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Graded Lie algebras and q-commutative and r-associative parameters

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Abstract. We study graded Lie algebras whose transformation parameters are graded q-commutativive and r-associative. We study first some graded algebras over a field, with no zero divisors at the level of monomials in their graded algebra generators. These generators are q-commutative and r-associative. We address the cohomology of the q-function and r-functions, in particular we study quaternions and octonions. We then define algebras whose transformation parameters are q-commutative and r-associative. We address a generalization of a theorem by Scheunert on its relation to Lie (super)algebras. We show finally that for the cases studied by Scheunert there is always a real and faithful transformation parameter basis with the required q-commutativity. We use this basis to perform a transformation on the graded Lie algebra that relates it to a plain Lie (super)algebra while respecting the self-adjoint character of generators and preserving the group grading.
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1. Introduction

We want to define and study a generalization of Lie algebras to include transformation parameters with noncommutative and/or nonassociative properties. In order to do so, we study first a particular type of algebras called finite perfect algebras. These algebras together with the Grassmann algebras will provide the necessary ingredients to construct a wide variety of graded noncommutative nonassociative algebras called finite quasi–perfect algebras. We define then graded Lie algebras whose transformation parameters generate a finite quasi–perfect algebra. We address finally the possibility of relating such graded algebras with ordinary Lie (super)algebras, trying to generalize results by Scheunert[3] on epsilon or color Lie (super)algebras. We study finally the existence of such relations respecting self-adjointness.

2. Finite perfect and quasi-perfect algebras

Let G be a finite abelian group. Let A be a G-graded algebra over a commutative field K, which as a vector space can be generated by a basis set $\{v_a | a \in G\}$, with $v_a \neq 0 \ \forall a \in G$. We assume that A has no zero divisors at the level of monomials in such basis elements (i.e. no finite product of v_a's is zero). We will call an algebra A (which is not necessarily commutative and associative) a *finite perfect algebra* [5],[9]. From this definition it follows that the structure constants $C_{a,b} \in K$ attached to the basis $\{v_a | a \in G\}$, where

$$v_a \cdot v_b = C_{a,b}\, v_{a+b}, \tag{1}$$

are all nonzero (since no zero divisors) i.e.

$$C : G \times G \to K^*, \qquad (a,b) \mapsto C_{a,b} \neq 0. \tag{2}$$

In order to analyze noncommutativity and nonassociativity in this type of algebras we define the following parameters:

$$\begin{aligned} q_{a,b} &= C_{a,b}\,(C_{b,a})^{-1} && (3)\\ r_{a,b,c} &= C_{b,c}\,(C_{a+b,c})^{-1}\,C_{a,b+c}\,(C_{a,b})^{-1}, && (4) \end{aligned}$$

where the exponents "-1" denote the multiplicative inverses in K.

From (1-4) we obtain

$$\begin{aligned} v_a \cdot v_b &= q_{a,b}\, v_b \cdot v_a, && (5)\\ v_a \cdot (v_b \cdot v_c) &= r_{a,b,c}\,(v_a \cdot v_b) \cdot v_c. && (6) \end{aligned}$$

Accordingly, the q- and r-functions

$$q : G \times G \to K^*, \qquad (a,b) \mapsto q_{a,b} \neq 0, \tag{7}$$

$$r : G \times G \times G \to K^*, \qquad (a,b,c) \mapsto r_{a,b,c} \neq 0, \tag{8}$$

will characterize the noncommutative and nonassociative properties (of the finite perfect algebra) respectively. These particular types of noncommutativity and nonassociativity are called diagonal noncommutativity and diagonal nonassociativity respectively, since they involve just the exchange of factors in the former, and just the alteration of parentheses in the latter case.

A finite perfect algebra is called *unital or with unit "1"* if we can adopt a basis such that for $o \in G$, the neutral element, $v_o = 1$, and $1 \cdot v_a = v_a \cdot 1 = v_a$ $\forall a \in G$. This leads to:

$$q_{o,a} = q_{a,0} = 1, \quad r_{o,a,b} = r_{a,o,b} = r_{a,b,o} = 1, \ \forall a,b \in G. \tag{9}$$

If each basis element v_a produces a different set of q- and r-factors under exchange of factors or rearrangement of parenthesis as any other basis element, the algebra is called *faithful* [5], i.e.

$$\begin{aligned} &\forall a,b \in G, \ a \neq b, \ \exists\, c,u \in G \quad \text{such that} \\ &q_{a,c} \neq q_{b,c} \ \text{ or } \ q_{c,a} \neq q_{c,b} \ \text{ or} \\ &r_{a,c,u} \neq r_{b,c,u} \ \text{ or } \ r_{c,a,u} \neq r_{c,b,u} \ \text{ or } \ r_{c,u,a} \neq r_{c,u,b}. \end{aligned} \tag{10}$$

We will study some basic properties of the q- and r-functions first and then we will analyze them using group cohomology.

We consider constrains arising from quadratic monomial in the generators:

$$v_a \cdot v_b = q_{a,b}\, v_b \cdot v_a = q_{a,b}\, q_{b,a}\, v_a \cdot v_b. \tag{11}$$

Since A has no zero divisors at the level of monomials, we have that

$$q_{a,a} = 1, \quad \text{(for the case } b = a\text{);} \tag{12}$$

$$q_{a,b}\, q_{b,a} = 1, \quad \text{(for the case } b \neq a\text{).} \tag{13}$$

We consider now cubic monomial generators and rearrangements using exchange of factors (in the same parenthesis) or alternation of parentheses:

$$v_a \cdot (v_b \cdot v_c) = q_{a,b+c}\, (v_b \cdot v_c) \cdot v_a, \tag{14}$$

$$v_a \cdot (v_b \cdot v_c) = r_{a,b,c}\, q_{a,b}\, (r_{b,a,c})^{-1}\, q_{a,c}\, r_{b,c,a}\, (v_b \cdot v_c) \cdot v_a, \tag{15}$$

$$v_a \cdot (v_b \cdot v_c) = r_{a,b,c}\, q_{a+b,c}\, q_{a,b}\, r_{c,b,a}\, q_{c,b}\, (v_b \cdot v_c) \cdot v_a. \tag{16}$$

The order in which the rearrangements are done is the same as the order in which the q- and r-factors appear. From the previous identities, we obtain the constraint:

$$\begin{aligned} q_{b,c}\, q^{-1}_{a+b,c}\, q_{a,b+c}\, q^{-1}_{a,b} &= r_{a,b,c}\, r_{c,b,a} = \\ &= q_{a,c}\, q_{b,c}\, q^{-1}_{a+b,c}\, r_{a,b,c} (r_{b,a,c})^{-1} r_{b,c,a}. \end{aligned} \tag{17}$$

