Virtual Knot Groups and Combinatorial Knots

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Abstract. Kauffman [16] and Kim [17] defined the group of a virtual knot by extending, in a natural way, the Wirtinger presentation of the fundamental group of classical knot. In this paper we present the group of a virtual knot by using the concept of combinatorial knot, introduced by Toro [21]. We show the advantages of this approach, that provides natural algorithms. We present examples of combinatorial knots whose groups have properties that are false, or unknown, in the category of the classical knots.
Keywords: Knots, Virtual Knots, Combinatorial Knots, Knot Groups, Virtual Knot Groups.

1. Introduction

In this paper we study the group of a virtual knot by using the concept of combinatorial knot. The group of a virtual knot was introduced by Kauffman [16] as a generalization of the group of a classical knot. This definition relies on a virtual knot diagram, so we do not have easy algorithms and computations are cumbersome. One of the advantages of using combinatorial knots to approach virtual knots is that we are not required to use diagrams and the definitions and proofs are presented in algorithmic form, easy to implement using a symbolic computation software such as *Mathematica.* We used combinatorial knots, instead of Gauss codes, introduced by Kauffman [16], because the combinatorial knot theory gives us an easy way to extend definitions and invariants from the classical knot theory to virtual knot theory. The second author has used combinatorial knots

to obtain an algorithm to construct an n-butterfly of a classical knot, see [11] and [21]. In our research, we use combinatorial knots, not only as a computational tool, but also as theoretical one, for instance, using combinatorial knots we found an easy way to compute an invariant that provides a necessary condition for a virtual knot to be a classical knot, see [22].

The paper is organized as follows. In section 2 we present a brief review of virtual knots [16] and combinatorial knots [21] and we sketch the proof that the virtual knot category is equivalent to the combinatorial knot category. Using this fact, in the rest of the paper we use combinatorial knots to present all the definitions and results and we do not differentiate between virtual knots and combinatorial knots. Section 3 provides the definition of the group of a combinatorial knot by using a presentation that extends the Wirtinger presentation of a classical knot. Given the correspondence between combinatorial knots and virtual knots, we define the group of a virtual knot. This definition is the same as the one given by Kauffman. In Section 4 we present properties of combinatorial knot groups and give several relevant examples. We present Kim's result [17], in which he gave conditions to determine when G is the fundamental group of a virtual knot. In the case that G corresponds to the group of a virtual knot, we give an algorithm to construct a combinatorial knot from the group presentation. Our proof of this result and the algorithm we get, show the advantages and power of working with combinatorial knots.

In the paper we work only with combinatorial knots and virtual knots, but there exists the concept of combinatorial link that corresponds to virtual link, and all our work can be extended to the group of a virtual link. It is important to note that there are properties that are different in the case of virtual link groups and this topic is the subject of our current research.

2. **Virtual knots and combinatorial knots**

The concept of a virtual knot was introduced by Kauffman [16] as a generalization of classical knot diagram. A *virtual knot diagram* is an oriented 4-valent planar connected graph, whose crossings are classified according to Figure 1a. An example of a virtual knot diagram is shown in Figure 1b.

The moves of virtual knot diagrams, illustrated in Figure 2 are called *generalized Reidemeister moves.* Two virtual knot diagrams D and D' are *equivalent* if D can be transformed into D' by a finite number of generalized Reidemeister moves. A *virtual knot* is an equivalence class of virtual knot diagrams.

Goussarov, Polyak and Viro [8] showed that the entire theory of classical knots is contained in the theory of virtual knots. Kauffman [16] gave examples of non classical virtual knots. In this sense virtual knot theory is
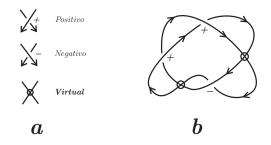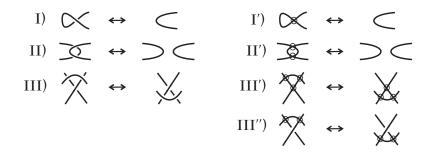
FIGURE 1. Cossing types.

FIGURE 2. I), II) and III) are the classical Reidemeister moves.

a nontrivial extension of the classical theory. Many invariants of classical knots have been extended to invariants of virtual knots, among them, the group of a classical knot.

Combinatorial knot theory was proposed by Toro [21] as a computational extension of knot theory. Figure 3 shows a diagram of a classical knot and its combinatorial knot.
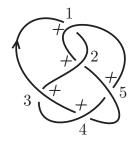

FIGURE 3. $((-1, 2, -3, 4, -5, 1, -2, 5, -4, 3), (1, 1, 1, 1, 1))$

Definition 1. *A knot code of n crossings is defined as*

$$((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m)),$$

where, as sets, $\{i_1, i_2, \cdots, i_{2n}\} = \{a_1, \cdots, a_n\} \cup \{-a_1, \cdots, -a_n\}$, $a_1, ..., a_n \in \mathbb{N}$, $m \geq \max\{a_1, a_2, \cdots, a_n\}$ *and* $e_j \in \{1, -1\}$, $j = 1, 2, \cdots, m$. *The sequences* $(i_1, i_2, \cdots, i_{2n})$ *and* $(e_1, \cdots, e_m)$ *are called crossing list and sign list, respectively.*

Two knot codes K_1 and K_2 are said to be equivalent if K_1 can be transformed into K_2 by a finite number of the following moves:

TYPE A: (Rotation)
$((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m)) \approx ((i_2, i_3, \cdots, i_{2n}, i_1), (e_1, \cdots, e_m)) \approx$
$\approx \cdots \approx ((i_{2n}, i_1, i_2, \cdots, i_{2n-1}), (e_1, \cdots, e_m)).$

TYPE B: (Renumbering)
$((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m)) \approx ((j_1, j_2, \cdots, j_{2n}), (\varepsilon_1, \cdots, \varepsilon_l))$
if there exist a bijective map $\varphi : \{i_1, \cdots, i_{2n}\} \to \{j_1, \cdots, j_{2n}\}$, such that, for every $k = 1, 2, \cdots, n$, $\varphi(-i_k) = -\varphi(i_k)$ and $\varepsilon_{|\varphi(i_k)|} = e_{|i_k|}$.

