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Linear systems and Multiplicity of ideals

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in memory of my friend Sevin Recillas

Introduction

A result of P. Samuel ([17] p. 186, Chap.II, Théorème 5) says that in a local noetherian ring $(\mathcal{O}, \mathfrak{M})$ of Krull dimension d in which the residual field k is infinite, the multiplicity of a $\mathfrak{M}$-primary ideal I is equal to the multiplicity of an ideal $(x_1, \ldots, x_d)$ generated by some parameter sequence $x_1, \ldots, x_d$ contained in I. By a theorem of Rees ([16] p.142 Theorem 9.44), this implies that the ideals I and $(x_1, \ldots, x_d)$ have the same integral closure in the ring $\mathcal{O}$.

In fact Samuel's proof shows that the elements of the parameter sequence can be chosen to be general elements of I, namely *superficial elements* of I.

An interesting consequence of Samuel's result is that, in the case the local ring $\mathcal{O}$ is a Cohen-Macaulay ring, e.g. a regular or a local complete intersection ring, the multiplicity of the ideal I in $\mathcal{O}$ is the length of the $\mathcal{O}$-module

$$\mathcal{O}/(x_1, \ldots, x_d)$$

Using a geometric interpretation of the multiplicity by C. P. Ramanujam ([15]), we shall give a geometric way to calculate the multiplicity. We shall consider the particular case of a non-singular complex surface and give an example with a geometric proof of a result of Mumford, as it was suggested to the author by M.S. Narasimhan.

Most of this note is written in the language of complex analytic spaces (see [2] and [1]), but the results can be stated and proved in the case of schemes of finite type (see definition in [3] Chap. IV 1.6.1) over an infinite field with equicharacteristic local rings.

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1. Integral closures and blowing-ups.

Let $(\mathcal{O}, \mathfrak{M})$ be a reduced complex analytic local ring and let J be an ideal of $\mathcal{O}$. We say that an element x of $\mathcal{O}$ is integral over the ideal J if there is a relation

$$x^n + \sum_{i=1}^{n} a_i x^{n-i} = 0$$

where $a_i \in J^i$.

Elements of $\mathcal{O}$ which are integral over J form an ideal $\overline{J}$ in $\mathcal{O}$ which contains J. This ideal is called the *integral closure* of J in $\mathcal{O}$.

We know that an ideal J of $\mathcal{O}$ defines an order function ν_J defined by

$$\nu_J(x) := \sup\{k\,,\, x \in J^k\} \in \mathbb{N} \cup +\infty$$

for any $x \in \mathcal{O}$.

We can define

$$\overline{\nu}_J(x) := \liminf \frac{\nu_J(x^k)}{k} \in \mathbb{N} \cup +\infty$$

Notice that one can show that $\overline{\nu}_J(x)$ is in $\mathbb{Q}$ and not in $\mathbb{R}$, as its definition suggests.

Then, we have the important following theorem (see [11] Théorème 2.1 or [7]):

1.1. **Theorem.** *Let $(\mathcal{O}, \mathfrak{M})$ be a reduced analytic local ring, J be an ideal of $\mathcal{O}$ and $x \in \mathcal{O}$. Denote by (Z, z) a germ of complex analytic space such that $\mathcal{O}_{Z,z} = \mathcal{O}$. The following conditions are equivalent:*

i) The element x is integral over the ideal J;

ii) We have $\overline{\nu}_J(x) \geq 1$;

iii) There is a modification $\pi : \tilde{Z} \to (Z, z)$ such that the space $\tilde{Z}$ is normal and $J\mathcal{O}_{\tilde{Z}}$ is principal and $x \circ \pi$ is a section of $J\mathcal{O}_{\tilde{Z}}$.

iv) Let $\pi : \tilde{Z} \to (Z, z)$ be the normalized blowing-up of J, then $x \circ \pi$ is a section of $J\mathcal{O}_{\tilde{Z}}$.

On the other hand we have the following consequence of a theorem of D. Rees ([16] p.142 Theorem 9.44):

1.2. **Theorem.** *Let $(\mathcal{O}, \mathfrak{M})$ be an analytic local ring which is an integral domain. Let $I \subset J$ be $\mathfrak{M}$-primary ideals of $\mathcal{O}$. Then, these ideals have the same multiplicity if and only if they have the same integral closure in $\mathcal{O}$.*

The preceding theorems give us the important corollary:

1.3**. Corollary.** *Let I be a $\mathfrak{M}$-primary ideal of a reduced analytic local ring $(\mathcal{O}, \mathfrak{M})$ and let $x_1, \ldots, x_d$ a sequence of parameters in I which generates an ideal $(x_1, \ldots, x_d)$ having the same multiplicity as the one of I. The normalized blowing-up of I equals the normalized blowing-up of $(x_1, \ldots, x_d)$.*

Proof: Let (Z, z) be a germ of reduced complex analytic space such that $\mathcal{O}_{Z,z} = \mathcal{O}$. From the theorem of Rees, it is enough to prove that the normalized blowing-up $\pi : \tilde{Z} \to (Z, z)$ of I is also the normalized blowing-up of the integral closure $\overline{I}$ of I in $\mathcal{O}$. We have $\overline{I} \supset I$, so $\overline{I}\mathcal{O}_{\tilde{Z}} \supset I\mathcal{O}_{\tilde{Z}}$. Theorem 1.1 implies that $\overline{I}\mathcal{O}_{\tilde{Z}} \subset I\mathcal{O}_{\tilde{Z}}$, so

$$\overline{I}\mathcal{O}_{\tilde{Z}} = I\mathcal{O}_{\tilde{Z}}.$$

Therefore $\overline{I}\mathcal{O}_{\tilde{Z}}$ is invertible and π factorizes uniquely by σ through the normalized blowing-up $\overline{\pi} : \overline{Z}' \to (Z, z)$ of $\overline{I}\mathcal{O}$:

$$\pi = \overline{\pi} \circ \sigma.$$

$$\begin{tikzcd} \tilde{Z} \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & \overline{Z}' \arrow[dl, "\overline{\pi}"] \\ & (Z, z) & \end{tikzcd}$$

