Central polynomials for matrix algebras over the Grassmann algebra

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Abstract. In this work, we describe a method to construct central polynomials for F-algebras where F is a field of characteristic zero. The main application deals with the T-prime algebras $M_n(E)$, where E is the infinite- dimensional Grassmann algebra over F, which play a fundamental role in the theory of PI-algebras. The method is based on the explicit decomposition of the group algebra FS_n.
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1. **Introduction**

Let F be a field of characteristic zero. The set of polynomial identities satisfied by a F-algebra A, denoted by $Id(A)$, is a T-ideal of the free algebra $F\langle X\rangle$ of polinomials over F, *i.e.*, $Id(A)$ is invariant over all endomorphisms of $F\langle X\rangle$. One of the main goals of the theory of PI-algebras, which are algebras satisfying non-trivial polynomial identities, is to determine their T-ideals. For each one of the T-prime algebras, classified by Kemer in [7], the T-ideal has been insistently studied and there are few known results. For example, we have the complete description just for $Id(E)$, $Id(M_2(F))$ and $Id(M_{1,1}(E))$ (see [9], [2] and [11], respectively). Remember that Kemer showed that the only non-trivial T-prime algebras in characteristic zero are the full matrix algebra $M_k(F)$, the algebra $M_k(E)$ of matrices over the the Grassmann algebra E and $M_{r,s}(E)$, a particular subalgebra of $M_{r+s}(E)$.

Trying to get new identities for an algebra A we can consider the set $C(A)$ of central polynomials of A which consists of $f(x_1, \cdots, x_n) \in F\langle X\rangle$ such that $f(a_1, \cdots, a_n)$ belongs to the center of A, $\forall a_1, \cdots, a_n \in A$. The existence of central polynomails is of great interest not only for the development of the structure theory of PI-algebras, but also for the combinatorial PI-theory (as a reference, see[4]). Since $Id(A) \subset C(A)$ it is interesting to construct central polynomials which are non identities of A, called non-trivial central polynomials. Once we have a non-trivial central polynomial $f = f(x_1, \cdots, x_n)$ of degree k of A we get a new polynomial from the commutator $[f, x]$ of degree $k + 1$ which belongs to $Id(A)$.

For the T-prime algebras, the existence of non-trivial central polynomials was proved by Kemer in [8] but in spite of the importance of the these algebras, the concrete form of the set of their central polynomials is far from being known, the only case which is completely settled is that of $M_2(F)$ (see [10]). Several attempts of getting new results have been considered. For example, in [1], Bondari developed a computational method to find all central polynomials of degree less than 9 for $M_3(F)$. In [3], V. Drensky constructed an element in $C(M_k(F)) \setminus Id(M_k(F))$ of degree $(k-1)^2 + 4$ for any $k \geq 3$ and it is the minimal degree known for a non-trivial central polynomial of $M_k(F)$ until this moment. Non-trivial central polynomials for the algebras $M_{k,l}(E)$ were constructed by Razmyslov in [12].

In this work we present a method to construct elements in $C(M_k(E)) \setminus Id(M_k(E))$ based on the explicited decomposition of the group algebra FS_k. The technics developed here were described from an algorithmic process which was implemented by using the free software GAP [5] for the case $k = 2$. We were able to show that the minimal degree of a non-trivial central polynomial in $M_2(E)$ is ≥ 9.

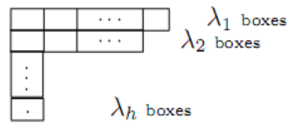
2. The algebra FS_n, PI-algebras and central polynomials

The representation theory of the symmetric group S_n over a field of characteristic zero is a very useful tool in the developing of the theory of PI-algebras. We start this section with some basic facts on this topic.

By fixing a partition $\lambda = (\lambda_1, \ldots, \lambda_h)$ of n, we associate to it the Young diagram D_λ which consists of n boxes $\square$ in the following way

We observe that the lengths of the columns form a new partition of n, $\lambda' := (\lambda'_1, \ldots, \lambda'_r)$, where $\lambda'_i = \sum_{\substack{\nu \\ \lambda_\nu \geq i}} 1$, called conjugate partition of λ.

$D_\lambda :=$



A Young tableau T_λ of a diagram D_λ is a filling of boxes of D_λ with the integers $1, 2, ..., n$. We say that T_λ is a standard Young tableau of shape λ if the integers in each row and in each column increase from left to right and from top to bottom, respectively. For example, for the partition $\lambda = (2, 1)$ of $n = 3$ we have 6 Young tableaux of shape λ and only two of them are standard:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

In general, the number of standard Young tableaux of shape λ is given by the hook formula (see [6]). During this text we consider d_λ the number of Young standard tableau of shape λ and the set $\{T_1, T_2, \cdots, T_{d_\lambda}\}$ of such tableaux in lexicographic order, that is, $T_1 < T_2 < \cdots < T_{d_\lambda}$.