Observe that if all r-factors are trivial (r=1) then we get a concrete constraint on the q-function just asserting that the q-function of jumping over a product of two factors is the same as jumping the first and then jumping the second. A q-function of such characteristics was defined by Scheunert [3]. A function q is called a *"commutation factor on an abelian group G"* if the following conditions are fulfilled:

$$q(a,b)q(b,a) = 1, \tag{18}$$
$$q(a,b+c) = q(a,b)q(a,c), \tag{19}$$
$$q(a+b,c) = q(a,c)q(b,c), \quad \forall a,b,c \in G. \tag{20}$$

This includes the constraint (13) from quadratic monomials, but not (12). This will allow to include the q-factors characterizing exterior or Grassmann algebras. In the finite perfect algebra case, if q is a "commutation factor on G" then (17) becomes:

$$1 = r_{a,b,c}\, r_{c,b,a}, \tag{21}$$
$$1 = r_{a,b,c}\, r_{c,a,b}\, r_{b,c,a}. \tag{22}$$

We observe that these identities result from handling the constraints on q-factors and r-factors separately, so for that a q satisfying (19-20) is also called a *"separated" q-function*, and the corresponding r is called a *"separated" r-function* (thus separated r-functions fulfill (21-22)).

We could consider a weaker condition than "commutation factors on G" or "separatedness". We call a q-function a *2-cocycle* if

$$q_{b,c}\, q^{-1}_{a+b,c}\, q_{a,b+c}\, q^{-1}_{a,b} = 1. \tag{23}$$

This name will become clear soon. Clearly, every "commutation factor on G" or separated q-function is a 2-cocycle. The identity (17) for 2-cocycles becomes:

$$1 = r_{a,b,c}\, r_{c,b,a}, \tag{24}$$
$$1 = (q_{a,c}\, q_{b,c}\, q^{-1}_{a+b,c})\,(r_{a,b,c}\, r_{c,a,b}\, r_{b,c,a}). \tag{25}$$

We call $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. Scheunert [3] shows that a general "commutation factor on G" over real or complex numbers can be generated by factors of the form

$$q_2(a,b) = (-1)^{ab}, \text{ for } a,b \in \{0,1\} = \mathbb{Z}_2 \quad (\text{Super} - \text{grading}), \quad (26)$$

$$q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi \mathrm{i}}{N}(nm' - n'm)\right\}, \quad (27)$$

$$\text{for } N \geq 2, \ (n,m),(n',m') \in \mathbb{Z}_N \oplus \mathbb{Z}_N,$$

and replications of them using diverse $\mathbb{Z}_N$–factors of the decomposition of the finite abelian G into

$$G = \mathbb{Z}_{N_1} \bigoplus \cdots \bigoplus \mathbb{Z}_{N_s}. \quad (28)$$

For further properties of q-functions in regards to flexibility, Jordan admissibility, weak alternativity, right/left alternativity, alternative, Moufang associativity see [5].

We consider finally a constraint from monomial of generators of order four

$$(v_a \cdot v_b) \cdot (v_c \cdot v_d) \ = \ r_{a+b,c,d} \left((v_a \cdot v_b) \cdot v_c\right) \cdot v_d, \quad (29)$$

$$(v_a \cdot v_b) \cdot (v_c \cdot v_d) \ = \ (r_{a,b,c+d})^{-1} \, r_{b,c,d} \, r_{a,b+c,d} \, r_{a,b,c} \left((v_a \cdot v_b) \cdot v_c\right) \cdot v_d, \quad (30)$$

and to avoid zero divisors, we obtain

$$r_{b,c,d} \, (r_{a+b,c,d})^{-1} \, r_{a,b+c,d} \, (r_{a,b,c+d})^{-1} \, r_{a,b,c} = 1. \quad (31)$$

This identity resembles the pentagon identity satisfied by the associator in a ring or algebra [7], which have more general counterparts on the *associahedra* structures satisfied by the associator [8]. The identity (31) is remarkable since much like (3-4) which involve only q-factors, this involves only r-factors. We call an r-factor obeying (31) a *3-cocycle*. To understand this we will involve a transformation that converts products into sums, and adopting a trivial action of the group on the parameters,as follows.

Let f be a generic function

$$f : G \times \cdots \times G \to K^*. \quad (32)$$

We can use an inclusion map L from image(f) into the abelian group G' generated by image(f), which can be finite and can be written additively, and it is clearly a subgroup of the abelian multiplicative group K^*,

$$L : \text{image}(f) \to G' = \text{gen}(\text{image}(f)) \subset K^*. \quad (33)$$

In the case of $K^* = \mathbb{C}^*$ (the non-zero complex numbers), the additive notation in G' can be obtained by using a logarithm mod 2πi, by selecting a cut that avoids any of the logarithms of the elements in image(f). We can use the map L to convert f into a function $\hat{f}$ between abelian groups (with additive operation), where we can consider cohomological properties of such maps:

$$\hat{f} = L \circ f : G \times \cdots \times G \to G'. \tag{34}$$

In this way we define the functions $\hat{C}$, $\hat{q}$, and $\hat{r}$.

The coboundary (or coderivative) of the function $\hat{q}$ is given by [1][2]:

$$(\delta^{(2)}\hat{q})[a,b,c] \;=\; a\,\hat{q}(b,c) - \hat{q}(a+b,c) + \hat{q}(a,b+c) - \hat{q}(a,b). \tag{35}$$

Now, equation (23) in terms of $\hat{q}$ becomes

$$\hat{q}(b,c) - \hat{q}(a+b,c) + \hat{q}(a,b+c) - \hat{q}(a,b) = 0. \tag{36}$$

Hence, by assuming the trivial action of G on $\hat{G}$ (in the term $a\,\hat{q}(b,c)$ in (35)), the equation (36) implies that $\hat{q}$ has vanishing coboundary, and thus it explains the name of 2-cocylce for q-functions satisfying (23).