Now we show that $I\mathcal{O}_{\overline{Z}'} = \overline{I}\mathcal{O}_{\overline{Z}'}$. First, notice that $I\mathcal{O}_{\overline{Z}'} \subset \overline{I}\mathcal{O}_{\overline{Z}'}$, and, for $k \geq 0$ $I^k\mathcal{O}_{\overline{Z}'} \subset \overline{I}^k\mathcal{O}_{\overline{Z}'}$. By definition we have that $\overline{I}\mathcal{O}_{\overline{Z}'}$ is locally principal. Since $\mathcal{O}_{Z,z}$ is noetherian, the ideal $\overline{I}$ is finitely generated. Let $f_1, \ldots, f_k$ be generators of $\overline{I}$. Let $y \in \overline{\pi}^{-1}(z)$. Since $\overline{I}\mathcal{O}_{\overline{Z}',y}$ is principal, one of the $f_i \circ \overline{\pi}$'s, say $f_1 \circ \overline{\pi}$, generates $\overline{I}\mathcal{O}_{\overline{Z}',y}$. On the other hand f_1 is integral over I, there is a relation:

$$f_1^N + \sum_1^N a_k f_1^{N-k} = 0$$

where $a_k \in I^k$. Therefore in $\overline{I}\mathcal{O}_{\overline{Z}',y}$, we have:

$$(f_1 \circ \overline{\pi})^N + \sum_1^N (a_k \circ \overline{\pi}) p (f_1 \circ \overline{\pi})^{N-k} = 0$$

and by dividing by $(f_1 \circ \overline{\pi})^N$:

$$1 + \sum_1^N \frac{(a_k \circ \overline{\pi})}{(f_1 \circ \overline{\pi})^k} = 0,$$

which yields

$$f_1 \circ \overline{\pi} = -\sum_1^N \frac{(a_k \circ \overline{\pi})}{(f_1 \circ \overline{\pi})^{k-1}}.$$

Since $a_k \circ \overline{\pi}$ belongs to I^k, $a_k \circ \overline{\pi} \in \overline{I}^{k-1} I\mathcal{O}_{\overline{Z}'}$ and we have, for $1 \leq k \leq N$,

$$\frac{(a_k \circ \overline{\pi})}{(f_1 \circ \overline{\pi})^{k-1}} \in I\mathcal{O}_{\overline{Z}',y},$$

so $f_1 \circ \overline{\pi} \in I\mathcal{O}_{\overline{Z}',y}$ and at y:

$$I\mathcal{O}_{\overline{Z}',y} = \overline{I}\mathcal{O}_{\overline{Z}',y} = (f_1 \circ \pi)\mathcal{O}_{\overline{Z}',y}.$$

Therefore the sheaf $I\mathcal{O}_{\overline{Z}'}$ is invertible. It follows that $\overline{\pi}$ factorizes uniquely by $\tau : \overline{Z}' \to \tilde{Z}$ through the morphism π:

$$\overline{\pi} = \pi \circ \tau$$

$$\begin{tikzcd} \overline{Z}' \arrow[rr, "\tau"] \arrow[dr, "\overline{\pi}"'] & & \tilde{Z} \arrow[dl, "\pi"] \\ & (Z,z) & \end{tikzcd}$$

The uniqueness of the morphism implies that necessarily σ is the inverse morphism of τ, which shows that the normalized blowing-ups of I and its integral closure $\overline{I}$ in $\mathcal{O}$ are the same.

2. Geometry of Multiplicities

In [15], C. P. Ramanujam gave an interesting geometrical interpretation of the multiplicity.

First recall that for an invertible sheaf $\mathcal{L}$ on a proper scheme X (resp. on a compact analytic space), the Euler characteristic $\chi(\mathcal{L}^n)$ of the cohomology on X of the n-th power $\mathcal{L}^n$ of $\mathcal{L}$ is a function of n which coincides with a polynomial $P_{\mathcal{L}}(n)$ of degree $m \leq d := \dim X$ in n. The coefficient of n^d in this polynomial is

$$\frac{1}{d!} d(\mathcal{L})$$

and $d(\mathcal{L})$ is called the degree of $\mathcal{L}$.

In the case of local analytic rings the result of C.P. Ramanujam (see [15] Theorem p. 64 and Remark (1) p. 66) can be stated in the following way:

2.1**. Theorem.** *Let* $(\mathcal{O}, \mathfrak{M})$ *be a reduced local analytic local ring and* I *a* $\mathfrak{M}$*-primary ideal of* $\mathcal{O}$*. Let* (Z, z) *be a germ of analytic space such that* $\mathcal{O}_{Z,z} = \mathcal{O}$*. Let* $\pi : Z' \to (Z, z)$ *be a bimeromorphic map such that* $\pi^* I$ *is an invertible sheaf on* Z'*. The degree of the restriction of* $\pi^* I$ *to the space defined by* $\pi^* I$ *is equal to the multiplicity of the ideal* I*.*

Considering the space defined by the coherent ideal sheaf $\pi^* I = I\mathcal{O}_{Z'}$, we have the exact sequence

$$0 \to I^{n+1}\mathcal{O}_{Z'} \to I^n\mathcal{O}_{Z'} \to I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'} \to 0$$

which yields that the degree of the restriction of $\pi^* I$ to the space $< \pi^* I >$ defined by $\pi^* I$ itself equals the degree of $I\mathcal{O}_{Z'}$ because

$$\chi(I^n\mathcal{O}_{Z'}) - \chi(I^{n+1}\mathcal{O}_{Z'}) = \chi(I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'})$$

and $\chi(I^n\mathcal{O}_{Z'}) - \chi(I^{n+1}\mathcal{O}_{Z'}) = P_{I\mathcal{O}_{Z'}}(n) - P_{I\mathcal{O}_{Z'}}(n+1)$ is a polynomial of degree $d-1$ with a term of degree $d-1$ equal to

$$-\frac{1}{(d-1)!} d(I\mathcal{O}_{Z'}) n^{d-1}.$$

Since $\chi(I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'})$ has a term of degree $d-1$ equal to

$$\frac{1}{(d-1)!} d(I\mathcal{O}_{Z'}|_{<\pi^* I>}) n^{d-1},$$

Ramanujam's theorem implies

2.2**. Corollary.** *The multiplicity of* I *equals:*

$$d(I\mathcal{O}_{Z'}|_{<\pi^* I>}) = -d(I\mathcal{O}_{Z'}).$$

Ramanujam's theorem in particular applies to the cases when π is the normalized blowing-up of I or a resolution of (Z, z) in which $\pi^* I$ is an invertible sheaf.