The row-stabilizer of a Young tableau T_λ is defined as

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, \cdots, a_{1\lambda_1}) \times \ldots \times S_{\lambda_h}(a_{h1}, \cdots, a_{h\lambda_h})$$

where $S_{\lambda_i}(a_{i1}, \cdots, a_{i\lambda_i})$ denotes the symmetric group on the integers $a_{i1}, \cdots, a_{i\lambda_i}$. Analogously, the column-stabilizer of T_λ is

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, \cdots, a_{\lambda'_1 1}) \times \ldots \times S_{\lambda'_r}(a_{1\lambda_1}, \cdots, a_{\lambda'_r \lambda_1})$$

where $\lambda' = (\lambda'_1, \cdots, \lambda'_r)$ is the conjugate partition of λ.

Furthermore, we define the essential idempotent associated to T_λ by

$$e_{T_\lambda} := \sum_{\rho \in R_{T_\lambda}} \sum_{\sigma \in C_{T_\lambda}} (\text{sgn } \sigma)\rho\sigma.$$

We will use e_i (or e_i^λ if necessary) to denote the essential idempotent e_{T_i}. In general, the $e_j's$ are not orthogonal but we have that $e_j e_i = 0$, when $T_i < T_j$.

We observe that $e_{\gamma T_\lambda} = \gamma e_{T_\lambda} \gamma^{-1}$ for all $\gamma \in S_n$. Moreover it is possible to prove that there exists a non-zero integer q such that $e_{T_\lambda}^2 = q e_{T_\lambda}$ for all Young tableaux T_λ; it follows that the element $e = \frac{1}{q}\, e_{T_\lambda} \in FS_n$ is idempotent. As an important property of the essential idempotents, we have that the left FS_n-modules $FS_n e_{T_\lambda}$ and $FS_n e_{\widetilde{T}_\lambda}$ are isomorphic when T_λ and $\widetilde{T}_\lambda$ are Young tableaux of same shape λ. Whereas T_λ and T_μ are of different shape, the modules $FS_n e_{T_\lambda}$ and $FS_n e_{T_\mu}$ are not isomorphic. The next theorem shows that the standard Young tableaux come into play if one wants to find, amoung the $n!$ essential idempotents arising from tableaux of shape λ, some orthogonal ones (see [6]).

Theorem 2.1. *We have*

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda, \ \textit{with } I_\lambda = \bigoplus_{T_\lambda \ standard} FS_n e_{T_\lambda}$$

where I_λ is a two-sided ideal with $\dim\ I_\lambda = d_\lambda^2$. *Moreover, each left module $FS_n e_{T_\lambda}$ is minimal and* $\dim FS_n e_{T_\lambda} = d_\lambda$.

It follows from the theorem above that $e_{T_\lambda} e_{T_\mu} = 0$, when λ and μ are different partitions of n.

By considering $\lambda \vdash n$ a fixed partition, we denote by S_{ij} (or S_{ij}^λ) the permutation of S_n which takes the tableau T_j to the tableau T_i, that is, $S_{ij}T_j = T_i$. Obviously, $S_{ij}^{-1} = S_{ji}$. And as we have observed above, it follows that $S_{ij}e_j = e_i S_{ij}$ and the following is true.

Lemma 2.2. *By a fixed partition λ of n, we have*

(1) *For each $k \in \{1, \cdots, d_\lambda\}$, the set $\{e_i S_{ik}\}_{i=1}^{d_\lambda}$ is a F-basis of $FS_n e_k$.*
(2) *The set $\{e_i S_{ij}\}_{i,j=1}^{d_\lambda}$ is a F-basis of I_λ.*

Proof. Since $e_i S_{ik} = S_{ik} e_k$ for a fixed k, we have J_i =span$\{e_i S_{ik}\}$ is a unidimensional space of $FS_n e_k$ for all $i = 1, \cdots, d_\lambda$. As we have observed, $e_j e_i = 0$ when $T_i < T_j$ in the set $\{T_1, T_2, \cdots, T_{d_\lambda}\}$ of standard Young tableaux of shape λ; thus the sum $J_1 + \cdots + J_{d_\lambda}$ is direct. So, $\dim(J_1 \oplus \cdots \oplus J_{d_\lambda}) = d_\lambda$ and then $J_1 \oplus \cdots \oplus J_{d_\lambda} = FS_n e_k$, from Theorem 2.1.

