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Trivial extensions, iterated tilted algebras and cluster-tilted algebras

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1. Introduction

Classical results in representation theory establish very interesting connections between iterated tilted algebras and trivial extensions of finite dimensional algebras, particularly deep and useful in the Dynkin case. Recent results show that there are also interesting connections betweeen iterated tilted algebras and cluster tilted algebras. The aim of this article is to describe these connections, and show that, though the situations are quite different there are strong analogies between them.

The relation between the properties of the trivial extension of a finite dimensional algebra Λ and those of the algebra itself have been the object of study by many mathematicians. Early work in this direction was done by H. Tachikawa (1980) who proved that the hereditary algebra Λ is of finite representation type if and only if the trivial extension $T(\Lambda)$ of Λ is of finite representation type. On the other hand, K. Yamagata proved when Λ has oriented cycles then $T(\Lambda)$ is of infinite representation type (1981).

The connections with tilting theory are given by two theorems, due to Hughes and Waschbüsch (1983) and to Assem, Happel and Roldán

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(1984), respectively. The first proves that the trivial extension of Λ is of finite representation type and Cartan class Q if and only if there exists a tilted algebra Λ' of Dynkin type Q such that $T(\Lambda) \simeq T(\Lambda')$. The second establishes that $T(\Lambda)$ is of finite representation type and Cartan class Q if and only if there exists a tilted algebra Λ' of Dynkin type Q such that $T(\Lambda) \simeq T(\Lambda')$.

A combinatorial method to decide wether two schurian algebras Λ and Λ' have isomorphic trivial extensions was given by Fernández in [23] (1999) using the notion of admissible cut. A subset Δ of the set of arrows of a quiver Q is called an admissible cut if each oriented minimal (or chordless) cycle of Q contains exactly one arrow of Δ. A quotient by an admissible cut of an algebra Λ is defined as the quotient of Λ by the ideal generated in Λ by an admissible cut in the quiver of Λ. It is proven in [23] that two schurian algebras Λ and Λ' have isomorphic trivial extensions if and only if Λ' is the quotient of $T(\Lambda)$ by an admissible cut. Thus iterated tilted algebras of Dynkin type coincide with quotients of trivial extensions of finite representation type by admissible cuts.

Combining these results Fernández classified all trivial extensions of finite representation type, giving a simple method to decide if an algebra is iterated tilted of a given Dynkin type. Moreover, she obtained, under a unified approach, the classification results for iterated tilted algebras of types A_n, D_n and E_6 obtained by different auhors with other methods ([4], [10], [34], [43]).

In connection with cluster algebras, defined and studied by Fomin and Zelevinski in 2000, cluster categories were defined by Buan, Marsch, Reinecke, Reiten and Todorov and a tilting theory was developed for them [16]. To each hereditary algebra a cluster algebra can be associated, in such a way that cluster variables correspond to indecomposable rigid (or exceptional) objects and clusters to cluster tilting objects in the cluster category of H.

Since then, the theory has had an extraordinary developement in different directions, with interesting connections to several areas of mathematics. We are interested here in the relation between cluster tilted algebras, relation extensions and iterated tilted algebras. Cluster tilted algebras are defined as endomorphism rings of cluster tilting objects in the cluster category of a hereditary algebra H.

The connection of cluster tilted algebras with tilted algebras was studied by Assem, Brüstle and Schiffler, and is given using the notion of relation extension. Given an algebra Λ of global dimension at most two, the relation extension of Λ is the trivial extension $\mathcal{R}(\Lambda) = \Lambda \ltimes \mathrm{Ext}^2_\Lambda(D\Lambda, \Lambda)$. They prove that an algebra is cluster tilted if and only if it is ismorphic to the relation extension of a tilted algebra. This result resembles the first of the results connecting tilting theory and trivial extensions above mentioned, but it is in some sense more general, since no assumption about representation type is made.