TYPE C: (Simplification)

$\quad ((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m, e_{m+1})) \approx ((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m)),$

if $m \geq |i_j|$ for all $j = 1, \cdots, 2n$.

TYPE I:
$((i_1, \cdots, i_{r-1}, i_r, -i_r, i_{r+2}, \cdots, i_{2n}), (e_1, \cdots, e_m)) \approx$
$((i_1, \cdots, i_{r-1}, i_{r+2}, \cdots i_{2n}), (e_1, \cdots, e_m)).$

TYPE II:
$((i_1, i_2, \cdots, i_{k-1}, h, j, i_{k+2}, \cdots, i_{t-1}, -h, -j, i_{t+2}, \cdots, i_{2n}), (e_1, \cdots, e_m))$
$\approx ((i_1, i_2, \cdots, i_{k-1}, i_{k+2}, \cdots, i_{t-1}, i_{t+2}, \cdots i_{2n}), (e_1, \cdots, e_m))$

or

$((i_1, i_2, \cdots, i_{k-1}, h, j, i_{k+2}, \cdots, i_{t-1}, -j, -h, i_{t+2}, \cdots, i_{2n}), (e_1, \cdots, e_m))$
$\approx ((i_1, i_2, \cdots, i_{k-1}, i_{k+2}, \cdots, i_{t-1}, i_{t+2}, \cdots i_{2n}), (e_1, \cdots, e_m))$

if and only if $j.h > 0$ and $e_{|j|} = -e_{|h|}$.

TYPE III:
$((i_1, \cdots, j, h, \cdots, p, q, \cdots, v, w, \cdots, i_{2n}), (e_1, \cdots, e_m)) \approx$
$\approx ((i_1, \cdots, h, j, \cdots, q, p, \cdots, w, v, \cdots, i_{2n}), (e_1, \cdots, e_m))$ if and only if
$j.h > 0, p \neq q, w \neq v$ and $\{p, q, v, w\} = \{-j, -h, k, -k\}$, for some crossing k.

The relations Type I, II and III are denominated *Reidemeister moves,* in a similar way to the classical case.

Note that in the type I and II moves we do not change the list of signs. In this way, we do not need to renumber the crosses and it could be information

about crossings that do not exist anymore. This is a special feature of combinatorial knots, that is extremely useful in computations. Note also that we have only a set of Reidemeister moves.

A combinatorial knot is defined as an equivalence class of knot codes under the relation $\approx$. A combinatorial knot is called *trivial or unknot* if it is equivalent to $((), ())$.

The category of combinatorial knots is equivalent to the category of virtual knots (and also to the category of Gauss diagrams and abstract knots), see [22], [15], [8] and [16].

We describe briefly a biyection between the set of virtual knot diagrams and the set of combinatorial knots. This correspondence extends to a biyective correspondence between the category of virtual knots and the category of combinatorial knots, see [22] for details.

Let K be a virtual knot diagram. We label its classical crossings with the numbers $1, 2, ..., n$, where n is the number of classical crossing of K. We take a point over K, not a crossing, and we follow the diagram writing down the list of crossing labels, with the convention that if we pass through an undercrossing i, we add $-i$ to the list, but if we pass through an overcrossing i, we add i to the list. The resulting collection is called *crossing list* of K. We construct also the list $(e_1, e_2, ..., e_n)$, where $e_i = 1$ if the crossing i is positive or $e_i = -1$ if it is negative. This list is called *list of signs* of K. The list formed by these two lists is a *knot code* that represents the virtual knot diagram K. It is denoted σ_K.

Now, we present a sketch of a construction of a virtual knot diagram from a knot code. Let $\sigma = ((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_m))$ be a knot code, where $\{i_1, i_2, \cdots, i_{2n}\} = \{a_1, \cdots, a_n\} \cup \{-a_1, \cdots, -a_n\}$. Let I_σ be a graph with vertices given by the set $\{a_1, \cdots, a_n\}$ and its edges given by the set $\{(i_1, i_2), ..., (i_n, i_1)\}$, where (i_k, i_{k+1}) connects the vertices a_i and a_j if and only if $|i_k| = a_i$ and $|i_{k+1}| = a_j$. We classify the vertices of I_σ according to Figure 4a.

Now, we consider the transformations shown in Figure 4b. If we take a plane realization of I_σ and apply these transformations to each vertex, we obtain a virtual knot diagram corresponding to the combinatorial knot σ. It is denoted K_σ.

As an example, for the combinatorial knot $\sigma = ((-1, -2, 3, 1, -3, 2), (-1, 1, 1))$, we get the diagram K_σ shown in Figure 5.

Definition 2. *A combinatorial knot σ is called admissible or classical if K_σ is equivalent to a diagram without virtual crossings.*

The combinatorial knot $((1, -2, 7, -4, 5, -1, 2, -5, 4, -7)), (-1, -1, 1, -1, 1, 1, -1))$ is classical, because it represents the classical knot of Figure 6.

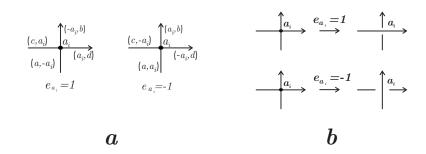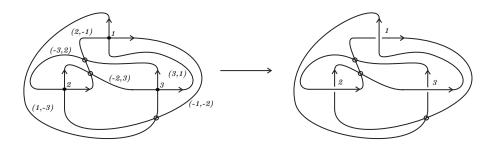FIGURE 4. Transforming vertices to classical crossing.