The question then arises on whether $q_{N\oplus N}$ in (27) has a trivial cohomology, i.e. if $\hat{q}_{N\oplus N}$ itself is the coboundary of a cochain $\hat{\phi}$ (leading automatically to its cocycle property and thus belonging to a trivial cohomology class). We obtain first $\hat{q}_{N\oplus N}$ concretely:

$$q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi \mathrm{i}}{N}\hat{q}_{N\oplus N}((n,m),(n',m'))\right\}, \tag{37}$$

$$\begin{aligned}&\hat{q}_{N\oplus N} : (\mathbb{Z}_N \oplus \mathbb{Z}_N)^2 \to \mathbb{Z}_N,\\ &\hat{q}_{N\oplus N}((n,m),(n',m')) = (nm' - n'm) \bmod N.\end{aligned} \tag{38}$$

Let us now assume that $\hat{q}_{N\oplus N}$ is a 1-coboundary, i.e. there exist a function:

$$\hat{\phi} : \mathbb{Z}_N \oplus \mathbb{Z}_N \to \mathbb{Z}_N, \tag{39}$$

such that

$$\hat{q}_{N\oplus N}(a,b) = (\delta^{(1)}\hat{\phi})[a,b] = a\,\hat{\phi}(b) - \hat{\phi}(a+b) + \hat{\phi}(a) = \hat{\phi}(b) - \hat{\phi}(a+b) + \hat{\phi}(a). \tag{40}$$

(again, the action of $\mathbb{Z}_N \oplus \mathbb{Z}_N$ on $\mathbb{Z}_N$ is trivial). Now, from $\hat{q}_{N\oplus N}(a,0) = \hat{q}_{N\oplus N}(0,a) = 0$ we verify $\hat{\phi}(0) = 0$. From $\hat{q}_{N\oplus N}(a,a) = 0$ we obtain $\hat{\phi}(2a) = 2\hat{\phi}(a)$. Proceeding in this manner, from $\hat{q}_{N\oplus N}(a,(n-1)a) = 0$ we obtain $\hat{\phi}(na) = n\hat{\phi}(a)$. Let $\hat{\phi}((1,0)) = k_1$ and and $\hat{\phi}((0,1)) = k_2$. From

$$\begin{aligned}&\hat{q}_{N\oplus N}((n,0),(0,m)) = nm\\ &= \hat{\phi}((0,m)) - \hat{\phi}((n,m)) + \hat{\phi}((n,0)) = m\,k_2 - \hat{\phi}((n,m)) + n\,k_1, \quad (41)\\ &\hat{q}_{N\oplus N}((0,m),(n,0)) = -nm\\ &= \hat{\phi}((n,0)) - \hat{\phi}((n,m)) + \hat{\phi}((0,m)) = n\,k_1 - \hat{\phi}((n,m)) + m\,k_2, \quad (42)\end{aligned}$$

it follows that $2nm = 0 \mod N$. This is a contradiction for $N > 2$. While for $N = 2$, the $\hat{q}_{2\oplus 2}$is actually the 1-coboundary of:

$$\hat{\phi}_{2\oplus 2}((n,m)) = n\,k_1 - nm + m\,k_2, \tag{43}$$

where $k_1,\ k_2 \in \mathbb{Z}/2\mathbb{Z}$ are arbitrary. This leads to the following proposition [6][9]:

Proposition 1.

$$\begin{aligned} &q_{N\oplus N} : (\mathbb{Z}_N \oplus \mathbb{Z}_N)^2 \to \mathbb{C}^*, \\ &q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi\mathrm{i}}{N}(nm' - n'm)\right\}. \end{aligned} \tag{44}$$

is a 2-cocycle. For $N > 2$ it has nontrivial cohomology (i.e. it is not a 1-coboundary). For $N = 2$, $\hat{q}_{2\oplus 2}$ is the 1-coboundary of the function in equation (43).

The quaternion algebra H is a faithful $\mathbb{Z}_2 \oplus \mathbb{Z}_2$–graded finite perfect algebra with unit [5] with:

$$q_H((n,m),(n',m')) : = \exp\{\pi\mathrm{i}(nm' - n'm)\}, \tag{45}$$

$$r_H((n,m),(n',m'),(n'',m'')) : = 1, \tag{46}$$

$$\forall (n,m),(n',m'),(n'',m'') \in \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Accordingly, its q-function has trivial cohomology class, and its r-function has trivial cohomology class as well.

The octonion algebra O is a $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$—graded faithful perfect algebra with unit [5] with:

$$\begin{aligned} &q_O((n,m,s),(n',m',s')) := \\ &= e^{\pi\mathrm{i}\{(nm'-n'm)+(ns'-n's)+(ms'-m's)+n'ms-nm's'+nm's-n'ms'+nms'-n'm's\}} \end{aligned} \tag{47}$$

$$\begin{aligned} &r_O((n,m,s),(n',m',s'),(n'',m'',s'')) := \\ &= e^{-\pi\mathrm{i}\{nm's''+nm''s'+n'ms''+n'm''s+n''ms'+n''m's\}} \end{aligned} \tag{48}$$

$$\forall (n,m,s),(n',m',s'),(n'',m'',s'') \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

By considering the function

$$\hat{\phi}_O((n,m,s)) = nm + ns + ms + nms, \tag{49}$$

we can check that $\hat{q}_O$ is the 1-coboundary of $\hat{\phi}_O$, and thus it is a 2-cocycle of trivial cohomology class.

A similar analysis can be done for the binary sedenion algebra [5] and further sedenion algebras. This will be presented elsewhere [12].

Let us discuss the cohomology of the r-function of a finite perfect algebra. Since r satisfies (31), it is a 3-cocycle. Now, the defining equation (4) establishes that $\hat{r}$ is the 2-coboundary of $\hat{C}$. Therefore, it has always trivial cohomology class. We obtain [9]:

Proposition 2. *All finite perfect algebras have r-functions with trivial cohomology class. In fact, the r-function is the 2-coboundary of the structure constant function C for the chosen basis $\{v_a|a \in G\}$.*

As a corollary we obtain [9]:

Corollary 1. *The quaternion and octonion algebras are faithful finite perfect algebras with unit whose q- and r-functions have trivial cohomology class.*

As we observe, these algebras over the reals leading to (normed, alternative and composition) division algebras turned out to be finite perfect algebras with trivial cohomology in their q- and r-functions. Hence, nontrivial noncommutative and nonassociative properties can follow from q- and r-functions with trivial cohomology class. In other words, there are algebras with nontrivial properties in regard to their lack of commutativity and/or associativity (such as quaternion and octonion algebras) which have nevertheless trivial cohomology associated with their q- and r-functions. The cohomological properties are going to be instrumental in the determination of novel perfect algebras with interesting properties. We are going to see below that finite perfect algebras with trivial or nontrivial cohomology in their q- and r-functions have trivial contributions as parameters of graded Lie algebras in the sense that they are equivalent to Lie (super)algebras.