For instance, when the bimeromorphic map π of the preceding theorem is a resolution of singularities π of (Z, z) for which $\pi^* I$ is invertible, we have:

2.3**. Corollary.** *Assume that the map* π *of the preceding theorem is a resolution of singularities for which* $\pi^* I$ *is invertible and* $(\mathcal{O}, \mathfrak{M})$ *is an integral domain, then the multiplicity of* I *equals* $(-1)^{d-1}(D)^d$*, where* d *is the Krull dimension of* $\mathcal{O}$*,* D *is the divisor defined by* $\pi^* I$ *on* Z' *and* $(D)^d$ *the* d*-th self-intersection of* D*.*

Proof: According to Ramanujam's theorem the multiplicity of I equals

$$d(I\mathcal{O}_{Z'}|_{<\pi^* I>}).$$

The preceding corollary gives

$$d(I\mathcal{O}_{Z'}|_{<\pi^*I>}) = -d(I\mathcal{O}_{Z'}).$$

Let $D =< \pi^* I >$ be the divisor of Z' defined by the invertible sheaf $I\mathcal{O}_{Z'}$. Hirzebruch-Riemann-Roch theorem (see [H] Theorem 4.1 Appendix A) gives that the degree $d(I\mathcal{O}_{Z'})$ of $I\mathcal{O}_{Z'}$ equals $(-1)^d(D)^d$. Precisely,

$$\chi(I^n\mathcal{O}_{Z'}) = ch(I^n\mathcal{O}_{Z'})Todd(\mathcal{T}_{Z'}) \cap [Z']$$

where $[Z']$ is the fundamental class of Z' and $ch(I^n\mathcal{O}_{Z'})$ is the Chern character and $Todd(\mathcal{T}_{Z'})$ is the Todd class of the tangent bundle of Z':

$$Todd(\mathcal{T}_{Z'}) = 1 + \frac{1}{2}c_1(\mathcal{T}_{Z'}) + \ldots,$$

and, since $I^n\mathcal{O}_{Z'}$ is invertible, we have:

$$ch(I^n\mathcal{O}_{Z'}) = 1 + nc_1(I\mathcal{O}_{Z'}) + \ldots + \frac{1}{d!}n^d c_1^d(I\mathcal{O}_{Z'}).$$

By comparing the terms of degree d in n, for $n \gg 0$, on each side of the equality of Hirzebruch-Riemann-Roch theorem, we have:

$$d(I\mathcal{O}_{Z'}) = c_1^d(I\mathcal{O}_{Z'}) \cap [Z'].$$

Since $I\mathcal{O}_{Z'}$ is $\mathcal{O}(-D)$ we have:

$$c_1^d(I\mathcal{O}_{Z'}) \cap [Z'] = (-D)^d$$

and the multipilicity of I is $-d(I\mathcal{O}_{Z'}) = -(-D)^d = (-1)^{d-1}(D)^d$.

3. **Linear Systems**

Let $(y_1, \ldots, y_k)$ be generators of an ideal J of the reduced analytic local ring $\mathcal{O}_{Z,z}$. We can construct the blowing-up of J in the following way.

Let Z be a representative of the germ (Z, z) such that the germs y_i $(1 \leq i \leq k)$ are defined by holomorphic functions defined on Z also denoted by y_k and let Y be a representative of the support of J in Z. Then on $Z \setminus Y$ we define the map λ into the complex projective space $\mathbb{P}^{k-1}$ by:

$$\lambda(z') = (y_1(z') : \ldots : y_k(z'))$$

for any $z' \in Z \setminus Y$.

The graph G of λ is an analytic subspace of $Z \times \mathbb{P}^{k-1}$. The topological closure $\overline{G}$ of G is naturally an analytic subspace of $Z \times \mathbb{P}^{k-1}$, because G is the difference of the analytic set defined by

$$(y_1 : \ldots : y_k) = (u_1 : \ldots : u_k)$$

in $Z \times \mathbb{P}^{k-1}$, the u_i's are the homogeneous coordinates of $\mathbb{P}^{d-1}$, and the analytic set $Y \times \mathbb{P}^{k-1}$ (use e.g. Lemma 3.9 of [19]). One can show that the restriction to $\overline{G}$ of the first projection onto Z is a representative of the blowing-up $p : Z_J \to (Z, z)$ of the ideal $J = (y_1, \ldots, y_k)$ in (Z, z). Notice that Z_J is reduced.

Let n be the normalization of $\overline{G}$, then by corollary 1.3 the composition $p \circ n$ is also the normalized blowing-up $\pi : \tilde{Z} \to (Z, z)$ of I in (Z, z).

Consider the special case J is generated by $d \geq 2$ generators where d is the Krull dimension of $\mathcal{O}_{Z,z}$ and $Y = \{z\}$. The blowing-up Z_J of J is given in $Z \times \mathbb{P}^{d-1}$ by the equations

$$u_{i+1}y_i - u_i y_{i+1} = 0$$

where $1 \leq i \leq d-1$. Therefore the second projection induces a map

$$\lambda_J : Z_J \to \mathbb{P}^{d-1}$$

which can be viewed as the family of curves defined by the linear system generated by $y_1, \ldots, y_d$. On the other hand the underlying set $|p^{-1}(z)|$ of the exceptional divisor of the blowing-up $p : Z_J \to (Z, z)$ is contained in $\{z\} \times \mathbb{P}^{d-1}$, so

$$|p^{-1}(z)| = \{z\} \times \mathbb{P}^{d-1}.$$

Let $\mathbf{a} = (a_1, \ldots, a_d)$ be a general point of $\mathbb{P}^{d-1}$. Since both Z_J and $\mathbb{P}^{d-1}$ are reduced, the general fiber $\lambda_J^{-1}(\mathbf{a})$ is a general reduced curve in the linear system of curves generated by $y_1, \ldots, y_d$ (see [13]). Therefore, after normalization, the inverse image $n^{-1}(\lambda_J^{-1}(\mathbf{a}))$ is a non-singular (reduced) curve transverse to the exceptional divisor of the normalized blowing-up $p \circ n$. Since all the components of the exceptional divisor of $p \circ n$ project onto $\mathbb{P}^{d-1}$, the curve $n^{-1}(\lambda_J^{-1}\mathbf{a}))$ intersects all these components.