FIGURE 5. Diagram K_σ for $\sigma = ((-1, -2, 3, 1, -3, 2), (-1, 1, 1))$.

This combinatorial knot is equivalent to the combinatorial knot
$((-1, 2, -3, 1, -4, 5, -2, 3, -5, 4), (-1, -1, -1, -1, -1))$.

FIGURE 6. Clasical knot.

We will show in Example 1 that the combinatorial knot in Figure 5 is
not classical, using the group of a combinatorial knot.

3. Definition of Combinatorial Knot Group

In this section we introduce the concept of combinatorial knot group and by using the correspondence between virtual knots and combinatorial knots, we recover Kaufmann's definition of virtual knot group.

Let $\sigma = ((i_1, i_2, \cdots, i_{2n}), (e_1, e_2, \cdots, e_m))$ be a combinatorial knot and let $a_1, ..., a_n \in \mathbb{Z}^+$ be such that $\{i_1, i_2, \cdots, i_{2n}\} = \{-a_1, -a_2, ..., -a_n\} \cup \{a_1, a_2, ..., a_n\}$. Without lost of generality we suppose that

$$\sigma = ((-a_1, i_2, \cdots i_r, -a_2, i_{r+2}, ..., -a_n, i_{t+2}, ..., i_{2n}), (e_1, e_2, \cdots, e_m)),$$
$$(3.1)$$

and that between $-a_i$ and $-a_{i+1}$, $i = 1, ..., n-1$, there are no numbers or only positive numbers. We define the arcs of σ as the subsequences $S_{a_i} = (-a_i, \cdots, -a_{i+1})$, $i = 1, 2, ..., n-1$ and $S_{a_n} = (-a_n, i_{t+2} \cdots, i_{2n}, -a_1)$.

For example, for the combinatorial knot $\sigma = ((-1, 2, -3, -4, 1, -2, 4, 3), (1, -1, 1, 1))$ the arcs are $S_1 = (-1, 2, -3)$, $S_3 = (-3, -4)$, $S_4 = (-4, 1, -2)$ and $S_2 = (-2, 4, 3, -1)$.

Definition 3. *Let $\sigma = ((i_1, i_2, \cdots, i_{2n}), (e_1, e_2, \cdots, e_m))$ be a non trivial knot code as in (3.1) and let $S_{a_1}, ..., S_{a_n}$ to be its arcs. We define the group of σ as*

$$G(\sigma) = \langle S_{a_1}, S_{a_2}, \cdots, S_{a_n} : r_1, r_2, \cdots, r_n \rangle,$$

where $r_1 = S_{t_1}^{-e_{a_1}} S_{a_n} S_{t_1}^{e_{a_1}} S_{a_1}^{-1}$, $r_j = S_{t_j}^{-e_{a_j}} S_{a_{j-1}} S_{t_j}^{e_{a_j}} S_{a_j}^{-1}$, for $j = 2, ..., n$, and $a_j \in S_{t_j}$, $t_j \in \{a_1, ..., a_n\}$, $j = 1, 2, ..., n$.

We define the group of $((), ())$ as $\mathbb{Z}$.

It is straightforward to prove that the group is invariant under moves A, B, C and Reidemeister moves. This allows us to define the group of a combinatorial knot.

To simplify notation, in the rest of the paper we will take all combinatorial knots to be of the form

$$((i_1, i_2, \cdots, i_{2n}), (e_1, \cdots, e_n)),$$

with $\{i_1, i_2, \cdots, i_{2n}\} = \{1, \cdots, n\} \cup \{-1, \cdots, -n\}$.

Definition 4. *The group of a combinatorial knot K is defined by $G(K) = G(\sigma)$, where σ is any knot code in the class of K. For a virtual knot K we define the group of K as $G(K) = G(\sigma_K)$, where σ_K is a knot code that represents K.*

This definition of the group of a virtual knots corresponds to the definition given by Kauffman and Kim, see [16] and [17]. In the case of a classical

knot K, it corresponds to the fundamental group of the knot complement, $\pi\left(S^3 - K\right)$.

4. **Properties of Combinatorial Knot Groups**

In this section we establish some properties of the group of a combinatorial knot and we give several examples.

Lemma 1. *Let K be a combinatorial knot. If $G = G(K)$ then $G_{ab} \cong \mathbb{Z}$.*

Proof. We know that if K is a combinatorial knot then $G(K)$ has presentation

$$G(K) = \langle S_1, S_2, \cdots, S_n : r_1, r_2, \cdots, r_n \rangle,$$

where $r_j = S_{t_j}^{-e_j} S_{j-1} S_{t_j}^{e_j} S_j^{-1}$, $j = 2, ..., n$ and $r_1 = S_{t_1}^{-e_1} S_n S_{t_1}^{e_1} S_1^{-1}$. A presentation for G_{ab} is

$$\langle S_1, S_2, \cdots, S_n : r_1, r_2, \cdots, r_n, [S_i, S_j], i, j = 1, 2, ..., n \rangle,$$

where $[S_i, S_j] = S_i S_j S_i^{-1} S_j^{-1}$. Therefore,

$$
\begin{aligned}
G_{ab} &\cong \langle S_1, S_2, \cdots, S_n : r_1, r_2, \cdots, r_n, S_i S_j = S_j S_i, i, j = 1, 2, ..., n \rangle \\
&\cong \langle S_1, S_2, \cdots, S_n : S_1 = S_2, S_2 = S_3, ..., S_{n-1} = S_n \rangle \\
&\cong \langle S_i \rangle, i = 1, 2, ..., n.
\end{aligned}
$$

$\square$

For a classical combinatorial knot K, the group $G(K)$ has a known and important topological interpretation.

Theorem 1. *If K is a classical combinatorial knot and K^* is a knot in S^3 that is represented by K then $G(K)$ is the fundamental group $\pi_1(S^3 - K^*)$.*

In the rest of the paper we do not distinguish between a classical combinatorial knot and the corresponding knot in S^3.