Now since

$$I_\lambda = \bigoplus_{T_\lambda \ standard} FS_n e_{T_\lambda}$$

we have the set $\{e_i S_{ij}\}_{i,j=1}^{d_\lambda}$ generates I_λ and using Theorem 2.1, we get $\dim\ I_\lambda = d_\lambda^2$. Then $\{e_i S_{ij}\}_{i,j=1}^{d_\lambda}$ forms a basis of I_λ. □

Now, if a F-algebra A satisfies a non trivial polynomial identity $f(x_1, \cdots, x_n) \in F\langle X \rangle$, i.e. $f(a_1, \cdots, a_n) = 0$, for all $a_1, \cdots, a_n \in A$ then denote it by $f \equiv 0$ in A and say that A is a PI-algebra. In characteristic zero, it is well known that the ideal $Id(A)$ of identities satisfied by A is finitely generated by its multilinear ones (which are linear in each of its variables) so we consider the F-space of multilinear polynomials in the first

n variables $x_1, \cdots, x_n$, that is, $P_n = \text{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\}$ and define the following left action of S_n over P_n

$$\sigma f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), \tag{1}$$

where $\sigma \in S_n$. We observe that FS_n and P_n are isomorphic (as F-vector spaces) from the linear isomorphism $\psi : FS_n \to P_n$ given by

$$\psi(\sigma) = x_{\sigma^{-1}(1)} x_{\sigma^{-1}(2)} \cdots x_{\sigma^{-1}(n)}.$$

Recording that $f(x_1, \cdots, x_n)$ is a central polynomial of A if $f(a_1, \cdots, a_n)$ belongs to the center of A for all $a_1, \cdots, a_n \in A$, it is well known that the set

$$C(A) = \{f \in F\langle X\rangle \mid [f, x] \in Id(A)\}$$

formed by all central polynomials of A is generated, as a T-space, by its multilinear polynomials.

In this work, we are interested in constructing polynomials in $C(A) \cap P_n$ which are not identities of A for a particular algebra A. Then by using the isomorphism ψ, we have to take elements $\alpha \in FS_n$ such that $\psi(\alpha) \in C(A) \backslash Id(A) \cap P_n$.

Definition 2.3. *We say that $\alpha \in FS_n$ is an element (or multilinear element) of degree n of A if and only if $\psi(\alpha) \in C(A)$.*

We observe the following fact.

Lemma 2.4. (1) *If $\alpha \in FS_n$ is an element of degree n of A then $\alpha.\beta$ is also an element of degree n of A for any $\beta \in FS_n$.*

(2) *If λ is a partition of n and $g_\lambda \in I_\lambda$ is such that $g_\lambda = g_\lambda^1 + g_\lambda^2 + \cdots + g_\lambda^{d_\lambda} \in I_\lambda$ where $g_\lambda^k \in FS_n e_k$, for all $k = 1, \cdots, d_\lambda$ and q is the non zero integer such that $e_k^2 = qe_k$, then we have*

(a) $g_\lambda^k(\frac{1}{q} e_k S_{kk}) = g_\lambda^k$ *for all $k = 1, \cdots, d_\lambda$.*

(b) $g_\lambda^k(\frac{1}{q} e_r S_{rr}) = 0$ *if $k > r$.*

(c) *Recursively defining θ_k^λ by $\theta_1^\lambda = \frac{1}{q} e_1 S_{11}$ and $\theta_k^\lambda = (1 - \theta_1^\lambda - \theta_2^\lambda - \cdots - \theta_{k-1}^\lambda)\frac{1}{q} e_k S_{kk}$, for all $k = 2, \cdots, d_\lambda$ we have*

(i) $g_\lambda.\theta_k^\lambda = g_\lambda^k$*, for all $k = 1, \cdots, d_\lambda$.*