Apply these results to the case of a $\mathfrak{M}_{Z,z}$-primary ideal I of the reduced analytic local ring $\mathcal{O}_{Z,z}$. The result of P. Samuel tells us that the ideal I is integral over a ideal J generated by d general elements $x_1, \ldots, x_d$ of I, where d is the Krull dimension of $\mathcal{O}_{Z,z}$. We have seen that the normalized blowing-up $\overline{\pi} : \tilde{Z} \to (Z, z)$ of I coincides with the normalized blowing-up of the ideal generated by $x_1, \ldots, x_d$. Let Γ be a general curve in the linear system of curves generated by $x_1, \ldots, x_d$. From what precedes we observe that Γ is reduced and the strict transform $\tilde{\Gamma}$ of Γ by $\overline{\pi}$ is a non-singular curve which intersects transversally all the components D_α, $\alpha \in A$ of $|\overline{\pi}^{-1}(z)|$. This strict transform of a general curve in the linear system of curves generated by $x_1, \ldots, x_d$ can be obtained in the following way:

- Let $\pi_J : Z_J \to (Z, z)$ be the blowing-up of the ideal J. We have a map $\lambda_J : Z_J \to \mathbb{P}^{d-1}$ defined by the generators $x_1, \ldots, x_d$ of J.

- Consider a general point $\mathbf{m}$ of $\mathbb{P}^{d-1}$, it is defined by $d-1$ linear equations $\sum_{i=1}^d \alpha_i^j \xi_i = 0$, $1 \leq j \leq d-1$, where $\xi_1, \ldots, \xi_d$ are the homogeneous coordinates of $\mathbb{P}^{d-1}$.
- The fiber $\lambda_J^{-1}(\mathbf{m})$ of λ_J over $\mathbf{m}$ is the strict transform by the blowing-up π_J of the curve Γ on (Z, z) defined by $\sum_{i=1}^d \alpha_i^j x_i = 0$, $1 \leq j \leq d-1$. Since $\mathbf{m}$ is a general point of $\mathbb{P}^{d-1}$, the germ of curve (Γ, z) is a general curve in the linear system of curves generated by $x_1, \ldots, x_d$. The strict transform of Γ by the normalized blowing-up $\overline{\pi}$ is $n^{-1}(\lambda_J^{-1}(\mathbf{m}))$.

Let d_α be the number of components of the strict transform $\tilde{\Gamma}$ which have a non-empty intersection with D_α. Let e_α be the multiplicity of D_α in the divisor defined by $I\mathcal{O}_{\tilde{Z}}$. Then, we have:

3.1. **Theorem.** *The multiplicity of the ideal I equals $\sum_{\alpha \in A} e_\alpha d_\alpha$.*

Proof: Let $\varphi : \mathcal{Z} \to \tilde{Z}$ be a resolution of singularities of $\tilde{Z}$. The sheaf $(\overline{p} \circ \varphi)^* I\mathcal{O}_{\mathcal{Z}} = I\mathcal{O}_{\mathcal{Z}}$ generated by I on $\mathcal{Z}$ is invertible. Let D be the divisor of $\mathcal{Z}$ defined by $I\mathcal{O}_{\mathcal{Z}}$. According to corollary 2.3, the multiplicity of I equals $(-1)^{d-1}(D)^d$. We shall prove:

3.2. **Lemma.**
$$(-1)^{d-1}(D)^d = \sum_{\alpha \in A} e_\alpha d_\alpha.$$

Using Ramanujam's result, this lemma obviously implies our theorem.

Proof of the lemma: First we observe that, since the image of D by the map $\overline{p} \circ \varphi$ is a point $\{z\}$, we have
$$D.div(f \circ \overline{p} \circ \varphi) = 0,$$
for any germ of functions $f \in \mathfrak{M}_{Z,z} \subset \mathcal{O}_{Z,z}$. In particular, if f is a general element of the ideal I, we have:
$$div(f \circ \overline{p} \circ \varphi) = D + H(f)$$
where $H(f)$ is the strict transform of $\{f = 0\}$.

Now let us choose $\alpha_i^j \in \mathbb{C}$, such that the $d-1$ linear equations $\sum_{i=1}^d \alpha_i^j \xi_i = 0$, $1 \leq j \leq d-1$, are general and define a general point of $\mathbb{P}^{d-1}$. Let $f_j := \sum_{i=1}^d \alpha_i^j x_i = 0$, $1 \leq j \leq d-1$. The functions f_j, $1 \leq j \leq d-1$, are general elements of the ideal I. The curve Γ on Z defined by $\{f_1 = \ldots = f_{d-1}\}$ is a general curve in the linear system of curves generated by $x_1, \ldots, x_d$. The strict transform of Γ by $\overline{p} \circ \varphi$ is the curve $H(f_1) \cap \ldots \cap H(f_{d-1})$.

The lemma will be consequence of the equality
$$(-1)^{d-1}(D^d) = (D.H(f_1).\dots.H(f_{d-1})).$$
In fact, since $(D.D + H(f_i)) = 0$, for $1 \leq i \leq d-1$, we have
$$(D.H(f_1).\dots.H(f_{d-1})) = -(D.H(f_1).\dots.H(f_{d-2}).D)$$
Therefore, by induction we can prove
$$(D.H(f_1).\dots.H(f_{d-1})) = (-1)^{d-2}(D.H(f_1).D.\dots.D) = (-1)^{d-1}(D^d).$$