The following result is central in classical knot theory, but it is not true in combinatorial knot theory, as Example 1 shows.

Theorem 2. [6]*If K is a classical combinatorial knot such that $G(K)$ is isomorphic to $\mathbb{Z}$, then K is trivial.*

Example 1. *For the combinatorial knot $L = ((-1, -2, 3, 1, -3, 2), (-1, 1, 1))$ we have*

$$G(L) = \left\langle S_1, S_2, S_3 : S_2 S_3 S_2^{-1} S_1^{-1}, S_3^{-1} S_1 S_3 S_2^{-1}, S_2^{-1} S_2 S_2 S_3^{-1} \right\rangle \cong \mathbb{Z}.$$

However, it is known that L is not equivalent to the trivial knot, see [10] and [23]. For the proof they use the polynomial invariant P, see also [22]. So, L is not a classical combinatorial knot.

Example 2. *Consider* $K = ((-1, 2, 3, -4, -3, 4, 1, -2), (1, 1, -1, -1))$. *The arcs of* K *are:* $S_1 = (-1, 2, 3, -4)$, $S_2 = (-2, -1)$, $S_3 = (-3, 4, 1, -2)$ *and* $S_4 = (-4, -3)$. *The relators for* $G(K)$ *are:*

$$r_1 : S_3^{-1} S_2 S_3 S_1^{-1}, \qquad r_2 : S_1^{-1} S_3 S_1 S_2^{-1},$$

$$r_3 : S_1 S_4 S_1^{-1} S_3^{-1}, \qquad r_4 : S_3 S_1 S_3^{-1} S_4^{-1},$$

therefore,

$$G(K) = \langle S_2, S_3 : S_2 S_3 S_2 = S_3 S_2 S_3 \rangle \cong \langle y, z : z^3 = y^2 \rangle.$$

As $G(K)$ *is not isomorphic to* $\mathbb{Z}$, *see* [1], *we obtain that* K *is not the trivial knot. We will probe in the next section that* K *is not a classical knot.*

4.1. The Over Presentation.

Definition 5. *Let* $\sigma = ((i_1, i_2, \cdots, i_{2n})), (e_1, e_2, \cdots, e_n)$ *be a non trivial knot code and let* $S_1, S_2, ..., S_n$ *be the arcs of* σ. *We denote by* $|S_i|$ *the cardinality of* S_i. *We say that* S_i *is a **bridge** of* σ *if* $|S_i| > 2$. *For a knot code* σ *the number of bridges is called the **bridge number**, and is denoted by* $br(\sigma)$. *For a combinatorial knot* K *its bridge number is defined by* $br(K) = \min\{br(\sigma) : [\sigma] = K\}$.

Remark 1.

(b) If K *is a classical combinatorial knot and* $br(K) = 1$, *then* K *is trivial.*

(c) In general, if $br(K) = 1$, *not necessary* K *is trivial. For example, for* $K = ((1, -2, -1, 2), (1, 1))$ *we have that* $br(K) = 1$ *but it is possible to prove that* K *is not trivial, see* [22].

Let K a combinatorial knot and $y_t = (-t, ..., -(t+1))$, $y_{t+1} = (-(t+p+1), ..., -(t+p+2))$ be the bridges of K. Without loss of generality we may assume that K is given by

$K = ((i_1, \cdots, -t, \cdots, -(t+1), \cdots, -(t+p), -(t+p+1), \cdots, -(t+p+$

$2), \cdots, i_{2n}), (e_1, \cdots, e_n)),$

where there is no crossing between $-(t+i)$ and $-(t+i+1)$, for $i = 1, 2, ..., p$. We construct the relator

$$r_t = w_t^{-1} y_t w_t y_{t+1}^{-1}, \qquad w_t = y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}} ... y_{a_p}^{e_{t+p}} y_{a_{p+1}}^{e_{t+p+1}}, \qquad (4.1)$$

where $y_{a_1}, y_{a_2}, ..., y_{a_{p+1}}$ are bridges of K such that $-(t+i) \in y_{a_i}$, $i = 1, 2, ..., p+1$.

The following theorem provides another presentation for $G(K)$ that is called *the over presentation*.

Theorem 3. *Let K be a non trivial combinatorial knot and let y_1, y_2,..., y_m be the bridges of K, then*

$$G(K) \cong \langle y_1, y_2, ..., y_m : r_1, ..., r_m \rangle,$$

where r_1,...,r_m are as in (4.1).

Proof. Let K be a combinatorial knot, $G(K) = \langle S_1, S_2, \cdots, S_n : r_1, r_2, \cdots, r_n \rangle$ the group of K, and $S_{i_1}, S_{i_2}, ..., S_{i_v}$ arcs of K such that $\left| S_{i_j} \right| = 2$, $j = 1, 2, ..., v$. To simplify notation, $x_j = S_{i_j}$, $j = 1, 2, ..., v$ and $y_u = S_{v+u}$, $u = 1, ..., m = n - v$ are the bridges of K. By Type A movements, we may assume that

$$K = ((i_1, \cdots, -t, \cdots, -(t+1), \cdots, -(t+p), -(t+p+1), \cdots, -(t+p+2), \cdots, i_{2n}), (e_1, ..., e_m)),$$

$y_t = (-t, \cdots, -(t+1))$, $x_i = (-(t+i), -(t+i+1))$, $i = 1, ..., p$, and $y_{t+1} = (-(t+p+1), ..., -(t+p+2))$. Let $y_{a_1}, y_{a_2}, ..., y_{a_{p+1}}$ be the bridges of K such that $-(t+i) \in y_{a_i}$, $i = 1, 2, ..., p+1$. The relators of $G(\sigma)$ for the crossings $-(t+1), -(t+2), ..., -(t+p), -(t+p+1)$ are:

$$r_{t+1} = y_{a_1}^{-e_{t+1}} y_t y_{a_1}^{e_{t+1}} x_1^{-1}, \quad r_{t+i} = y_{a_i}^{-e_{t+i}} x_{i-1} y_{a_i}^{e_{t+i}} x_i^{-1}, i = 2, ..., p, \tag{4.2}$$

$$r_{t+p+1} = y_{a_{p+1}}^{-e_{t+p+1}} x_p y_{a_{p+1}}^{e_{t+p+1}} y_{t+1}^{-1}.$$