As for trivial extensions, there is also a connection between cluster tilted algebras and iterated tilted algebras, but in this case only with those of global dimension at most two [1, 11, 31]. Given an iterated tilted algebra B, then $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$, where $\mathrm{D^b}(H)$ denotes the derived category of a hereditary algebra H and T is a tilting complex in $\mathrm{D^b}(H)$. When $\mathrm{gldim} B \leq 2$, then T defines a cluster tilting object in the cluster category $\mathcal{C}(H)$ and $C = \mathrm{End}_\mathcal{C}(T)$ is a cluster tilted algebra. Moreover, there exists a sequence of algebra homomorphisms $B \to C \xrightarrow{\pi} \mathcal{R}(B) \to B$ whose composition is the identity of B and $\mathrm{Ker}(\pi) \subseteq \mathrm{rad}^2 C$. In particular, C and $\mathcal{R}(B)$ have the same quivers.

In contrast with the situation for trivial extensions, it is not required here that B is of Dynkin type. However, it is not true in general that $C \simeq \mathcal{R}(B)$, not even in the Dynkin case.

Finally, we turn our attention to admissible cuts of cluster tilted algebras of finite representation type, where the following result, proven in [11], holds: An algebra B with $\mathrm{gldim}\, B \leq 2$ is iterated tilted of Dynkin type Q if and only if it is the quotient of a cluster-tilted algebra of type Q by an admissible cut.

These results can be applied to classify cluster tilted algebras of finite type. In fact, combining them Bordino, Fernández and Trepode classified those of type E_p, as communicated by Fernández in ICRA XII, 2008.

2. Trivial extensions and relation extensions

2.1. **Quivers and path algebras.** Given a quiver Q, we will denote by Q_0 the set of vertices, and by Q_1 the set of arrows of Q. For an arrow α, $s(\alpha)$ and $t(\alpha)$ denote the starting and terminating vertices of α, respectively. For a field k and a quiver Q, let kQ be the *path algebra* of Q, whose underlying k-vector space has the set of all paths as a basis and with multiplication induced linearly by the concatenation of paths, that is, if $\delta = \beta_s \cdots \beta_1$ and $\gamma = \alpha_r \cdots \alpha_1$ then $\delta\gamma = \beta_s \cdots \beta_1 \alpha_r \cdots \alpha_1$ if $s(\beta_1) = t(\alpha_r)$, and $\delta\gamma = 0$ otherwise. For each vertex $i \in Q_0$, let e_i be the associated trivial path of length 0. The *radical* is the ideal of kQ generated by all paths of positive length and will be denoted by $\operatorname{rad} kQ$.

We will assume in all that follows that k is an algebraically closed field. By an algebra we mean a finite dimensional k-algebra, which we also assume to be basic and indecomposable. Such an algebra Λ is isomorphic to the quotient of a path-algebra by an *admissible* ideal I, that is $\Lambda \simeq kQ/I$, where Q is a quiver, I is contained in $\operatorname{rad}^2 kQ$ and the quotient kQ/I is finite-dimensional. The pair (Q, I) is called a *presentation* for Λ. Given an element x in kQ, we will indicate also by x the corresponding element in Λ. By a *relation* ρ of kQ/I we mean an element of I which is a linear combination of paths starting at the same vertex $s(\rho)$ and stopping at the same vertex $t(\rho)$. For each $i \in Q_0$, S_i will denote the simple Λ-module associated to i, and P_i and I_i the projective cover and injective envelope of S_i, respectively.

All modules considered are finitely generated left modules, $\operatorname{mod}\Lambda$ denotes the category of finitely generated modules, and $\operatorname{ind}\Lambda$ is the full subcategory of $\operatorname{mod}\Lambda$ consisting of one copy of each indecomposable Λ-module. Moreover, if M is a Λ-module, we denote by $\operatorname{add}M$ the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the direct sums of summands of M.

A subquiver Q' of a quiver Q is called a *chordless* (or *minimal*) *cycle* if Q' is full, connected and in every vertex of Q' exactly two arrows of Q' incide (starting or stopping there). In case exactly one arrow stops and the other starts the cycle is called *oriented*.