(ii) $g_\mu.\theta_k^\lambda = 0$*, for all partition μ of n different from λ.*

Proof. To show the first item, let $\alpha = \sum \alpha_\sigma \sigma$, $\beta = \sum \beta_\tau \tau$ be elements of FS_n and for each τ, consider the endomorphism φ_τ of $F\langle X\rangle$ such that $x_i \mapsto x_{\tau^{-1}(i)}$ for all $i = 1, \cdots, n$ and fixes all the remaining variables. Thus for each $\tau \in S_n$ we have

$$\begin{aligned}\psi(\alpha.\tau) &= \psi\left(\left(\sum \alpha_\sigma \sigma\right).\tau\right) = \sum \alpha_\sigma \psi(\sigma.\tau) \\ &= \sum \alpha_\sigma x_{\tau^{-1}(\sigma^{-1}(1))} x_{\tau^{-1}(\sigma^{-1}(2))} \cdots x_{\tau^{-1}(\sigma^{-1}(n))} \\ &= \sum \alpha_\sigma \varphi_\tau (x_{\sigma^{-1}(1)} x_{\sigma^{-1}(2)} \cdots x_{\sigma^{-1}(n)}) \\ &= \sum \alpha_\sigma \varphi_\tau(\psi(\sigma)) = \varphi_\tau(\psi(\alpha))\end{aligned}$$

and for a fixed variable x,

$$\begin{aligned}[\psi(\alpha.\beta), x] &= \left[\psi\left(\alpha.\left(\sum \beta_\tau \tau\right)\right), x\right] \\ &= \sum \beta_\tau [\psi(\alpha.\tau), x] \\ &= \sum \beta_\tau [\varphi_\tau(\psi(\alpha)), \varphi_\tau(x)] \\ &= \sum \beta_\tau \varphi_\tau([\psi(\alpha), x]).\end{aligned}$$

It follows that if $\psi(\alpha) \in C(A)$, *i.e.*, $[\psi(\alpha), x] \in Id(A)$ then $\psi(\alpha.\beta) \in C(A)$.

Now we fix a partition λ and consider $g_\lambda = g_\lambda^1 + g_\lambda^2 + \cdots + g_\lambda^{d_\lambda} \in I_\lambda$ such that $g_\lambda^k \in FS_n e_k$, for all $k = 1, \cdots, d_\lambda$. For a fixed $k \in \{1, \cdots, d_\lambda\}$, since $\{e_i S_{ik}\}_{i=1}^{d_\lambda}$ is a basis for $FS_n e_k$ it follows that there exist α_i's in F such that $g_\lambda^k = \sum_{i=1}^{d_\lambda} \alpha_i e_i S_{ik}$. On the other hand for all $i = 1, \cdots, d_\lambda$ we have

$$e_i S_{ik} \left(\frac{1}{q} e_k S_{kk}\right) = \frac{1}{q} e_i e_i S_{ik} S_{kk} = \frac{1}{q} e_i^2 S_{ik} = \frac{1}{q} q e_i S_{ik} = e_i S_{ik}$$

that implies $g_\lambda^k(\frac{1}{q} e_k S_{kk}) = g_\lambda^k$. Moreover if $k > r$ then

$$e_i S_{ik} \left(\frac{1}{q} e_r S_{rr}\right) = \frac{1}{q} S_{ik} e_k e_r S_{rr} = 0.$$

In this way we have proved 2.(a) and 2.(b). To see item 2.(c).ii, it is enough to note that $g_\mu \in I_\mu$ and $\theta_k^\lambda \in I_\lambda$ so that $I_\mu I_\lambda = \{0\}$ if μ and λ are different partitions of n. The item 2.(c).i. can be proved by induction on k and we are done. □

The Theorem 2.1 and last lemma show that we can write each element g of degree n of A as a sum

$$g = \bigoplus_{\lambda \vdash n} g_\lambda \tag{2}$$

where $g_\lambda = g_\lambda^1 + g_\lambda^2 + \cdots + g_\lambda^{d_\lambda} \in I_\lambda$ and $g_\lambda^k \in FS_n e_k$ is also an element of degree n of A, for each partition λ of n. In fact, from item 1 of Lemma 2.4, we have that $g.(\theta_1^\lambda + \theta_2^\lambda + \cdots + \theta_{d_\lambda}^\lambda)$ is an element of degree n of A. By using item 2.(c), we have

$$g.(\theta_1^\lambda + \theta_2^\lambda + \cdots + \theta_{d_\lambda}^\lambda) = g_\lambda.(\theta_1^\lambda + \theta_2^\lambda + \cdots + \theta_{d_\lambda}^\lambda) = g_\lambda^1 + g_\lambda^2 + \cdots + g_\lambda^{d_\lambda} = g_\lambda.$$

Furthermore, since $g_\lambda = g_\lambda^1 + g_\lambda^2 + \cdots + g_\lambda^{d_\lambda}$ where d_λ is the number of standard Young tableaux of shape λ and $g_\lambda^k \in FS_n e_k$, it follows from items 1 and 2.(c) of the last lemma that g_λ^k is also an element of degree n of A.