It remains to prove that $(D.H(f_1).\dots.H(f_{d-1})) = \sum_{\alpha \in A} e_\alpha d_\alpha$. The curve Γ being a general curve in the linear system of curves generated by $x_1, \dots, x_d$ the strict transform $\tilde{\Gamma}$ of Γ by $\overline{\pi}$ is non-singular and transverse to the components of $|\overline{\pi}^{-1}(z)|$. Since φ is a resolution of singularities $\tilde{Z}$ and
$$\varphi^{-1}(\tilde{\Gamma}) = H(f_1) \cap \dots \cap H(f_{d-1})$$
the intersection points of $H(f_1) \cap \dots \cap H(f_{d-1})$ and D are the inverse images by φ of the intersection points of $\tilde{\Gamma}$ and $|\overline{\pi}^{-1}(z)|$ and the multiplicity e_α of D_α in $\tilde{Z}$ equals the multiplicity of the corresponding component in $\mathcal{Z}$. Since the intersection of $\tilde{\Gamma}$ with the divisor of $\tilde{Z}$ defined by $I\mathcal{O}_{\tilde{Z}}$ is $\sum_{\alpha \in A} e_\alpha d_\alpha$, we have
$$(D.H(f_1).\dots.H(f_{d-1})) = \sum_{\alpha \in A} e_\alpha d_\alpha.$$

4. An example

Let us consider the simple case when $\mathcal{O}_{Z,z}$ is a regular local ring of Krull dimension 2. The multiplicity $e(I)$ of a $\mathfrak{M}_{Z,z}$-primary ideal I is the multiplicity of an ideal generated (f, g) by two general elements of I. Since $\mathcal{O}_{Z,z}$ is regular, it is Cohen-Macaulay, so:
$$e(I) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{Z,z}}{(f,g)}.$$
Therefore, the multiplicity of I is the intersection number of $f = 0$ and $g = 0$ at z.

The blowing-up π_J of the ideal $J := (f, g)$ gives the surface Z_J defined by $\beta f - \alpha g = 0$ in $Z \times \mathbb{P}^1$. The projection onto Z restricted to Z_J is the blowing-up π_J and the projection onto $\mathbb{P}^1$ restricted to Z_J extends to Z_J the map λ from $Z \setminus \{z\}$ into $\mathbb{P}^1$ defined by $\lambda(z') = (f(z') : g(z'))$, for $z' \in Z \setminus \{z\}$.

In [18] M. Spivakovsky shows that the singularities of the normalization $\tilde{Z}$ of Z_J are rational. He calls these singularities *Sandwich singularities* (see also [8]).

Let $\varphi : \mathcal{Z} \to \tilde{Z}$ be the minimal resolution of $\tilde{Z}$. The map $\pi_J \circ n \circ \varphi$, where n is the normalization of Z_J, is a bimeromorphic map from a non-singular surface $\mathcal{Z}$ onto Z:

$$\mathcal{Z} \xrightarrow{\varphi} \tilde{Z} \xrightarrow{n} Z_J \xrightarrow{\pi_J} (Z, z).$$

Therefore, it is the composition of a sequence of point blowing-ups. In fact, since the strict transforms $H(f)$ and $H(g)$ of $f = 0$ and $g = 0$ by $\pi_J \circ n \circ \varphi$ are non-singular, distinct and transverse to $|(\pi_J \circ n)^{-1}(z)|$, the map $\pi_J \circ n \circ \varphi$ is an embedded resolution of the plane curve $fg = 0$.

Conversally let $\sigma : \mathcal{Z}' \to (Z, z)$ be the minimal embedded resolution of the germ of curve $\{fg = 0\}$ in (Z, z). Let $\mathcal{D}_\alpha$, $\alpha \in A$, be the components of the exceptional divisor $\mathcal{E}$ of σ which intersect the strict transform of the curve $\{fg = 0\}$. Consider the connected components of the closure of $\mathcal{E} \setminus \cup_{\alpha \in A} \mathcal{D}_\alpha$ and the singular surface $\tilde{Z}'$ obtained from $\mathcal{Z}'$ by contracting these components:

$$\varphi' : \mathcal{Z}' \to \tilde{Z}'.$$

Since σ is the minimal embedded resolution of the germ of curve $\{fg = 0\}$ in (Z, z), the only components of $\mathcal{E}$ which might be of self-intersection -1 are among the components $\mathcal{D}_\alpha$, $\alpha \in A$. Therefore, the contraction φ' is the minimal resolution of $\tilde{Z}'$.

The contraction of the components $\mathcal{D}_\alpha$, $\alpha \in A$, defines a holomorphic map:

$$\overline{\pi}' : \tilde{Z}' \to (Z, z)$$

$$\begin{tikzcd} \overline{Z}' \arrow[rr, "\varphi'"] \arrow[dr, "\sigma"'] & & \tilde{Z} \arrow[dl, "\overline{\pi}'"] \\ & (Z, z) & \end{tikzcd}$$

We have:

4.1**. Lemma.** *The ideal sheaf* $(f, g)\mathcal{O}_{\tilde{Z}'} = \overline{\pi}'^*(f, g)\mathcal{O}_Z$ *is invertible.*

Proof: Let $\tilde{H}(f)$ and $\tilde{H}(g)$ be the strict transforms of $f = 0$ and $g = 0$ by $\overline{\pi}'$, then the valuation along $\varphi'(D_\alpha)$ of any function $h = uf + vg$ of $\mathcal{O}_{Z,z}$ being more that the one of f or g, at any non-singular point y of $\cup_{\alpha \in A} \varphi'(D_\alpha)$ which is neither a singular point of $\tilde{Z}'$ nor a point of $\tilde{H}(f)$ (resp. a point of $\tilde{H}(g)$), $f \circ \overline{\pi}'$ (resp. $g \circ \overline{\pi}'$) is a generator of $(f, g)\mathcal{O}_{\tilde{Z}',y}$.

On the other hand $f \circ \overline{\pi}'$ (resp. $g \circ \overline{\pi}'$) does not vanish on

$$\tilde{Z}' \setminus (\tilde{H}(f) \cup_{\alpha \in A} \varphi'(D_\alpha)) \quad (resp.\, \tilde{Z}' \setminus (\tilde{H}(g) \cup_{\alpha \in A} \varphi'(D_\alpha))).$$

Therefore, for any function h in (f,g), the meromorphic function $(h/f)\circ\overline{\pi}'$ is bounded on

$$\tilde{Z}' \setminus (\tilde{H}(f) \cup \Sigma \cup \Sigma_{\tilde{Z}'}),$$

where Σ is the finite set of singular points of $\cup_{\alpha\in A}\varphi'(D_\alpha)$ and $\Sigma_{\tilde{Z}'}$ is the finite set of singular points of $\tilde{Z}'$. Since $\tilde{Z}'$ is normal, this implies that $(h/f)\circ\overline{\pi}'$ is holomorphic on $\tilde{Z}'\setminus\tilde{H}(f)$. Similarly $(h/g)\circ\overline{\pi}'$ is holomorphic on $\tilde{Z}'\setminus\tilde{H}(g)$. It shows that the ideal sheaf $(f,g)\mathcal{O}_{\tilde{Z}'} = \overline{\pi}'^{*}(f,g)\mathcal{O}_Z$ is invertible.

