of the three copies of $\mathbb{Z}_2$ (in that order). G acts on V as follows: b and c act as $diag(-1,-1,1,1)$ and $diag(1,1,-1,-1)$, respectively, on $\{x_1, x_2, x_3, x_4\}$ and e interchanges $x_1 \leftrightarrow x_3, x_2 \leftrightarrow x_4$. Clearly $\mathbb{C}[V]^G$ is the set of all polynomials $\sum \lambda_{i_1,i_2,i_3,i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$ such that $\lambda_{i_1,i_2,i_3,i_4} \neq 0$ implies $i_1 + i_2$, $i_3 + i_4$ are even and $\lambda_{i_1,i_2,i_3,i_4} = \lambda_{i_3,i_4,i_1,i_2}$.

Using Magma (http://magma.maths.usyd.edu.au) we find that the ring of invariant polynomials is generated, as an algebra, by $f_1 = x_1^2 + x_3^2$, $f_2 = x_2^2 + x_4^2$, $f_3 = x_1^4 + x_3^4$, $f_4 = x_2^4 + x_4^4$, $h_1 = x_1x_2 + x_3x_4$, $h_2 = x_1^2x_2^2 + x_3^2x_4^2$, $h_3 = x_1^2x_3x_4 + x_1x_2x_3^2$, $h_4 = x_1x_2x_4^2 + x_2^2x_3x_4$; with relations

$$-f_1f_2h_1 + f_1h_4 + f_2h_3 - h_1^3 + 2h_1h_2,$$
$$\frac{1}{2}f_1^2f_2 + \frac{1}{2}f_1h_1^2 - \frac{1}{2}f_1h_2 - \frac{1}{2}f_2f_3 - h_1h_3,$$
$$\frac{1}{2}f_1f_2^2 - \frac{1}{2}f_1f_4 + \frac{1}{2}f_2h_1^2 - \frac{1}{2}f_2h_2 - h_1h_4,$$
$$-\frac{1}{2}f_1^2f_4 + f_1f_2h_2 - \frac{1}{2}f_2^2f_3 + f_3f_4 - h_2^2,$$
$$-\frac{1}{2}f_1^2h_4 + \frac{1}{2}f_1f_2h_3 + \frac{1}{2}f_1h_1h_2 - \frac{1}{2}f_2f_3h_1 + f_3h_4 - h_2h_3,$$
$$\frac{1}{2}f_1f_2 * h_4 - \frac{1}{2}f_1f_4h_1 - \frac{1}{2}f_2^2h_3 + \frac{1}{2}f_2h_1h_2 + f_4h_3 - h_2h_4,$$
$$\frac{1}{2}f_1^3f_2 + \frac{1}{2}f_1^2h_1^2 - f_1^2h_2 - \frac{1}{2}f_1f_2f_3 - \frac{1}{2}f_3h_1^2 + f_3h_2 - h_3^2,$$
$$\frac{1}{2}f_1^2f_2^2 - 3/4f_1^2f_4 + \frac{1}{2}f_1f_2h_1^2 - 3/4f_2^2f_3 + f_3f_4 - \frac{1}{2}h_1^2h_2 - h_3h_4,$$
$$\frac{1}{2}f_1f_2^3 - \frac{1}{2}f_1f_2f_4 + \frac{1}{2}f_2^2h_1^2 - f_2^2h_2 - \frac{1}{2}f_4h_1^2 + f_4h_2 - h_4^2.$$

Proposition 2. The Poisson bracket on $\mathbb{C}[V]^G$ cannot be extended to a Poisson structure on $\mathbb{C}[V]\#G$ for V and G as defined above.

Proof. Take $\phi = x_1^2 + x_3^2, \psi = x_1x_2 + x_3x_4 \in \mathbb{C}[V]^G$ and $g = b$. In this case $\{\phi, \psi\} = 2x_1^2 + 2x_3^2$, $\langle \varphi - {}^g\varphi \,:\, \varphi \in \mathbb{C}[V]\rangle = \langle x_1, x_2\rangle$ and $G_b = \{1, b, c, bc\}$. Hence,

$$\left(\frac{\mathbb{C}[V]}{\langle \varphi - {}^g\varphi \,:\, \varphi \in \mathbb{C}[V]\rangle}\right)^{G_g} = \mathbb{C}[x_3, x_4]^{\{1,b,c,bc\}} = \mathbb{C}[x_3^2, x_3x_4, x_4^2],$$

so $P_i\left(\left(\{\phi,\psi\}\right)b\right) = 2x_3^2b$

On the other hand, $P_i\left(\left(\psi\sigma_{\phi,g}\right)b\right) = P_i\left(\left(\left(x_1x_2 + x_3x_4\right)\sigma_{\phi,g}\right)b\right) =$
$P_i\left(\left(\left(x_3x_4\right)\sigma_{\phi,g}\right)b\right)$ and none of the terms here can be equal to $-2x_3^2$ since they all contain x_4. This contradicts (1). □

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