2.2. **Trivial extensions.** Let Λ be an artin algebra and M a $\Lambda-\Lambda$-bimodule. We recall that the *trivial extension* $\Lambda \ltimes M$ of Λ by M is the algebra whose underlying k-vector space is $\Lambda \times M$ with multiplication $(\lambda, m) \cdot (\lambda', m') = (\lambda\lambda', \lambda m' + m\lambda')$. We will be interested in the following two cases.

When $M = D\Lambda$ is the dual of the algebra Λ, then $T(\Lambda) = \Lambda \ltimes D\Lambda$ is called the *trivial extension* of Λ. This is a symmetric algebra, and therefore it is selfinjective. Thus all indecomposable non-projective $T(\Lambda)$-modules have infinite projective dimension, so the homological properties of Λ and $T(\Lambda)$ are very different. However, there are deep connections between the representation theories of Λ and $T(\Lambda)$.

When $\operatorname{gldim}\Lambda \leq 2$, the trivial extension $\mathcal{R}(\Lambda) = \Lambda \ltimes \operatorname{Ext}^2_\Lambda(D\Lambda, \Lambda)$ is called the *relation extension* of Λ. This algebra was considered and studied by Assem, Brüstle and Schiffler in [2], where they prove that cluster tilted algebras are relation extensions of tilted algebras.

Trivial extensions are particular cases of split extensions. We recall that an algebra Γ is a *split extension of an algebra* Λ *by the ideal* M *of* Γ if there exist a split surjective algebra homomorphism $\pi : \Gamma \to \Lambda$ with kernel M, and the ideal M is nilpotent. In this case, $M \subseteq \operatorname{rad}\Gamma$, and there is an algebra homomorphism $\sigma : \Lambda \to \Gamma$ such that $\pi\sigma = id_\Lambda$. Relations for split extensions have been studied in [3]. In particular, if Λ is identified with a subalgebra of Γ through σ, a presentation for Γ can be chosen so that the arrows of Λ are arrows of Γ and the relations for Λ are also relations for Γ.

To describe the quiver of the trivial extension $\Lambda \ltimes M$ we will need the following facts, whose proof is straightforward.

Lemma 2.1. *Let* Λ *be an algebra and* M *a* $\Lambda - \Lambda$*-bimodule. Then*

(a) $\operatorname{rad}\Lambda \ltimes M = (\underline{\mathrm{r}}, M)$*, where* $\underline{\mathrm{r}}$ *denotes the radical of* Λ.

(b) $\operatorname{rad}^2 \Lambda \ltimes M = (\underline{\mathrm{r}}^2, \underline{\mathrm{r}}M + M\underline{\mathrm{r}})$

(c) $\operatorname{rad}\Lambda \ltimes M / \operatorname{rad}^2 \Lambda \ltimes M$ *and* $(\underline{\mathrm{r}}/\underline{\mathrm{r}}^2, M/(\underline{\mathrm{r}}M + M\underline{\mathrm{r}}))$ *are isomorphic vector spaces.*

Let $\Lambda^e = \Lambda \otimes \Lambda^{op}$. Then M is a Λ^e-module, with radical $\underline{\mathrm{r}}M + M\underline{\mathrm{r}}$ and $M/(\underline{\mathrm{r}}M + M\underline{\mathrm{r}}) = \operatorname{top}_{\Lambda^e} M$. Thus we get from (b) that the vertices of $Q_{\Lambda \ltimes M}$ are the vertices of Q_Λ. On the other hand, it follows from (c) that

$$\dim_k((ej, 0). \operatorname{rad}\Lambda \ltimes M / \operatorname{rad}^2 \Lambda \ltimes M.(e_i, 0) = \dim_k(e_j.\underline{\mathrm{r}}/\underline{\mathrm{r}}^2.e_i + e_j. \operatorname{top}_{\Lambda^e} M .e_i).$$

Thus the arrows from i to j of $Q_{\Lambda \ltimes M}$ are obtained by adding $\dim_k(e_j. \operatorname{top}_{\Lambda^e} M .e_i)$ arrows to the arrows from i to j of Λ.