Thus, the contraction $\overline{\pi}'$ factorizes uniquely through the normalized blowing-up

$$\bar{\pi} := \pi_J \circ n : \tilde{Z} \to (Z,z)$$

of the ideal (f,g):

$$\overline{\pi}' = \overline{\pi}\circ\overline{\theta}$$

$$\begin{tikzcd} \tilde{Z}' \arrow[rr, "\bar{\theta}"] \arrow[dr, "\bar{\pi}'"'] & & \tilde{Z} \arrow[dl, "\bar{\pi}"] \\ & (Z,z) & \end{tikzcd}$$

Since $\sigma : \mathcal{Z}' \to (Z,z)$ is the minimal embedded resolution of $\{fg=0\}$ in Z, the map $\overline{\pi}\circ\varphi$ factorizes uniquely through σ:

$$\overline{\pi}\circ\varphi = \sigma\circ\eta.$$

The map $\varphi'\circ\eta$ is constant on the exceptional fibers of φ and the space $\tilde{Z}$ is normal, so it gives a unique holomorphic map

$$\overline{\eta} : \tilde{Z} \to \tilde{Z}'$$

such that $\overline{\eta}\circ\varphi = \varphi'\circ\eta$.

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "\eta"'] \arrow[dr, "\varphi"] & \\ \mathcal{Z}' \arrow[r, "\varphi'"] \arrow[dr, "\sigma"'] & \tilde{Z}' \arrow[d, "\bar{\pi}'"] & \tilde{Z} \arrow[l, "\bar{\eta}"'] \arrow[dl, "\bar{\pi}"] \\ & (Z,z) & \end{tikzcd}$$

Necessarily, because of the uniqueness of the factorizations, $\overline{\eta}$ is the inverse of $\overline{\theta}$.

So, we have proved:

4.2. **Lemma.** *The map $\overline{\pi}' : \tilde{Z}' \to (Z, z)$ obtained by contraction is the normalized blowing-up of I.*

The preceding results yields:

4.3. **Theorem.** *Let $(\mathcal{O}_{Z,z}, \mathfrak{M}_{Z,z})$ be the analytic local ring of the germ of a non-singular complex surface (Z, z). Let I be a $\mathfrak{M}_{Z,z}$-primary ideal of $\mathcal{O}_{Z,z}$. Consider f and g, such that, the Milnor number of f and g at z is minimum among the Milnor numbers at z of the elements of I and assume that the ideal I is integral over the ideal (f, g) generated by f and g. The normalized blowing-up of I in (Z, z) is obtained from the minimal embedded resolution of the curve $fg = 0$ by contracting the exceptional components which do not intersect the strict transform of $fg = 0$.*

Recall that the Milnor number of f at z (introduced in [12]) is a topological invariant of the germ of $f = 0$ at z (see [6] p. 261). It is the number of vanishing cycles of $f = 0$ at z and equals the first Betti number of $\{f = t\} \cap B_\varepsilon(z)$, where $B_\varepsilon(z)$ is a sufficiently small ball centered at z and $\varepsilon \gg |t| > 0$.

Elements f with the minimum Milnor number in I have the same topology by using the results of [9]. Moreover, since f and g belongs the linear system $\lambda f + \nu g$ and have the minimum Milnor number in this linear system, because $\lambda f + \nu g \in I$, one can show that they have the same embedded resolution (see e.g. [10] §2).

Theorem 4.3 indicates that one can choose superficial elements (f, g) of I to be elements of I with minimum Milnor number at z and such that I and the ideal (f, g) have the same multiplicity.

In this context Theorem 3.1 tells us that the multiplicity of I equals the intersection number of the strict transform $\tilde{H}(f)$ of f by the normalized blowing-up of I and the exceptional divisor of this normalized blowing-up.

The preceding discussion also gives the following result:

Let f be an element of I having the smallest Milnor number at z. Let $\tau : \mathcal{Z} \to (Z, z)$ be the minimal embedded resolution of $f = 0$.

4.4. **Corollary.** *The ideal sheaf is $\tau^* I$ is invertible except possibly at the points where the strict transform of $f = 0$ intersects the exceptional divisor of τ. It becomes invertible after a sequence of blowing-ups which separates non-singular branches at these points.*

Proof: As we indicate above, elements of I with the minimum Milnor number have the same embedded resolution. So, the minimal embedded resolution of f is also the minimal embedded resolution of g. However the minimal embedded resolution of f might not be the minimal embedded resolution of $fg = 0$, if the strict transforms of $f = 0$ and $g = 0$ in the minimal embedded resolution of $f = 0$ have common points on the exceptional divisor, in which case one has to separate the strict transforms of $f = 0$ and $g = 0$, by a sequence of point blowing-ups to separate tangent non-singular branches.

We have already seen above that, in the embedded resolution of $fg = 0$, the pull-back of I is invertible. In fact, one can check that the points of the embedded minimal resolution of $f = 0$, where the strict transforms of $f = 0$ and $g = 0$ have common points, are precisely the points where the pull-back of I is not invertible.

In summary, the ideal $\tau^*(f, g)$ is invertible on the minimal embedded resolution of $f = 0$ or on the modification of this minimal embedded resolution obtained by a sequence of point blowing-ups to separate the branches of $f = 0$ and $g = 0$ passing through common points on the exceptional divisor. Since this embedded resolution of $fg = 0$ is non-singular, it is normal. So, it dominates the normalized blowing-up of (f, g) which is also the normalized blowing-up of I by corollary 1.3.