Let A_1 and A_2 be graded algebras with bases $\{v_a|a \in G_1\}$ and $\{u_b|b \in G_2\}$ respectively. We call the *direct product algebra* $A_1 \times A_2$ which is generated by products $\{v_a \cdot u_b|(a,b) \in G_1 \oplus G_2\}$ where the generators of A_1 commute with those of A_2: $v_a \cdot u_b = u_b \cdot v_a, \forall (a,b) \in G_1 \oplus G_2$, and the generators of A_1 associate with those of A_2: $v_a \cdot (u_b \cdot u_d) = (v_a \cdot u_b) \cdot u_d$ and $v_a \cdot (v_c \cdot u_b) = (v_a \cdot v_c) \cdot u_b$ for all $a, c \in G_1$, and $b, d \in G_2$.

Let A_1 and A_2 be graded algebras with bases $\{v_a|a \in G\}$ and $\{u_b|b \in G\}$ respectively, and graded over the same finite abelian group G. We call $A_1 \odot A_2$ the *merged product algebra* which is generated by the products $\{v_a \cdot u_a|a \in G\}$, which generate a subalgebra of the product algebra $A_1 \times A_2$.

Observe that direct and merged products are defined for graded algebras with a base $\{v_a|a \in G\}$. That is, for algebra factors $A = \bigoplus_{g \in qG} A_g$ such that $\dim_K(A_g) = 1$.

It is immediate to prove that both the direct product as well as the merged product algebras with all its factors being finite perfect algebras are also finite perfect algebras. Furthermore, the q- and r-functions of the direct product $A_1 \times A_2$ with finite perfect algebra factors are given by:

$$q_{A_1\times A_2}((a,b),(c,d)) \;:=\; q_{A_1}(a,c)\, q_{A_2}(b,d), \qquad (50)$$

$$r_{A_1\times A_2}((a,b),(c,d),(f,g)) \;:=\; r_{A_1}(a,c,f)\, r_{A_2}(b,d,g). \qquad (51)$$

The q- and r-functions of the merged product $A_1 \odot A_2$ with finite perfect algebra factors are given by:

$$q_{A_1\odot A_2}(a,b) \;:=\; q_{A_1}(a,b)\, q_{A_2}(a,b), \qquad (52)$$

$$r_{A_1\odot A_2}(a,b,c) \;:=\; r_{A_1}(a,b,c)\, r_{A_2}(a,b,c). \qquad (53)$$

These products will provide enough tools to amalgamate diverse simpler finite perfect algebras into more entangled ones.

Finally we consider the Grassmann algebras. Let $\mathbf{A}_1$ be an associative $\mathbb{Z}_2$-graded algebra generated as a vector space by $\{1,\theta_1\}$, with 1 of degree $0 \in \mathbb{Z}_2$ or "even," and θ_1 with degree $1 \in \mathbb{Z}_2$ or "odd", and $\theta_1\theta_1 = -\theta_1\theta_1$. This leads clearly to zero divisors at the level of monomials in generators. This is equivalent to say that its q-function is given by (26). Observe that in this case the q-function is not defined by (3) as it is done for finite perfect algebras since here some structure constants vanish. The structure constants of the Grassmann algebra with base $\{v_0 = 1, v_1 = \theta_1\}$ are: $C_{0,0} = 1, C_{0,1} = C_{1,0} = 1, C_{1,1} = 0$. In this case the q-function is defined with the choice (26) that respects (5). Analogously, its r-function is not defined by (4), but we take $r = 1$, since the algebra is associative, in consonance with (6). As a vector space over K, the Grassmann algebra satisfies:

$$\mathbf{A}_1 = K\,1 \oplus K\,\theta_1. \qquad (54)$$

This concept is extended to Grassmann algebras $\mathbf{A}_M$ with several mutually associative and mutually anti-commutative $\theta_i\theta_j = -\theta_j\theta_i$ algebra generators $\{\theta_1,\theta_2,\cdots,\theta_M\}$. As a vector space over K, $\mathbf{A}_M$ has dimension 2^M.

We can consider now the direct product or the merged product between a finite perfect algebra A_1 and a Grassmann algebra $\mathbf{A}_M$. The algebras resulting from these type of products and further direct or merged products with finite perfect algebras are called *finite quasi-perfect algebras*. If all the algebras involved in the products to create the finite quasi-perfect algebra have unit (the Grassmann algebra do have unit 1), then it is called *unital or with unit*.

The remarkable feature of finite quasi-perfect algebras is that the corresponding q- and r-functions can be split into factor contributions (analogous to (50-51) or to (52-53)) in which one factor gives contributions from a finite perfect algebra and the other factor gives contributions to q-factors alone (since the Grassmann algebra is associative) from a (plain) Grassmann algebra.

The generators of a finite quasi-perfect algebra as a vector space over K can be expressed as a product $\mathbf{v}_a \cdot w_b$, where $\mathbf{v}_a$ is itself a product of generators of finite perfect algebras, and w_b is a basis element of a Grassmann algebra $\mathbf{A}_M$ (as vector space over K). This way of splitting each finite quasi-perfect algebra generator will be of fundamental importance since we will attempt to compensate all the contributions to the q- and r-factors originated from the finite perfect algebras leaving only the associative (plain) Grassmann algebra contributions.

3. Epsilon or color Lie (super)algebras and "discoloration"

Let G be a finite abelian group and q a commutation factor on G. A G-graded q-Lie algebra L (also called color, Epsilon [3] or (G,q)-graded Lie algebra [5]), is a G-graded vector space over K that has a binary operation $[\ ,\ ]$ satisfying:

$$L = \bigoplus_{a\in G} L_a, \qquad [L_a, L_b] \subset L_{a+b}, \tag{55}$$

$$[G_{ia}, G_{jb}] = -q(a,b)[G_{jb}, G_{ia}], \tag{56}$$

$$[G_{ia}, [G_{jb}, G_{kc}]] = [[G_{ia}, G_{jb}], G_{kc}] + q(a,b)[G_{jb}, [G_{ia}, G_{kc}]]. \tag{57}$$

where G_{ia}, G_{jb}, G_{kc} are any elements (for instance generators) in L of degree a, b, c respectively.