Thus

$$g_\lambda \text{ is an element of degree } n \text{ of } A \Leftrightarrow$$
$$\psi(g_\lambda) = \psi(g_\lambda^1) + \psi(g_\lambda^2) + \cdots + \psi(g_\lambda^{d_\lambda}) \in \psi(I_\lambda) \cap C(A) \Leftrightarrow$$
$$\psi(g_\lambda^k) \in \psi(FS_n e_{T_\lambda}) \cap C(A), \text{ for all } k = 1, \cdots, d_\lambda \Leftrightarrow$$
$$g_\lambda^k \text{ is an element of degree } n \text{ of } A, \text{ for all } k = 1, \cdots, d_\lambda.$$

In this way, by fixing a partition λ of n and taking $d = d_\lambda$; for each $k = 1, \cdots, d$, we will use that $\{e_i S_{ik}\}_{i=1}^d$ is abasis of $FS_n e_k$, and construct just the elements of degree n of A. In particular, we are interested in elements of type

$$\alpha_1 e_1 S_{1k} + \alpha_2 e_2 S_{2k} + \cdots + \alpha_d e_d S_{dk}, \ \alpha_i \in F.$$

Note that for each partition $\lambda \vdash n$, we can consider T_1 the standard Young tableau which the columns, from the first to the last one, were filling from the top to the bottom in increasing order with $1, \cdots, n$ and we call it canonical tableau.

Now the isomorphism between $FS_n e_1$ and $FS_n e_k$ can be given by $\gamma \mapsto \gamma S_{1k}$ and so for an element of degree n of A

$$f = \alpha_1 e_1 S_{11} + \alpha_2 e_2 S_{21} + \cdots + \alpha_d e_d S_{d1}$$

in $FS_n e_1$ we have, from item 1 of Lemma 2.4, an element of degree n of A

$$f S_{1k} = \alpha_1 e_1 S_{1k} + \alpha_2 e_2 S_{2k} + \cdots + \alpha_d e_d S_{dk}$$

in $FS_n e_k$ and vice-versa.

Definition 2.5. *The elements of degree n which are linear combinations of $e_i S_{i1}$, with $i = 1, \cdots, d_\lambda$ are called elements of type $\mathcal{T}$ of A.*

We conclude that in order to determine the multilinear elements of degree n of A it is enough to consider, for each partition λ of n, elements of type $\mathcal{T}$ of A and so we have proved the next result.

Theorem 2.6. *The elements of degree n of A for a fixed partition $\lambda \vdash n$ are linear combinations over F of elements of type $\mathcal{T}$*

$$\sum_{i=1}^{d} \alpha_i e_i S_{i1},$$

where $\alpha_i \in F$, for all $i = 1, \cdots, d$ and d is the number of standard Young tableaux of shape λ.

3. Elements of type $\mathcal{T}$ in $M_2(F)$ and central polynomials of $M_2(E)$

We consider the T-prime algebra $M_2(E)$, where E is the Grassmann algebra of infinite dimension over the field F generated by $\{1, v_1, v_2, \cdots | v_i v_j = -v_j v_i\}$ and want to apply the method developed in the last section to determine elements of type $\mathcal{T}$ of a specifical degree n which are non-trivial central polynomials of $M_2(E)$. In order to do it, for each partition λ of n, we inicially construct the elements of type $\mathcal{T}$ in $C(M_2(F))$. Since the minimal degree of an identity of $M_2(E)$ is 8 (see [13]), it is natural to start the calculations with $n = 7$.

The arguments used in this work produce a systematic process and now we describe how we constructed computational routines (CR) which were implemented in the software GAP [5] in order to determine central polynomials of $M_2(F)$.

CR "CentralCoeffTest(n)" : We know that if $f = f(x_1, \cdots, x_n)$ is an element of degree n of A of type $\mathcal{T}$ then there exist $\alpha_1, \ldots, \alpha_d \in F$ such that

$$f = \sum_{k=1}^{d} \alpha_k S_{k1} e_1 = \sum_{k=1}^{d} \alpha_k F_k(x_1, \cdots, x_n).$$

On the other hand, for each $k = 1, \ldots, d$,

$$F_k = F_k(x_1, \cdots, x_n) = \sum_{p \in R_{T_1}} \sum_{q \in C_{T_1}} (\text{sgn } q) S_{k1} pq$$

and from the isomorphism ψ we have

$$F_k = \sum_{p \in R_{T_1}} \sum_{q \in C_{T_1}} (\text{sgn } q) x_{(S_{k1}pq)^{-1}(1)} x_{(S_{k1}pq)^{-1}(2)} \cdots x_{(S_{k1}pq)^{-1}(n)}$$

where T_1 is the canonical Young tableau of shape λ. We observe that to determine F_k it is enough to construct R_{T_1}, C_{T_1} and S_{k1}. Since we work with multilinear polynomials, in the substitutions it is enough to consider the elementary matrices E_{ij}, that is, whose entries are equal to 0 except the (i, j) entry which is 1.