M.S. Narasimhan showed me the following result of D. Mumford (see [14] Lemma p. 91-92) which can be obtained by using this viewpoint.

Consider $(Z, z) = (\mathbb{C}^2, O)$ and the ideal I generated by the monomials $x^{r_0}y^{s_0}, \ldots, x^{r_n}y^{s_n}$. Let $\alpha = p/q$ (p and q being relatively prime). Denote by ν_α the discrete valuation of rank 1 on $\mathcal{O}_{\mathbb{C}^2,O}$ centered at O such that

$$\nu_\alpha(\sum_{i,j} a_{i,j}x^iy^j) = \min_{a_{i,j}\neq 0}(ip + jq).$$

4.5**. Proposition.** *The exceptional divisors of the normalized blowing-up of I are those prime divisors of the field of fractions of $\mathcal{O}_{\mathbb{C}^2,O}$ corresponding to valuations ν_α with $\alpha = p/q$ where the least integer in the sequence of integers $r_ip + s_iq$ $(0 \leq i \leq n)$ occurs at least twice.*

Proof: First, notice that the ideal I might not be primary for the maximal ideal $\mathfrak{M}_{\mathbb{C}^2,O}$ of $\mathcal{O}_{\mathbb{C}^2,O}$. However, there are unique integers a and b and a unique $\mathfrak{M}_{\mathbb{C}^2,O}$-primary ideal I, such that:

$$I = (x^a)(y^b)I'.$$

Now, it is clear that, since the ideal (x^a) and (y^b) are invertible, the normalized blowing-up of I' is also the normalized blowing-up of I.

Let $x^{r'_0}y^{s'_0}, \ldots, x^{r'_n}y^{s'_n}$ be the generators of I', so, for $0 \leq i \leq n$:

$$r'_i = r_i - a \text{ and } s'_i = s_i - b.$$

As we have seen in our example above, the components of the normalized blowing-up of the $\mathfrak{M}_{\mathbb{C}^2,O}$-primary ideal I' come from components of the minimal embedded resolution:

$$\pi : Z \to (\mathbb{C}^2, O)$$

of FG, where F and G are two linear combinations:

$$\sum_{i=o}^{i=n} \lambda_i^j x^{r'_i} y^{s'_i}, \text{ with } j = 0, 1$$

having the minimal Milnor number at O and such that I' and (F, G) have the same multiplicity. Precisely, consider the exceptional components D_α, $\alpha \in A$, of this embedded resolution which intersect the strict transforms of $FG = 0$. Now contract the exceptional components of π which are not among the D_α's. We obtain:

$$\gamma : Z \to Z_1$$

and π defines a unique morphism $\pi_1 : Z_1 \to (\mathbb{C}^2, O)$, such that $\pi = \pi_1 \circ \gamma$. We saw above that π_1 is the normalized blowing-up of I.

Corollary 4.4 suggests to consider first the minimal embedded resolution

$$Z' \to (\mathbb{C}^2, O)$$

of $F = 0$.

Consider the set B of exponents (r'_i, s'_i), for $0 \leq i \leq n$, in the real plane. The convex hull of B is called the *Newton Polyhedron* of the set of exponents B. The *Newton Polygon* $\mathcal{N}(B)$ of B is the set of faces viewed from the origin $(0, 0)$. Call $A_1, \ldots, A_\ell$ the sides of the Newton Polygon $\mathcal{N}(B)$ with respective slopes

$$-\frac{p_1}{q_1} \leq -\frac{p_2}{q_2} \leq \ldots \leq -\frac{p_\ell}{q_\ell}.$$

Notice that the slopes of the edges of $\mathcal{N}(B)$ are given by the linear forms $p\alpha + q\beta$ which attain their minimum at two exponents of B at least.

Denote $F_{A_j} := \sum_{(r'_i, s'_i) \in A_j} \lambda_i x^{r'_i} y^{s'_i}$ where the λ_i are general complex numbers.

4.6**. Lemma.** *In the linear family of polynomials*

$$F = \sum_{i=o}^{i=n} \lambda_i x^{r'_i} y^{s'_i}$$

where $\lambda_i \neq 0$, for $0 \leq i \leq n$, are general complex numbers, the plane curve singularity $F = 0$ at O is isolated and has the topological type at O of

$$F_1 = F_{A_1} \ldots F_{A_\ell}.$$

Proof: Consider a polynomial $F_0(x,y)=\sum_{i=o}^{i=n}\beta_i x^{r'_i}y^{s'_i}=G_{A_1}+\ldots+G_{A_\ell}$ of this linear family, where:

$$G_{A_j}=\sum_{(r'_i,s'_i)\in A_j}\beta_i x^{r'_i}y^{s'_i}.$$

The proof of Puiseux Theorem (see [19] Chap, IV §3) shows that the series:

$$y-cx^{\gamma_i}-\ldots$$

where $\gamma_i=q_i/p_i$ and c is a root of $P_i(t):=\sum_{(r'_j,s'_j)\in A_i}\beta_j t^{s'_j}$ divide F_0 in the ring $\cup_{n\geq 1}\mathbb{C}[y][[x^{1/n}]]$. For a general choice of the coefficients β_j, the polynomial P_i has b_i distinct solutions, where:

$$b_i:=\sup_{(r'_j,s'_j)\in A_i}s'_j-\inf_{(r'_j,s'_j)\in A_i}s'_j.$$

Therefore, for a general choice of the coefficients β_j, the Puiseux series are all distinct which implies that the product:

$$\prod_{i=1}^{i=\ell}\prod_{P_i(c)=0}(y-cx^{\gamma_i}-\ldots)$$

divides F_0:

$$F_0=u\prod_{i=1}^{i=\ell}\prod_{P_i(c)=0}(y-cx^{\gamma_i}-\ldots),$$

where u is a unit in $\mathbb{C}[[x,y]]$, because $\sum_{i=1}^{i=\ell}b_i=\sup_{(r'_i,s'_i)\in\mathcal{N}(B)}s'_i$. This shows that $F_0=0$ has an isolated singularity at O and:

$$G_{A_i}=u_i\prod_{P_i(c)=0}(y-cx^{\gamma_i}-\ldots),$$

where u_i is a unit in $\mathbb{C}[[x,y]]$.