Scheunert [3] shows that there is a bijection between a general G-graded q-Lie algebra and the ordinary Lie (super)algebra $\hat{L}$:

$$L \longrightarrow \hat{L}, \quad \hat{L}_a = v_{-a} \otimes L_a, \tag{58}$$

$$G_{ia} \mapsto \hat{G}_{ia} := v_{-a} \otimes G_{ia}, \tag{59}$$

where the v_a are the corresponding generators of the finite perfect algebra that produces all the contributions to q but those of a Grassmann algebra. Under such a transformation we move the structure constants $\mathbf{C}$ of L to the structure constants $\hat{\mathbf{C}}$ of $\hat{L}$:

$$[G_{ia}, G_{jb}] = \mathbf{C}_{ia,jb}^{k(a+b)} G_{k(a+b)}, \tag{60}$$

$$[\hat{G}_{ia}, \hat{G}_{jb}] = [v_{-a} \otimes G_{ia}, v_{-b} \otimes G_{jb}] = \hat{\mathbf{C}}_{ia,jb}^{k(a+b)} \hat{G}_{k(a+b)}, \tag{61}$$

$$\hat{\mathbf{C}}_{ia,jb}^{k(a+b)} = C(-b,-a)\, \mathbf{C}_{ia,jb}^{k(a+b)}, \tag{62}$$

where $C_{a,b}$ are the structure constants associated with the basis $\{v_a | a \in G\}$ of the finite perfect algebra.

This process of moving from a color Lie (super)algebra to an ordinary Lie (super)algebra is sometimes called "discoloration". We want to solve two questions:

1. Can we define a generalized graded Lie algebra whose parameters are in a finite quasi-perfect algebra, and if so, can we generalize Scheunert's theorem so as to find a bijection to an ordinary Lie (super)algebra?

2. Can we provide a basis choice for associative finite perfect algebra with q-function being a "commutation factor on G", such that self-adjoint generators are transformed into self-adjoint operators after the transformation (58-59)?

4. Graded Lie algebras with noncommutative and nonassociative parameters

We introduce a generalization of the Epsilon or color Lie (super)algebras to define algebras whose parameters are in a finite quasi-perfect algebra A, such that as a vector space over K it can be generated by the products $\mathbf{v}_a \cdot w_b$, where $\mathbf{v}_a$ is itself a product of generators of finite perfect algebras, and w_b is a basis element of a Grassmann algebra $\mathbf{A}_M$ (as vector space over K). The finite perfect algebra generated by the factors $\mathbf{v}_a$ will be called A_1.

Definition: The triple $(G, q_A : r_A)$ corresponding to a unital G-graded quasi-perfect algebra A over K is called a *finite quasi-perfect group grading over* K. If A were actually a finite perfect algebra (i.e. when the Grassmann algebra $\mathbf{A}_M$ contribution is trivial, $\mathbf{A}_M = \mathbf{A}_0 = K$) then the triple is called a *finite perfect group grading over* K.

Definition: We call $\mathbf{L}$ a $(G, q_A : r_A)$–graded Lie algebra over K [4][5] if $(G, q_A : r_A)$- is a finite perfect or quasi-perfect group grading over K, $\mathbf{L} = \bigoplus_{a \in G} \mathbf{L}_a$ is a G-graded vector space over K such that following fine axioms are fulfilled.

Axiom 1: The grading group G is generated by the elements $a \in G$ such that $\mathbf{L}_a \neq \{0\}$.

Axiom 2: There is a closed binary G-graded product [,] in $\mathbf{L}$:

$$[\cdot,\cdot] : \mathbf{L} \times \mathbf{L} \to \mathbf{L};\ (Q_a, Q'_b) \mapsto [Q_a, Q'_b] \in \mathbf{L}_{a+b}, \tag{63}$$

$$[\mathbf{L}_a, \mathbf{L}_b] \subset \mathbf{L}_{a+b}. \tag{64}$$

Axiom 3: The product [,] is bilinear with respect to the addition operation in each vector space component $\mathbf{L}_a$. That is, for all $Q_a, Q'_b, Q''_b \in \mathbf{L};\ y \in K$:

$$[Q_a, Q'_b + yQ''_b] = [Q_a, Q'_b] + y[Q_a, Q''_b], \tag{65}$$

$$[Q'_b + yQ''_b, Q_a] = [Q'_b, Q_a] + y[Q''_b, Q_a]. \tag{66}$$

Axiom 4: The product [,] is q- *antisymmetric*, i.e. for all $Q_a, Q'_b \in \mathbf{L}$:

$$[Q_a, Q'_b] = -q_A(a,b)[Q'_b, Q_a]. \tag{67}$$

Axiom 5: The product [,] is *(q,r)-Jacobi associative*, i.e. $\forall\ Q_a, Q'_b, Q''_c \in \mathbf{L}$:

$$\begin{aligned}&[Q_a,[Q'_b,Q''_c]] = r_A(a,b,c)[[Q_a,Q'_b],Q''_c] + \\ &+ r_A(a,b,c)q_A(a,b)(r_A(b,a,c))^{-1}[Q'_b,[Q_a,Q''_c]].\end{aligned} \tag{68}$$

This structure will allow among others, the definition of Lie algebras whose parameters are quaternions or octonions. This clearly depend on a careful matching between the admissible grades of such algebras and the ones of the aimed Lie algebraic structure. It is well known that some (exceptional) Lie algebras have a close relationship with these nonassociative (finite perfect) algebras.

The connection between plain Lie algebras and $(G, q_A : r_A)$–graded Lie algebras over K is given by the following theorem, which generalizes results from [6]:

Theorem 1. *If* $\mathbf{L}$ *is a* $(G, q_A : r_A)$*–graded Lie algebra over* K *and* A *is a* G*-graded finite quasi-perfect algebra with basis* $\{\mathbf{w}_a | a \in G\}$ *leading to the group grading* $(G, q_A : r_A)$*, then the algebra* $(A \otimes \mathbf{L})_o$ *generated by the products* $\{\mathbf{w}_{-a} \otimes Q_a | a \in G\}$ *with the bilinear product:*

$$\begin{aligned}&[\mathbf{w}_{-a} \otimes Q_a, \mathbf{w}_{-b} \otimes Q'_b] = \\ &= r_A(-b,-a,a)(r_A(-b-a,a,b))^{-1}(\mathbf{w}_{-b} \cdot \mathbf{w}_{-a}) \otimes [Q_a, Q'_b],\end{aligned} \tag{69}$$

constitutes a plain Lie algebra over K*.*

Proof: In order to prove this we just need to verify that the product [,] defined in (69) is a closed binary operation in $(A \otimes \mathbf{L})_o$ and satisfies anticommutativity and the Jacobi-identity. Showing closure under this product is straightforward. Let $\{G_{ia} | a \in G,\ i = 1, .., \dim \mathbf{L}_a\}$ be a generator basis for $\mathbf{L}$. Hence $\{\mathbf{w}_{-a} \otimes G_{ia} | a \in G,\ i = 1, \ldots, \dim \mathbf{L}_a\}$ constitutes a vector space generator basis for $(A \otimes \mathbf{L})_o$.