Now we have to find $\alpha_k' s$ such that $[f, x_{n+1}] \equiv 0$ in $M_2(F)$, that is

$$\sum_{k=1}^{d} \alpha_k [F_k(x_1, \cdots, x_n), x_{n+1}] \equiv 0 \text{ in } M_2(F).$$

In this way, for any set $\{A_1, \cdots, A_{n+1}\}$ of elementary matrices of $M_2(F)$ we have

$$\sum_{k=1}^{d} \alpha_k [F_k(A_1, \cdots, A_n), A_{n+1}] = 0.$$

By considering for each k

$$[F_k(A_1, \cdots, A_n), A_{n+1}] = \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} \in M_2(\mathbb{Q}) \subseteq M_2(F)$$

and replacing it in the equality above, we get

$$\sum_{k=1}^{d} \alpha_k \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix} = 0 \;\Leftrightarrow\; \underbrace{\begin{pmatrix} x_1 & \cdots & x_d \\ y_1 & \cdots & y_d \\ z_1 & \cdots & z_d \\ w_1 & \cdots & w_d \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_d \end{pmatrix}}_{X} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

• We have 4^{n+1} possibilities to choose the elementary matrices $E_{11}, E_{12}, E_{21}, E_{22}$ to form a colection of $n + 1$ matrices.

• After testing all of the possibilities we can form a system $BX = 0$ where B is the reduced echelon form of the matrix formed by the aglutination of matrices C in each possibility, having 4^{n+2} rows and d columns.

• The next step is solving the system.

• A new routine **"ProbablePolynomials (B, n)"** describes the possible polynomials to be f. This routine uses other routine, called **"MonomialTest (Partition)"**, to determine the polynomials $F_1, F_2, \cdots, F_d$ which works in the following way:

∗ If the rank of B is less than d we have nonzero values for α_k's which will form a list of candidates different from zero.

∗ If the rank of B is d we have $\alpha_1 = \alpha_2 = \cdots = \alpha_d = 0$ and so $f = 0$. It means that is there is no element different from zero of degree n of $M_2(F)$ for the partition λ.

∗ Finally, if $B = 0$ then for each one of 4^{n+1} possibilities, the matrix

$$[F_k(A_1, A_2, \cdots, A_n), A_{n+1}] = \begin{pmatrix} x_k & y_k \\ z_k & w_k \end{pmatrix}$$

is null, for all $k = 1, 2, \cdots, d$. Thus for all $k = 1, 2, \cdots, d$, we have that

$$[F_k(A_1, A_2, \cdots, A_n), A_{n+1}] \equiv 0 \text{ in } M_2(F)$$

and so the polynomials $F_1, \cdots, F_d$ are elements of degree n of type $\mathcal{T}$ of $M_2(F)$.

The next result guarantees that the elements of type $\mathcal{T}$ of $M_2(E)$ comes from the elements of type $\mathcal{T}$ of $M_2(F)$.

Proposition 3.1. *Let A and B be F-algebras such that $B \subseteq A$ and consider $\lambda \vdash n$ a fixed partition. If $\psi(I_\lambda) \cap C(B) = span_F \{\psi(f_1), \cdots \psi(f_r)\}$ where $f_j's$ are of form $\sum_{i=1}^{d} \alpha_i e_i S_{ik_j} \in FS_n e_{k_j}$ with $k_j \in \{1, ..., d\}$ and $j \in \{1, ..., r\}$ then $\psi(I_\lambda) \cap C(A) = span_F \{\psi(f_{i_1}), ..., \psi(f_{i_s})\}$ where $i_1, ..., i_s \in \{1, ..., r\}$.*

Proof. If $\psi(g) \in \psi(I_\lambda) \cap C(A)$ then for all $k = 1, ...d$ there exist $\psi(g_\lambda^k) \in \psi(FS_n e_k) \cap C(A) \subseteq \psi(FS_n e_k) \cap C(B)$ such that $\psi(g_\lambda) = \psi(g_\lambda^1) + ... + \psi(g_\lambda^d)$.