From the relator r_{t+1} we get $x_1 = y_{a_1}^{-e_{t+1}} y_t S_{a_1}^{e_{t+1}}$. Replacing into the relator r_{t+2} we have

$$x_2 = y_{a_2}^{-e_{t+2}} y_{a_1}^{-e_{t+1}} y_t y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}},$$

from the relator r_{t+3} we get

$$x_3 = y_{a_3}^{-e_{t+3}} y_{a_2}^{-e_{t+2}} y_{a_1}^{-e_{t+1}} y_t y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}} y_{a_3}^{e_{t+3}}$$

then

$$x_i = y_{a_i}^{-e_{t+i}} ... y_{a_3}^{-e_{t+3}} y_{a_2}^{-e_{t+2}} y_{a_1}^{-e_{t+1}} y_t y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}} y_{a_3}^{e_{t+3}} ... y_{a_i}^{e_{t+i}}, i = 1, 2, ..., p. \tag{4.3}$$

Now, if we replace x_p in the relator r_{t+p+1} we obtain the relator

$$y_{t+1} = y_{a_{p+1}}^{-e_{t+p+1}} y_{a_p}^{-e_{t+p}} ... y_{a_3}^{-e_{t+3}} y_{a_2}^{-e_{t+2}} y_{a_1}^{-e_{t+1}} y_t y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}} y_{a_3}^{e_{t+3}} ... y_{a_p}^{e_{t+p}} y_{a_{p+1}}^{e_{t+p+1}}.$$

If $w_t = y_{a_1}^{e_{t+1}} y_{a_2}^{e_{t+2}} y_{a_3}^{e_{t+3}} ... y_{a_p}^{e_{t+p}} y_{a_{p+1}}^{e_{t+p+1}}$, then $y_{t+1} = w_t^{-1} y_t w_t$. Since $x_1, ..., x_p$ appear only in the relators $r_{t+1}, ..., r_{t+p}$ and r_{t+p+1} respectively, then we have

$$G \cong \langle y_1, ..., y_m : \mathbf{r}_1, ..., \mathbf{r}_m \rangle, \text{ where } \mathbf{r}_t = w_t^{-1} y_t w_t y_{t+1}^{-1} \text{ for } t = 1, ..., m.$$

$\square$

Example 3. *Let K be the combinatorial knot $((1, 3, -4, -3, 4, 2, -1, -2),$
$(1, 1, 1, 1))$. Then*

$$G(K) \cong \left\langle y_1, y_2 : y_1^{(y_2 y_1)} y_2^{-1}, y_2^{(y_1 y_2)} y_1^{-1} \right\rangle \cong \left\langle y, z : z^3 = y^2 \right\rangle \not\cong \mathbb{Z}.$$

Given a combinatorial knot K, we define another group, denoted $G_u(K)$, as

$$G_u(K) = G(K^*),$$

where, if $K = ((i_1, i_2, \cdots, i_{2n}), (e_1, e_2, \cdots, e_n),$

$$K^* = ((-i_1, -i_2, \cdots, -i_{2n}), (-e_1, -e_2, \cdots, -e_n)).$$

This group $G_u(K)$ is called the *under group* of K and we sometimes call the group $G(K)$ the *over group* of K.

For combinatorial knots these two groups may not coincide. For the combinatorial knot in Example 2, we have

$$G_u(K) \cong \mathbb{Z} \text{ and } G(K) = \left\langle y, z : z^3 = y^2 \right\rangle \not\cong \mathbb{Z}.$$

In the classical case, however they do coincide.

Lemma 2. *If K is a classical knot then $G(K) \cong G_u(K)$.*

Proof. Let K be a classical *knot, then $\pi_1(S^3 - K) \cong \pi_1(S^3 - K^*)$. Since $G(K)$ and $G(K^*)$ are presentations of the same group, then they are isomorphic. $\square$

This lemma allows as to conclude that the combinatorial knot of Example 2 is not a classical combinatorial knot.

4.2. **Peripheral system.** Let $K = ((i_1, i_2, \cdots, i_{2n}), (e_1, e_2, \cdots, e_n))$ be a combinatorial knot and let $G = G(K) = \langle S_1, S_2, \cdots, S_n : r_1, r_2, \cdots, r_n \rangle$ be its group.

Let $l = S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n}$, where $i \in S_{k_i}$, $i = 1, 2, ..., n$ and $p = e_1 + ... + e_n$. Define $l_\alpha = S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n} S_\alpha^{-p}$ for all $\alpha = 1, 2, ..., n$.

Lemma 3. *With the above notation, $l_\alpha \in [G, G] = G'$ for all $\alpha = 1, 2, ..., n$.*

Proof. Given that $G_{ab} \cong \mathbb{Z}$, by Lemma 1, we may see G_{ab} as the cyclic group generated by the lateral left class $S_\alpha G'$ and $S_i G' = S_\alpha G'$, for every $i \in \{1, 2, ..., n\}$, therefore

$$S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n} G' = (S_{k_1}^{e_1} G')(S_{k_2}^{e_2})...(S_{k_n}^{e_n} G') = S_\alpha^{e_1 + e_2 + ... + e_n} G',$$

then $l_\alpha = S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n} S_\alpha^{-p} \in G'$. $\square$

Since $S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n}$ and $S_{k_t}^{e_t} S_{t+1}^{e_{t+1}} ... S_{k_n}^{e_n} S_{k_1}^{e_1} ... S_{t-1}^{e_{t-1}}$ represent the same lateral left class in G_{ab}, we assume that $l_\alpha = S_{k_1}^{e_1} S_{k_2}^{e_2} \cdots S_{k_n}^{e_n} S_\alpha^{-p} =$ $S_{k_t}^{e_t} S_{k_{t+1}}^{e_{t+1}} ... S_{k_n}^{e_n} S_{k_1}^{e_1} ... S_{k_{t-1}}^{e_{t-1}} S_\alpha^{-p}$, $t = 1, 2, ..., n$.