Now, from $\mathbf{w}_{-b} \cdot \mathbf{w}_{-a} = q_A(-b,-a)\, \mathbf{w}_{-a} \cdot \mathbf{w}_{-b}$, $[G_{ia}, G_{jb}] =$
$= -q_A(a,b)[G_{jb}, G_{ia}]$, and using an identity obtained by reorganizing the

product $(\mathbf{w}_{-a}\cdot\mathbf{w}_a)\cdot(\mathbf{w}_{-b}\cdot\mathbf{w}_b)=(\mathbf{w}_{-b}\cdot\mathbf{w}_b)\cdot(\mathbf{w}_{-a}\cdot\mathbf{w}_a)$, namely

$$\begin{aligned} & r_A(-b,-a,a)(r_A(-b-a,a,b))^{-1}q_A(-b,-a)q_A(a,b) \\ &\quad = r_A(-a,-b,b)(r_A(-a-b,b,a))^{-1}. \end{aligned} \tag{70}$$

(One comment is in order. This identity holds since the contribution of the Grassmann algebra produces matching q-contributions by itself since its q is "separated". The further contributions of the unital finite perfect algebras can not produce zero divisors at the level of monomials, so the product of their contributed q and r-factors after a reorganization of factors and parentheses returning to the initial monomial should be 1.) Using these results we obtain:

$$\begin{aligned} & [\mathbf{w}_{-a}\otimes G_{ia},\mathbf{w}_{-b}\otimes G_{jb}] \\ & = r_A(-b,-a,a)(r_A(-b-a,a,b))^{-1}(\mathbf{w}_{-b}\cdot\mathbf{w}_{-a})\otimes[G_{ia},G_{jb}] \\ & = r_A(-b,-a,a)(r_A(-b-a,a,b))^{-1}q_A(-b,-a)(\mathbf{w}_{-a}\cdot\mathbf{w}_{-b}) \\ & \quad \otimes - q_A(a,b)[G_{jb},G_{ia}] \\ & = -[\mathbf{w}_{-b}\otimes G_{jb},\mathbf{w}_{-a}\otimes G_{ia}], \end{aligned} \tag{71}$$

where in the last equality we made use of the identity (70) and definition (69). Now, the validity of the Jacobi identity

$$\begin{aligned} & [\mathbf{w}_{-a}\otimes G_{ia},[\mathbf{w}_{-b}\otimes G_{jb},\mathbf{w}_{-c}\otimes G_{kc}]] \\ & \quad = [[\mathbf{w}_{-a}\otimes G_{ia},\mathbf{w}_{-b}\otimes G_{jb}],\mathbf{w}_{-c}\otimes G_{kc}] \\ & \quad\ + [\mathbf{w}_{-b}\otimes G_{jb},[\mathbf{w}_{-a}\otimes G_{ia},\mathbf{w}_{-c}\otimes G_{kc}]], \end{aligned} \tag{72}$$

follows similarly using reiteratively the product in (69) in each product in (72). Then we require an identity obtained from the monomial equation $(\mathbf{w}_{-a}\cdot\mathbf{w}_a)\cdot((\mathbf{w}_{-b}\cdot\mathbf{w}_b)\cdot(\mathbf{w}_{-c}\cdot\mathbf{w}_c))=((\mathbf{w}_{-a}\cdot\mathbf{w}_a)\cdot(\mathbf{w}_{-b}\cdot\mathbf{w}_b))\cdot(\mathbf{w}_{-c}\cdot\mathbf{w}_c)=(\mathbf{w}_{-b}\cdot\mathbf{w}_b)\cdot((\mathbf{w}_{-a}\cdot\mathbf{w}_a)\cdot(\mathbf{w}_{-c}\cdot\mathbf{w}_c))$, namely:

$$r_A(-c,-b,b)(r_A(-b-c,b,c))^{-1}r_A(-b-c,-a,a) \tag{73}$$

$$\begin{aligned} & (r_A(-a-b-c,a,b+c))^{-1} \\ & = r_A(-b,-a,a)(r_A(-a-b,a,b))^{-1}r_A(-c,-a-b,a+b)\cdot \\ & \qquad \cdot(r_A(-a-b-c,a+b,c))^{-1}r_A(-c,-b,-a)(r_A(a,b,c))^{-1} \\ & = r_A(-c,-a,a)(r_A(-a-c,a,c))^{-1}r_A(-a-c,-b,b)\cdot \\ & \qquad \cdot(r_A(-a-b-c,b,a+c))^{-1}(r_A(-c,-a,-b))^{-1}r_A(-c,-b,-a)\cdot \\ & \qquad \cdot r_A(b,a,c)(r_A(a,b,c))^{-1}q_A(-a,-b)q_A(b,a). \end{aligned} \tag{74}$$

The first equality in (74) is used to transform the r-factors resulting from the first term in the right-hand side of equation (72), and the last equality allows to transform the r- and q-factors resulting in the second term in the right-hand side of (72). After such transformations we obtain the

(q,r)-Jacobi associativity equation (68) multiplied in both sides by the same monomial in the transformation parameters. And this completes the proof.□

Observe that likewise as in the proof of Scheunert, the Lie algebra $(A \otimes \mathbf{L})_o$ inherits the G-grading from the original Lie algebra.

A very interesting byproduct of this exploration is the formulation of the question: **"which are the relations between the q- and r-functions in a finite perfect algebra that generate all possible relations obtained form arbitrary monomials?** We clearly know partial or constrained solutions to this questions, when the algebra is commutative, or associative or separated. But it is not known to the authors if there is a universal finite basis of identities from which all the others follow. Obviously, the functions q, and r given by equations (3) and (4) provide a solution to this question in the case of perfect algebras.

Now, as we are going to see, that the defined $(G, q_A : r_A)$–graded Lie algebra over K follows a similar fate as the color or epsilon (super)algebras. This generalize once again the result by Scheunert [3], which has already obtained some generalization in [10] [11], for some novel sorts of noncommutativity, but without involving nonassociativity.