On the other hand, since $\psi(g) \in \psi(I_\lambda) \cap C(A) \subseteq \psi(I_\lambda) \cap C(B)$ there exist $\beta_j's \in F$ such that $\psi(g_\lambda) = \beta_1 \psi(f_1) + ... + \beta_r \psi(f_r)$. It implies

$$\psi(g_\lambda) = \psi(\beta_1 f_1) + ... + \psi(\beta_r f_r) = \psi(g_\lambda^1) + ... + \psi(g_\lambda^d).$$

Using that $\psi(\beta_j f_j) \in \psi(FS_n e_{k_j}) \cap C(B)$ by the uniqueness of the decomposition it follows that

$$\psi(g_\lambda^{k_j}) = \psi(\beta_j f_j) \text{ for all } j \in \{1, 2, ..., r\}$$

$$\psi(g_\lambda^k) = 0 \text{ if } k \notin \{k_1, ..., k_r\}.$$

So $\psi(f_j) \in C(A)$, $\forall j \in \{1, ..., r\}$ such that $\beta_j \neq 0$ and the result follows. □

Now we will see the results which are important to decide whether an element of degree n of type $\mathcal{T}$ of $M_2(F)$ is an element of degree n of type $\mathcal{T}$ of $M_2(E)$ or not.

We have the following remark from Vishne [13].

Remark 3.2. *Let $f \in P_n$. Then for all $k \geq 2$:*

$f \equiv 0$ in $M_k(E)$ if and only if for any choice of elementary matrices $A_i = E_{a_i b_i}$ and either $v_i^* = v_i$ or $v_i^* = 1$, the substitution $x_i \mapsto A_i v_i^*$ in f gives zero.

Next we have an important result to finish our algorithms.

Proposition 3.3. *Let $\{j_1, \cdots, j_q\} \subset \{1, \cdots, n\}$ with $j_1 < \cdots < j_q$ and $M = M(x_1, \cdots, x_n) = \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)}$ a monomial in P_n. If $r_1 < \cdots < r_q$ such that $\sigma(r_i) = j_{l_i}$ where $l_i \in \{1, \cdots, q\}$ for all $i \in \{1, \cdots, q\}$ then the* *substitution* $x_i \mapsto A_i v_i^*$ *in the monomial* M *where* A_i *is an unitary matrix and* $v_i^* = v_i$ *if* $i \in \{j_1, \cdots, j_q\}$ *and* $v_i^* = 1$ *if* $i \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_q\}$ *we have*

$$M(A_1 v_1^*, \cdots, A_n v_n^*) = (sgn\ \tau_\sigma)\alpha_\sigma A_{\sigma(1)} \ldots A_{\sigma(n)}(v_{j_1} \ldots v_{j_q})$$

where $\tau_\sigma \in S_q$ *with* $\tau_\sigma(1) = l_1, \cdots, \tau_\sigma(q) = l_q$.

Proof. In fact the substitution $x_i \mapsto A_i v_i^*$ in the monomial M where $v_i^* = v_i$ if $i \in \{j_1, ..., j_q\}$ and $v_i^* = 1$ if $i \in \{1, ..., n\} \setminus \{j_1, ..., j_q\}$ is the same as $x_{\sigma(t)} \mapsto A_{\sigma(t)} v_{\sigma(t)}^*$ for all $t \in \{1, ..., n\}$, where $v_{\sigma(t)}^* = v_{\sigma(t)}$ if $t \in \{r_1, ..., r_q\}$ and $v_{\sigma(t)}^* = 1$ if $t \in \{1, ..., n\} \setminus \{r_1, ..., r_q\}$. Thus

$$\begin{aligned} M(A_1 v_1^*, ..., A_n v_n^*) &= \alpha_\sigma A_{\sigma(1)} v_{\sigma(1)}^* \ldots A_{\sigma(n)} v_{\sigma(n)}^* \\ &= \alpha_\sigma A_{\sigma(1)} \ldots A_{\sigma(n)}(v_{\sigma(r_1)} \ldots v_{\sigma(r_q)}) \\ &= \alpha_\sigma A_{\sigma(1)} \ldots A_{\sigma(n)}(v_{j_{\tau_\sigma(1)}} \ldots v_{j_{\tau_\sigma(q)}}) \\ &= (\text{sgn}\ \tau_\sigma)\alpha_\sigma A_{\sigma(1)} \ldots A_{\sigma(n)}(v_{j_1} \ldots v_{j_q}). \end{aligned}$$

□

Now let us explain how to construct elements in $C(M_2(E))$.