In particular, for a general choice of the β_i, each plane curve $G_{A_i}=0$ has an isolated singularity at O. Moreover, since G_{A_i} is a weighted homogeneous polynomial, each branch of $G_{A_i}=0$ is also defined by a weighted homogeneous polynomial with the same weights. This implies that in each Puiseux series above has the simple form $y-cx^{\gamma_i}$. The Milnor number of these branches is $(p_i-1)(q_i-1)$ and their pairwise intersection numbers are p_iq_i.

One can also prove:

4.7. **Lemma.** *The minimum Milnor number of a linear combination of elements of I' equals the Kushnirenko number $2S-a-b+1$, where S is the area below the Newton polygon $\mathcal{N}(B)$, a is $\sup_{\alpha_i\neq 0} r'_i$ and b is $\sup_{\alpha_i\neq 0} s'_i$.*

Proof: The number of branches of $G_{A_i}=0$ at O equals b_i/q_i and the Milnor number each branch of $G_{A_i}=0$ at O is $(p_i-1)(q_i-1)$. The pairwise intersection numbers of these branches are equal to p_iq_i.

Defining:

$$a_i := \sup_{(r'_j,s'_j)\in A_i} r'_j - \inf_{(r'_j,s'_j)\in A_i} r'_j,$$

the number of branches of $G_{A_i}=0$ at O also equals a_i/p_i.

The pairwise intersection numbers of branches of $G_{A_i}=0$ and $G_{A_j}=0$, for $i<j$, are equal to p_iq_j.

So, the Milnor number $\mu(F_0,O)$ at O of F_0 for a general choice of the coefficients β_i equals (see [12] Theorem 10.5 and Remark 10.10):

$$\sum_{i=1}^{i=\ell}\frac{b_i}{q_i}(p_i-1)(q_i-1)+\sum_{i=1}^{i=\ell}2p_iq_i\frac{b_i}{2q_i}(\frac{b_i}{q_i}-1)+2\sum_{i=1}^{i=\ell}\sum_{i<j}p_iq_j\frac{b_i}{q_i}\frac{b_j}{q_j}-\sum_{i=1}^{i=\ell}\frac{b_i}{q_i}+1.$$

On the other hand:

$$2S=\sum_{i=1}^{i=\ell}\frac{b_i^2}{q_i^2}p_iq_i+2\sum_{i=1}^{i=\ell-1}a_ib_{i+1}$$

$$a=\sum_{i=1}^{i=\ell}a_i \quad \text{and} \quad b=\sum_{i=1}^{i=\ell}b_i.$$

Using the equality:

$$\frac{a_i}{p_i}=\frac{b_i}{q_i}$$

we obtain:

$$\mu(F_0,O)=2S-a-b+1.$$

To finish the proof of Lemma 4.6, it is enough to notice that F and F_1 belong to the same linear system and the minimum Milnor number is the minimum Milnor number among the analytic functions having the support of their Newton principal part on $\mathcal{N}(B)$, i.e. the Kushnirenko number (see [5] 1.10), as stated in the main theorem of Kushnirenko [5].

Now, the minimal embedded resolution of $F_0=0$ for a general choice of the coefficients β_i, is also an embedded resolution for $G_{A_i}=0$. We have noticed that the branches of $G_{A_i}=0$ are weighted homogeneous curves $\lambda x^{q_i}+\nu y^{p_i}=0$. This implies that the multiplicities of the coordinates

x and y along the component intersected by the the strict transforms of the branches of $G_{A_i} = 0$ are respectively equal to p_i and q_i. Thus, this component defines a divisorial valuation of the field of fractions of $\mathcal{O}_{\mathbb{C}^2,O}$ given by $v_i(x) = p_i$ and $v_i(y) = q_i$. Therefore:

$$v_i(\sum c_{\alpha,\beta} x^\alpha y^\beta) = \inf_{c_{\alpha\,\beta}\neq 0} (p_i\alpha, q_i\beta).$$

Each slope $-p_i/q_i$ of the Newton Polygon of B defines such a valuation. By definition of the Newton Polygon, these valuations are defined by pairs of integers (p, q), for which the minimum of the linear form $p\alpha + q\beta$ is obtained for at least two distinct pairs among $\{(r'_i, s'_i)\}$. These valuations correspond to the ones given by Proposition 4.5. To prove that these are the divisorial valuations of the exceptional components of the normalized blowing-up of I', it remains to prove that the minimal embedded resolution of $F_0 = 0$ already gives after contraction the normalized blowing-up of I'.

As remarked before, we have to prove that the strict transform of a curve singularity $G = 0$ defined by a general element G of I', such that I' and the ideal (F_0, G) have the same multiplicity, is disjoint from the strict transform of $F_0 = 0$ in the minimal embedded resolution of $F_0 = 0$.

To obtain this last assertion, notice that, in the minimal embedded resolution of $F_0 = 0$, the strict transforms of the branches $\lambda x^{q_i} + \nu y^{p_i} = 0$ given by the edges of the Newton polygon for distinct $(\lambda : \nu)$ are disjoint. This implies that the strict transform of $G_0 = 0$, given by another general choice of the coefficients β_i, is disjoint from the strict transform of $F_0 = 0$.

As seen in §3 above, a general element G of I' to be considered can be chosen as $G = G_0 + H$, where H is a general linear combination of monomial of B which are not on $\mathcal{N}(B)$. The Puiseux series associated to G are of the type $y - cx^{\gamma_i} + \ldots$. This shows that the strict transforms of the branches of $G = 0$ intersect the strict transforms of the branches of $G_0 = 0$ in the minimal embedded resolution of $F_0 = 0$. This yields that the strict transforms of the branches of $G = 0$ are disjoint from the strict transforms of the branches of $F_0 = 0$ in the minimal embedded resolution of $F_0 = 0$.

So, the normalized blowing-up of I' is already obtained from the minimal embedded resolution of $F_0 = 0$.

Therefore, the components of the minimal embedded resolution of $F_0 = 0$ intersected by the strict transforms of the branches of $F_0 = 0$ give the components of the normalized blowing-up of I'. As we proved above, the divisorial valuations of the exceptional components of the normalized blowing-up of I' are effectively the valuations given in the Proposition 4.5.

This ends our proof.

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