Theorem 2. *There is a bijection between a each $(G, q_A : r_A)$–graded Lie algebra over K and a Lie (super)algebra whose parameters are in the Grassmann algebra contributing to the finite quasi-perfect algebra A. Let $\{\mathbf{v}_a | a \in G\}$ be a vector space basis of all the finite perfect algebra contributions to the finite quasi-perfect algebra A. These $\mathbf{v}_a$'s will generate a finite perfect algebra A_1. This algebra A_1 will produce all q- factor contributions of q_A with the exception of those provided by the contributing Grassmann algebra, and it will produce all the contributions to the r_A-factors (since the Grassmann algebra is associative, it has trivial contributions to r_A). The linear map*

$$B : \mathbf{L} \to (A_1 \otimes \mathbf{L})_o, \tag{75}$$

$$G_{ia} \mapsto \hat{G}_{ia} := \mathbf{v}_{-a} \otimes G_{ia}. \tag{76}$$

will provide the desired bijection. The space $(A_1 \otimes \mathbf{L})_o$ together with the bilinear product defined by:

$$[\hat{G}_{ia}, \hat{G}_{jb}] = \hat{\mathbf{C}}^{k(a+b)}_{ia,jb} \hat{G}_{k(a+b)}, \tag{77}$$

$$\hat{\mathbf{C}}^{k(a+b)}_{ia,jb} = C_{A_1}(-b,-a)\, \mathbf{C}^{k(a+b)}_{ia,jb} r_{A_1}(-b,-a,a)(r_{A_1}(-b-a,a,b))^{-1}, \tag{78}$$

satisfies:

$$[\hat{G}_{ia}, \hat{G}_{jb}] = -q_{\mathbf{A}_M}(a,b)[\hat{G}_{jb}, \hat{G}_{ia}]. \tag{79}$$

$$[\hat{G}_{ia}, [\hat{G}_{jb}, \hat{G}_{kc}]] = [[\hat{G}_{ia}, \hat{G}_{jb}], \hat{G}_{kc}] + q_{\mathbf{A}_M}(a,b)[\hat{G}_{jb}, [\hat{G}_{ia}, \hat{G}_{kc}]], \tag{80}$$

where the function $q_{\mathbf{A}_M}$ casts only the contribution to the q-factors from the Grassmann algebra $\mathbf{A}_M$ factor of the finite quasi-perfect algebra A in the sense of identities (50-51) and (52-53) for the contributions of the factors in direct or merged products.

Proof: The verification of (79-80) follow similar steps as the previous theorem, just that here the q-factors from the finite perfect algebra factor A_1 and those contributed by the Grassmann algebra factor $\mathbf{A}_M$ in the direct or merged product algebra are going to be managed separately (since they arise separately according to identities (50-51) and (52-53)), while all the r-factors come only from A_1 (since $\mathbf{A}_M$ is associative). Let us verify that the $\hat{G}_{ia}$ satisfy (79), From the definition (78) we have:

$$\begin{aligned}\hat{\mathbf{C}}_{ia,jb}^{k(a+b)} &= C_{A_1}(-b,-a)\,\mathbf{C}_{ia,jb}^{k(a+b)} r_{A_1}(-b,-a,a)(r_{A_1}(-b-a,a,b))^{-1} \\ &= C_{A_1}(-b,-a)\,(-q_{A_1}(a,b)q_{\mathbf{A}_M}(a,b))\mathbf{C}_{jb,ia}^{k(a+b)} r_{A_1}(-b,-a,a) \\ &\qquad (r_{A_1}(-b-a,a,b))^{-1} = -q_{\mathbf{A}_M}(a,b)\hat{\mathbf{C}}_{jb,ia}^{k(a+b)},\end{aligned} \tag{81}$$

where in the last equality we used (78) to express $\mathbf{C}_{jb,ia}^{k(a+b)}$ in terms of $\hat{\mathbf{C}}_{jb,ia}^{k(a+b)}$ since all other factors are invertible nonzero q-factors , r-factors or nonzero structure constants $C_{A_1}(-a,-b)$ factors. Now, using equation (3) we find for the finite perfect part: $C_{A_1}(-b,-a)(C_{A_1}(-a,-b))^{-1} = q_{A_1}(-b,-a)$ which together with identity (70) provides the last line in (82).

The proof of (80) follows similar lines using the Jacobi identity in terms of the structure constants $\mathbf{C}_{ia,jb}^{k(a+b)}$. $\square$

This theorem just states that although the parameters in an $(G, q_A : r_A)$–graded Lie algebra are nonassociative and noncommutative, all but the noncommutativity coming from a Grassmann algebra $\mathbf{A}_M$ can be compensated by a suited change of variables, giving thus no real fundamental contribution to the Lie algebraic structure. The G-grading structure remains intact under the change of variables map B. The reason why the contribution of the Grassmann algebra $\mathbf{A}_M$ cannot be compensated comes from the fact that due to the nilpotence of the θ_i's some structure constants are zero, and thus do not lead to a bijection (see (77-78)), and thus the Grassmann algebra structure constants C cannot be used to compensate their q-factors contributions. For instance, equation (3) does not hold for Grassmann algebras.

Further directions for study of finite quasi-perfect algebras include looking at how one might generalize the Poincare-Birkhoff-Witt Theorem to explicitly determine the structure of the enveloping algebra (see [3], Section 4C). A related direction of inquiry would be to determine the status of Ado's theorem in this context (see [3], Section 7); i.e., whether a $(G, q_A : r_A)$–graded Lie algebra over K can always be represented faithfully as a subalgebra of endomorphisms of a finite dimensional $(G, q_A : r_A)$–graded vector space. Some matrix representations of such $(G, q_A : r_A)$–graded Lie algebras, together with particular graded matrix products were presented in [6].