Definition 6. *Let K be a combinatorial knot. For $\alpha = 1, 2, ..., n$, each arc S_α of K is called a meridian of K and l_α is called a longitude of K. A peripheral pair of a combinatorial knot K is the pair (m_α, l_α).*

In the classical knot theory the peripheral pair has an important geometric role, but in combinatorial knot theory we do not have any geometric interpretation yet.

Example 4. *Let $K = ((-1, 2, -3, 1, -2, 3), (-1, -1, -1))$ a longitude for K is $l = S_1^{-1} S_2^{-1} S_3^{-1} m^3$. Figure 7 shows how we can get such longitude in the classical case.*
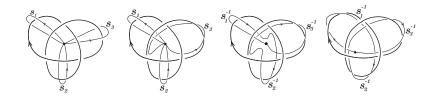

FIGURE 7. A longitude in a classical knot diagram.

We say that two peripheral pairs (m_i, l_i) and (m_j, l_j) are conjugate in $G(K)$ if there exists $w \in G(K)$ such that $m_i = w^{-1} m_j w$ and $l_i = w^{-1} l_j w$.

A *peripheral structure* of a combinatorial knot K is the conjugated class in $G(K)$ of a peripheral pair of K.

Proposition 1. *The peripheral structure of a combinatorial knot is unique up to conjugation.*

Proof. Let (S_i, l_i) and (S_j, l_j) be two peripheral pairs of a combinatorial knot K. We may assume that $j > i$. From the relators $r_{i+1}, ..., r_j$ we get that

$$S_j = S_{t_j}^{-e_j} S_{t_{j-1}}^{-e_{j-1}} ... S_{t_{i+1}}^{-e_{i+1}} S_i S_{t_{i+1}}^{e_{i+1}} ... S_{t_{j-1}}^{e_{j-1}} S_{t_j}^{e_j},$$

where $v \in S_{t_v}$, $v = i+1, ..., j$, thus, if $w_1 = w_1(i, j) = S_{t_{i+1}}^{e_{i+1}} ... S_{t_{j-1}}^{e_{j-1}} S_{t_j}^{e_j}$, then $S_j = w_1^{-1} S_i w_1$.

$$l_j = S_{t_{j+1}}^{e_{j+1}} ... S_{t_n}^{e_n} S_{t_1}^{e_1} ... S_{t_i}^{e_i} S_{t_{i+1}}^{e_{i+1}} ... S_{t_j}^{e_j} S_j^{-p} = w_2 w_1 S_j^{-p},$$

where $w_2 = S_{t_{j+1}}^{e_{j+1}}...S_{t_n}^{e_n}S_{t_1}^{e_1}...S_{t_i}^{e_i}$. Since $S_j^{-p} = w_1^{-1}S_i^{-p}w_1$, then $l_j = w_2 S_i^{-p}w_1 = w_1^{-1}w_1 w_2 S_i^{-p}w_1$. Now,

$$l_i = S_{t_{i+1}}^{e_{i+1}}...S_{t_{j-1}}^{e_{j-1}}S_{t_j}^{e_j}S_{t_{j+1}}^{e_{j+1}}...S_{t_n}^{e_n}S_{t_1}^{e_1}...S_{t_i}^{e_i}S_i^{-p} = w_1 w_2 S_i^{-p},$$

so $l_j = w_1^{-1}l_i w_1$, therefore (S_i, l_i) and (S_j, l_j) are conjugates in $G(K)$. $\square$

Let K be a combinatorial knot, $G(K)$ its group and $(m, l) = (S_i, l_i)$ the peripheral structure of K. The triplet $(G(K), m, l)$ is called peripheral system of K. We say that two peripheral systems $(G(K_1), m, l)$ and $(G(K_2), m', l')$ are isomorphic if and only if there exists an isomorphism $\varphi : G(K_1) \to G(K_2)$ such that $\varphi(m) = m'$ and $\varphi(l) = l'$. We have that the peripheral system is an invariant of combinatorial knots.

Waldhausen's theorem [24] implies that the equivalence class of classical knots is determined by the fundamental group and the peripheral system, see [9] for the proof.

Theorem 4. *(Waldhausen) Two classical combinatorial knots K_1 and K_2, with peripheral structures $(G(K_i), l_i, m_i)$, $i = 1, 2$, are equivalent if and only if there exists an isomorphism $\varphi : G(K_1) \to G(K_2)$ such that $\varphi(l_1) = l_2$ and $\varphi(m_1) = m_2$.*

Corollary 1. *Let K be a classical combinatorial knot with trivial longitude, then K is the trivial knot.*

Example 5. *For the combinatorial knot $K = ((-1, 2, 3, -4, -3, 4, 1, -2), (1, 1, -1, -1))$ we have, $G(K) \not\cong \mathbb{Z}$. The longitude l_1 satisfies*

$$l_1 = S_3 S_1 S_1^{-1} S_3^{-1} a_2^0 = e.$$

By Waldhausen's theorem K is not classical.

4.3. **Wirtinger Presentation.** Let G be a group. A Wirtinger presentation of G is a presentation of the form

$$G = \langle x_1, x_2, \cdots, x_p : r_1, r_2, \cdots, r_q \rangle,$$

where $r_k = x_j^{w_k}x_i^{-1}$, $1 \leq i, j \leq p$ and $w_1, \cdots, w_q$ are words in the free group $F(x_1, ..., x_p)$, not necessarily different in G.