CR "CentralPolynomialTest(h)": verifies if a polynomial $h(x_1, \cdots, x_n)$ belongs to $C(M_2(E))$ and it works from some steps.

(1) Let

$$f = f(x_1, \cdots, x_n) = \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \ldots x_{\sigma(n)} \in P_n.$$

We have:

• According to Remark 3.2, $f \in Id(M_2(E))$ if and only if for any choice $\{A_1 v_1^*, \cdots, A_n v_n^*\}$ where A_i's are elementary matrices and $v_i^* = v_i$ or 1, we have

$$f(A_1 v_1^*, \cdots, A_n v_n^*) = 0.$$

• There exist $4^n.2^n$ ways to make that choice: 4^n possibilities to the sequence $\{A_1, \cdots, A_n\}$ and 2^n possibilities to the sequence $v_1^*, \cdots, v_n^*$.

• We identify

$$\{v_1^*, \cdots, v_n^*\} \leftrightarrow \{j_1, \cdots, j_q\} \subseteq \{1, \cdots, n\}, \text{ with } j_1 < \cdots < j_q$$
$$\text{where } v_i^* = v_i \text{ if } i \in \{j_1, \cdots, j_q\} \text{ and}$$
$$v_i^* = 1 \text{ if } i \in \{1, \cdots, n\} \setminus \{j_1, \cdots, j_q\}.$$

- For all $4^n.2^n$ choices $\{A_1v_1^*, \cdots, A_nv_n^*\}$ we have

$$f(A_1v_1^*, \cdots, A_nv_n^*) = 0$$

$$\Leftrightarrow$$

$$\sum_{\sigma\in S_n} (\text{sgn } \tau_\sigma)\alpha_\sigma A_{\sigma(1)} \dots A_{\sigma(n)}(v_{j_1} \dots v_{j_q}) = 0$$

$$\Leftrightarrow$$

$$\sum_{\sigma\in S_n} (\text{sgn } \tau_\sigma)\alpha_\sigma A_{\sigma(1)} \dots A_{\sigma(n)} = 0.$$

(2) For

$$h = h(x_1, \cdots, x_n) = \sum_{\sigma\in S_n} \alpha_\sigma x_{\sigma(1)} \dots x_{\sigma(n)} \in P_n$$

and each sequence $\{A_1, \cdots, A_{n+1}\}$ of elementary matrices and each subset $\{j_1, \cdots, j_q\} \subseteq \{1, \cdots, n+1\}$, the routine constructs a new polynomial
$f = f(x_1, \cdots, x_n) = [h, x_{n+1}] = \sum_{\gamma\in S_{n+1}} \alpha_\gamma x_{\gamma(1)} \dots x_{\gamma(n+1)} \in P_{n+1}.$

(3) For each monomial of f the routine constructs the permutation τ_γ as in the Proposition 3.3 and computes the product $A_{\gamma(1)} \dots A_{\gamma(n)}A_{\gamma(n+1)}$ to form the matrix

$$A = \sum_{\gamma\in S_{n+1}} (\text{sgn } \tau_\gamma)\alpha_\gamma A_{\gamma(1)} \dots A_{\gamma(n)}A_{\gamma(n+1)}.$$

(4) If $A = 0$ for all sequences $\{A_1, \cdots, A_{n+1}\}$ and all subsets $\{j_1, \cdots, j_q\} \subseteq \{1, \cdots, n+1\}$ then $f \in Id(M_2(E))$. As a consequence, $h \in C(M_2(E))$.

To finish, we use a new routine **"PolynomialIdentitiesTest(h)"** to determine if a polynomial $h \in C(M_2(E))$ is or not an identity, since our interest is to find non-trivial central polynomials. The idea used in this routine is analogous to that one used in the previous, where the difference consists in considering h instead of f.

The implementation of the routines above guaranteed the following.

Theorem 3.4. *If f is a non-trivial central polynomial of $M_2(E)$ then degree of $f \geq 9$.*

4. **Final remarks**

The procedure was designed to find central polynomials of arbitrary degree of the matrix algebra $M_k(E)$, for any $k \geq 2$. The tests done using the software GAP showed that the algebra $M_2(E)$ doesn't contain central polynomials of degree 7. Since Vishne informed the only elements of degree 8 of $C(M_2(E))$ are the identities, to construct central polynomials of bigger degree, we have to consider $n = 9$. The work in this case requires a large number of computations and our next goal is improving the algorithms to finally determine the minimal degree of a non-trivial element in $C(M_2(E))$. The sources of the computional routines and procedures, together with some examples, are available upon request by e-mail.

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