5. **Transformations maintaining self-adjointness**

The only question that blocks the use of the transformations (58-59) or (75-76) is the question of the preservation of self-adjointness character under such transformations. We are going to tackle this question in the case of having associative transformation parameters with "commutation factor" given by (26-27) and replications of them involving diverse factors in the decomposition of G in (28). We will investigate if there is a basis choice of the corresponding finite perfect algebra that allows to have

$$\overline{\hat{G}_{ia}} = \hat{G}_{ia} \quad \text{whenever} \quad \overline{G_{ia}} = G_{ia}. \tag{82}$$

The adjoint operation should remain anti-involutive:

$$\overline{[\hat{G}_{ia}, \hat{G}_{jb}]} = [\overline{\hat{G}_{jb}}, \overline{\hat{G}_{ia}}] = [\hat{G}_{jb}, \hat{G}_{ia}], \tag{83}$$

and at the level of scalars in $K = \mathbb{C}$, the involution acts as complex conjugation. In order for that to be the case, we need

$$\overline{[\hat{G}_{ia}, \hat{G}_{jb}]} = [\hat{G}_{jb}, \hat{G}_{ia}] = (\hat{\mathbf{C}}^{k(a+b)}_{ia,jb})^* \hat{G}_{k(a+b)}. \tag{84}$$

where $(\cdot)^*$ denotes complex conjugation. We need thus

$$(\hat{\mathbf{C}}^{k(a+b)}_{ia,jb})^* = (C_{-b,-a})^* \, (\mathbf{C}^{k(a+b)}_{ia,jb})^* = C_{-a,-b} \, \mathbf{C}^{k(a+b)}_{jb,ia}. \tag{85}$$

Now, since the original adjointion was also anti-involutive, we just need to have

$$(C_{-b,-a})^* = C_{-a,-b}. \tag{86}$$

Accordingly, the array $C_{a,b}$ used by the transformation (58-59) needs to be hermitian, in order to transform self-adjoint generators into self-adjoint ones.

Since the transformation (78) in this case targets to eliminate the q-factors of the form (27) associated with finite perfect algebras, the question is thus: Is there a basis for the finite perfect algebra with q-factors as in (27) that has hermitian structure constants? We can put the question in a different manner. Let us assume that the basis $\{v_a | a \in G\}$ of the finite perfect algebra is real, i.e. $\overline{v_a} = v_a$, $\forall a \in G$. Then,

$$\overline{v_a \cdot v_b} = \overline{v_b} \cdot \overline{v_a} = (C_{a,b})^* \overline{v_{a+b}}. \tag{87}$$

¿From the reality of the basis it follows $(C_{a,b})^* = C_{b,a}$, which is just equation (86). We see that if we have a self-adjoint basis

$$\overline{v_a} = v_a, \quad \forall a \in \mathbb{Z}_N \oplus \mathbb{Z}_N, \tag{88}$$

then (82-83) are fulfilled. The question can be then rephrased: Can we define a self-adjoint basis for an finite perfect algebra with q-factor given by (27)?

Let $\epsilon_{(1,0)} = \overline{\epsilon_{(1,0)}}$, and $\epsilon_{(0,1)} = \overline{\epsilon_{(0,1)}}$ the self-adjoint basis to generate the algebra A_1. We define

$$\epsilon_{(n,m)} = \exp\left\{\frac{\pi \mathrm{i}}{N} g(n,m)\right\} (\epsilon_{(1,0)})^n (\epsilon_{(0,1)})^m. \tag{89}$$

Let us find the choice of $g(n,m)$ that makes all the $\epsilon_{(n,m)}$ self-adjoint:

$$\begin{aligned} \overline{\epsilon_{(n,m)}} &= \overline{\exp\left\{\frac{\pi \mathrm{i}}{N} g(n,m)\right\} (\epsilon_{(1,0)})^n (\epsilon_{(0,1)})^m} = \\ &= \exp\left\{-\frac{\pi \mathrm{i}}{N} g(n,m)\right\} (\epsilon_{(0,1)})^m (\epsilon_{(1,0)})^n = \\ &= \exp\left\{-\frac{\pi \mathrm{i}}{N} g(n,m)\right\} \exp\left\{-\frac{2\pi \mathrm{i}}{N} mn\right\} (\epsilon_{(1,0)})^n (\epsilon_{(0,1)})^m. \end{aligned} \tag{90}$$

where we used the q-factor in (27) to exchange the monomials. In order to have self-adjoitness we require $g(n,m) = -nm$. From this, we can find:

$$\epsilon_{(n,m)}\, \epsilon_{(n',m')} = \exp\left\{\frac{\pi \mathrm{i}}{N}(nm' - mn')\right\} \epsilon_{(n+n',m+m')}. \tag{91}$$

We clearly see that their structure constants are hermitean, and the algebra A_1 has a self-adjoint basis. Accordingly we can complement Scheunert's theorem by adding the following result following from the construction above:

Proposition 3. *Let q be a "commutation factor on G". The bijection (58-59) between a (color, epsilon or) G-graded q-Lie algebra and an ordinary Lie (super)algebra can be chosen so that the map maintains self-adjointness in the mapped algebra elements.*

References

[1] M. F. Atiyah and C. T. C. Wall, *Cohomology of Groups*, from J. W. S. Cassel and A Fröhlich, Algebraic Number Theory, Thompson Book Company, Washington D. C, (1967).

[2] A. Babakhanian, *Cohomological methods in group theory*, Marcel Dekker Inc., (1972).

[3] M. Scheunert, *Generalized Lie Algebras*, J. Math. Phys. **20**(4), 712-720, April (1979).

[4] L.A. Wills Toro, *Symmetry transformations with noncommutative and nonassociative parameters*, Internat. Jour. Theor. Phys. **36** No. 12, 2963-2996 (Dec. 1997).

[5] L.A. Wills–Toro, *Noncommutative and nonassociative gradings and their algebras.* M.Sc. Thesis, Universidad Nacional de Colombia Medellin, (1999).

[6] L.A. Wills-Toro, *Finite group graded Lie algebraic extensions and Trefoil symmetric relativity, Standard Model, Yang-Mills and gravity theories*, University of Hawaii Manoa, Ph.D. Thesis (2004).

[7] R.D. Schafer, *An Introduction to Nonassociative Algebras.* New York: Dover, p. 13, (1996).

[8] J. Stasheff, *H-spaces from a Homotopy Point of View*, Springer Lecture Notes in Mathematics 161, Springer-Verlag, New York, (1970).

[9] L.A. Wills–Toro, T. Craven, and J.D. Velez, *Cohomology of deformation parameters of diagonal noncommutative nonassociative graded algebras.* Acta Mathematica Academiae Paedagogiace Nyíregyháziensis, Vol. 24, No. 3, pp. 271-277 (2008).

[10] H. Pop, *A generalization of Scheunert's theorem on cocycle twisting of the Lie color algebras*, preprint q-alg 9703002 (1997).

[11] Y. Bahturin and S. Montgomery, *Pi-envelopes of the Lie superalgebras*, Proceedings of the AMS, Vol. 127, No. 10, 2829-2839 (1999).

[12] L.A. Wills–Toro, et al. In preparation.