A Wirtinger presentation is called cyclic if it is of the form

$$\langle x_1, x_2, \cdots, x_n : r_1, r_2, \cdots, r_m \rangle,$$

where $r_i = x_i^{w_i}x_{i+1}^{-1}$, $i = 1, 2, ..., m$, $w_1, \cdots, w_m \in F(x_1, ..., x_n)$, not necessarily different in G.

A Wirtinger presentation is called realizable if $w_i = x_{k_i}^{\varepsilon_i}$, for $i \in \{1, 2, ..., n\}$ and $\varepsilon_i \in \{1, -1\}$.

Note that the group of a combinatorial knot is given by a realizable Wirtinger presentation, see Definition 3.

Definition 7. *The deficiency of a finite group presentation is the number of generators minus the number of relators. The deficiency of a group G, $def(G)$, is defined as the maximum deficiency of all finite group presentation for G.*

If K is a combinatorial knot then $G(K)$ has a realizable Wirtinger presentation of deficiency 0 or 1.

The following result is central in the theory of virtual knots and its proof shows the advantage of working with combinatorial knots.

Theorem 5. *Let G be a group with realizable Wirtinger presentation of deficiency 0, then there exist a combinatorial knot K such that $G(K) \cong G$.*

Proof. Let G a group such that $G = \langle x_1, x_2, \cdots, x_n : r_1, r_2, \cdots, r_n \rangle$, with $r_i = x_{k_i}^{-e_i} x_i x_{k_i}^{e_i} x_{i+1}^{-1}$, $i = 1, 2, 3, ..., n$, $x_{n+1} = x_1$ and $e_i \in \{1, -1\}$. Let K the combinatorial knot given by $K = ((A_1, -1, A_2, -1, A_3, -3, ..., A_i, -i, A_{i+1}, ..., A_n, -n) (e_1, e_2, ..., e_n))$, where $\{A_1\}, ..., \{A_n\}$ are subset of $\{1, 2, ..., n\}$ defined by $\{A_i\} = \{j \in \{1, 2, ..., n\} : x_{k_j}^{e_j} = x_i^{\pm 1}\}$. We have that $\cup_{i=1}^{n} \{A_i\} = \{1, 2, ..., n\}$. Is possible that some of A_i are empty.

If $S_1 = (-n, A_1, -1)$, $S_2 = (-1, A_2, -2)$,..., $S_n = (-(n-1), A_n, -n)$, then $i \in S_{k_i}$ and

$$G(K) = \langle S_1, S_2, \cdots, S_n : \mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n \rangle,$$

where $\mathbf{r}_i = S_{k_i}^{-e_i} S_i S_{k_i}^{e_i} S_{i+1}^{-1}$, $i = 1, 2, ..., n$, $S_{n+1} = S_1$. So $G(K) \cong G$. $\square$

Example 6. *Let $G = \left\langle x_1, x_2, x_3 : x_1^{x_3} x_2^{-1}, x_2^{x_1^{-1}} x_3^{-1}, x_3^{x_2^{-1}} x_1^{-1} \right\rangle$, then: $r_1 = x_1^{x_3} x_2^{-1}$, $r_2 = x_2^{x_1^{-1}} x_3^{-1}$ and $r_3 = x_3^{x_2^{-1}} x_1^{-1}$, so*

$$A_1 = (2), \ A_2 = (3) \ and \ A_3 = (1).$$

therefore, if

$$K = ((-1, 2, -2, 3, -3, 1), (-1, 1, -1)),$$

then $G(K) = G$. Moreover, a peripheral structure of K is

$$(x_1, x_2^{-1} x_3 x_1^{-1} x_1) = (x_1, x_2^{-1} x_3).$$

Theorem 6. *Let G be a group with cyclic Wirtinger presentation of deficiency 0 or 1, then there exists a combinatorial knot K such that $G(K) \cong G$.*

Proof. Suppose that G has a presentation of the form $\langle x_1, x_2, \cdots, x_n : r_1,$ $r_2, \cdots, r_{n-1}\rangle$, where $r_i = w_i^{-1}x_iw_ix_{i+1}^{-1}$, $i = 1, 2, ..., n-1$. Let $w_n = (w_1w_2...w_{n-1})^{-1}$, then G is isomorphic to $\langle x_1, x_2, \cdots, x_n : r_1, r_2, \cdots, r_{n-1},$ $\mathbf{r}_n\rangle$, with $\mathbf{r}_n = w_n^{-1}x_nw_nx_1^{-1}$. So, without loss of generality we may assume that G has deficiency 0.

Suppose that $w_i = x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_1}...x_{i_k}^{\varepsilon_k}$, where $k = k(i)$, and $\varepsilon_1, ..., \varepsilon_k \in \{1, -1\}$, then

$$x_{i+1} = x_{i_k}^{-\varepsilon_k}...x_{i_2}^{-\varepsilon_1}x_{i_1}^{-\varepsilon_1}x_ix_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_1}...x_{i_k}^{\varepsilon_k}.$$

Let $y_{(i,0)} = x_i$, $y_{(i,1)} = x_{i_1}^{-\varepsilon_1}y_{(i,0)}x_{i_1}^{\varepsilon_1}, \cdots, y_{(i,k-1)} = x_{i_{k-1}}^{-\varepsilon_{k-1}}y_{(i,k-2)}x_{i_{k-1}}^{\varepsilon_{k-1}}$. By Tietze transformation, G is isomorphic to the presentation

$$\left\langle x_1, x_2, , x_n, y_{(1,0)}, \cdots, y_{(n,k(n))} : r_1, r_2, \cdots, r_n, R\right\rangle,$$

where $R = \{y_{(i,0)} = x_i, y_{(i,r)}^{-1}x_{i_r}^{-\varepsilon_r}y_{(i,r-1)}x_{i_r}^{\varepsilon_r}\}_{r=1,...,k(i)-1}^{i=1,...,n}$. thus,

$$G \cong \left\langle y_{(1,0)}, \cdots, y_{(i,0)}, \cdots, y_{(i,k(i)-1)}, \cdots, y_{(n,k(n))} : \mathbf{T}\right\rangle,$$

with,

$$\mathbf{T} = \left\{y_{(i,r)}^{-1}y_{(i_r,0)}^{-\varepsilon_r}y_{(i,r-1)}y_{(i_r,0)}^{\varepsilon_r}\right\}_{i=1,2,...,n}^{r=1,2,...,k(i)-1}.$$

which is a realizable Wirtinger presentation, by Theorem 5, there exists a combinatorial knot K such that $G(K) \cong G$. From Theorem 5, if G is a group with a realizable Wirtinger presentation of deficiency 0, then G is the group of a combinatorial knot. $\square$

Example 7. *Let* $G = \left\langle x_1, x_2, x_3 : x_1^{-1}x_3^{x_2^2}, x_2^{-1}x_1^{x_3^2}, x_3^{-1}x_2^{x_1^2}\right\rangle.$

Let $y_1 = x_3$, $y_2 = x_2^{-1}y_1x_2$, $y_3 = x_1$, $y_4 = x_3^{-1}y_3x_3$, $y_5 = x_2$, $y_6 = x_1^{-1}y_5x_1$. *Then* G *has the following presentation,*

$$\left\langle y_1, y_2, y_3, y_4, y_5, y_6 : y_3^{-1}y_2^{y_5}, y_5^{-1}y_4^{y_1}, y_1^{-1}y_6^{y_3}, y_2^{-1}y_1^{y_5}, y_4^{-1}y_3^{y_1}, y_6^{-1}y_5^{y_3}\right\rangle \cong$$

$$\cong \left\langle y_1, y_2, y_3, y_4, y_5, y_6 : y_1^{-1}y_6^{y_3}, y_2^{-1}y_1^{y_5}, y_3^{-1}y_2^{y_5}, y_4^{-1}y_3^{y_1}, y_5^{-1}y_4^{y_1}, y_6^{-1}y_5^{y_3}\right\rangle.$$

So, if $K = ((-1, 4, 5 - 2, -3, 1, 6 - 4, -5, 2, 3 \ -6), (1, 1, 1, 1, 1, 1))$ *then* $G(K) \cong G$.

Theorem 7. *If G is a group with Wirtinger presentation of deficiency 0 or 1 such that $G_{ab} \cong \mathbb{Z}$, then G is a combinatorial knot group.*

Proof. Let $G = \langle x_1, x_2, \cdots, x_p : r_1, r_2, \cdots, r_q\rangle$ *a Wirtinger presentation, where $q = p$ or $q = p - 1$. By doubling any relators, we may assume that $q = p$.*

Let $I(G)$ the graph such that its vertex are labeled with x_1, x_2, ..., x_p. x_ix_j is a edge of $I(G)$ if and only if there exist a relator in $\{r_1, r_2, ..., r_p\}$ of

the form $x_i^w x_j^{-1}$. Since $G_{ab} \cong \mathbb{Z}$, then all x_i are conjugated in G, so $I(G)$ is a connected graph. Consider the operation on the graph shown in Figure 8,
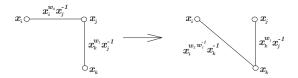

FIGURE 8. Operation on the graph $I(G)$

$I(G)$ can be transformed, using a finite number of the movements showed in Figure 8, in a cycle C_p such that $V(C_p) = V(I(G))$ and $E(C_p) = E(I(G))$, see [17]. Suppose that $C_p = x_{i_1} x_{i_2} ... x_{i_p} x_{i_1}$, if we denote $y_t = x_{i_t}$, $t = 1, 2, ..., p$, then $C_p = y_1 y_2 ... y_p y_1$ and the corresponding operations on the relators of G give a cyclic Wirtinger presentation. By Theorem 6 G is a combinatorial knot group. $\square$

Example 8. *The Baumslag-Solitar group* $G = \langle x, y : x^{-1} y^2 x = y^3 \rangle$ *is the group of a combinatorial knot. Since*

$$G \cong \langle x, y, y_1, y_2 : y = x^{-1} y^2 x y^{-2}, y_1 = x, y_2 = y^2 x y^{-2} \rangle ,$$

then $G \cong \langle y_1, y_2 : y_2 = y_1^{(y_1^{-1} y_2)^{-2}}, y_1 = y_2^{(y_1^{-1} y_2)^2} \rangle.$

If $x_1 = y_1$, $x_2 = y_2 x_1 y_2^{-1}$, $x_3 = y_1 x_2 y_1^{-1}$, $x_4 = y_2 x_3 y_2^{-1}$, $x_5 = y_2$, $x_6 = y_1^{-1} x_5 y_1$, $x_7 = y_2^{-1} x_6 y_2$ *and* $x_8 = y_1^{-1} x_7 y_1$, *then*

$G = \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 : x_8^{x_5} x_1^{-1}, x_1^{x_5^{-1}} x_2^{-1}, x_2^{x_1^{-1}} x_3^{-1}, x_3^{x_5^{-1}} x_4^{-1},$
$x_4^{x_1^{-1}} x_5^{-1}, x_5^{x_1} x_6^{-1}, x_6^{x_5} x_7^{-1}, x_6^{x_5} x_7^{-1} \rangle.$
So G *is the group of the combinatorial knot*
$K = ((-1, 3, 5, 6, 8, -2, -3, -4, -5, 1, 2, 4, 7, -6, -7, -8), (1, -1, -1, -1,$
$-1, 1, 1, 1)).$
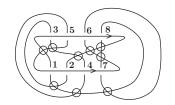
The corresponding virtual knot is given in Figure 9.

FIGURE 9. A virtual knot for the Baumslag-Solitar group

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