To describe the quiver of the trivial extension we observe that $e_j. \operatorname{top}_{\Lambda^e} D\Lambda .e_i \simeq e_i. D(\operatorname{soc}_{\Lambda^e}\Lambda) .e_j$.

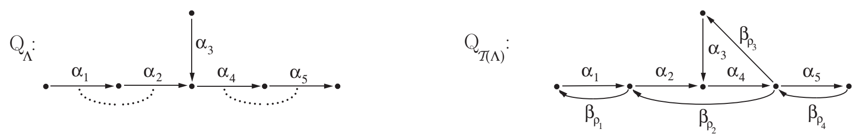
Proposition 2.2. [25, Proposition 2.2] *Let* Q_Λ *be the quiver of* Λ*, then the quiver of* $T(\Lambda)$ *is given by*

(a) $(Q_{T(\Lambda)})_0 = (Q_\Lambda)_0$

(b) $(Q_{T(\Lambda)})_1 = (Q_\Lambda)_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$, *where* $\{p_1, \ldots, p_t\}$ *is a* k*-basis of* $\mathrm{soc}_{\Lambda^e}\Lambda$ *consisting of linear combinations* p_i *of paths with the same origin* $s(p_i)$ *and the same end* $t(p_i)$, *and* β_{p_i} *is an arrow from* $t(p_i)$ *to* $s(p_i)$.

We say that a path γ in Λ is *maximal* if $\gamma \neq 0$ and $\gamma.\alpha = 0 = \alpha\gamma$ in Λ, for any arrow α of Q_Λ. We observe that maximal paths are in $\mathrm{soc}\Lambda$, so it follows from (b) in the the above proposition that all arrows in $Q_{T(\Lambda)}$ are in oriented cycles. Moreover, when Λ is *schurian*, that is, $\dim_k \mathrm{Hom}_\Lambda(P, P') \leq 1$ for every pair of indecomposable projective modules P, P', then $\mathrm{soc}\Lambda$ is generated by the maximal paths.

Example 2.3. For the algebra Λ we indicate the quiver of $T(\Lambda)$.



As usual, the dotted lines indicate the relations for Λ. In this case, $\{p_1 = \alpha_1,\ p_2 = \alpha_4\alpha_2,\ p_3 = \alpha_4\alpha_3,\ p_4 = \alpha_5\}$ is a basis of soc Λ consisting of maximal paths.

For algebras Λ whose quiver has no oriented cycles and such that gl.dim.$\Lambda \leq 2$ the ordinary quiver of the relation extension is described in [2]. Let Λ be such an algebra and let $M = \mathrm{Ext}^2_\Lambda(D\Lambda, \Lambda)$. Then $M/\underline{\mathrm{r}}M$
$\simeq \mathrm{Ext}^2_\Lambda(\mathrm{soc}D\Lambda, \Lambda/\underline{\mathrm{r}}\Lambda)$, and the canonical map $\Lambda \to \Lambda/\underline{\mathrm{r}}\Lambda$ and the inclusion soc $\Lambda \to \Lambda$ induce an epimorhism $M \to M/\underline{\mathrm{r}}M$. Moreover, $e_j.M/\underline{\mathrm{r}}M.e_i \simeq \mathrm{Ext}^2_\Lambda(S_i, S_j)$ (See [2, section 2]).

The dimension of $\mathrm{Ext}^2_\Lambda(S_i, S_j)$ has been described by Bongartz in [14, 1.2] in terms of the number of relations for Λ when the quiver of Λ has no oriented cycles, in the following way. Let $\mathcal{R}$ be a *system of relations for* Λ, that is, a minimal set of relations generating I as a two sided ideal of kQ_Λ. Then the cardinality of $\mathcal{R} \cap (e_j.I.e_i)$ are equal $\dim_k \mathrm{Ext}^2_\Lambda(S_i, S_j)$.

We describe next the quiver of $Q_{\mathcal{R}(\Lambda)}$.

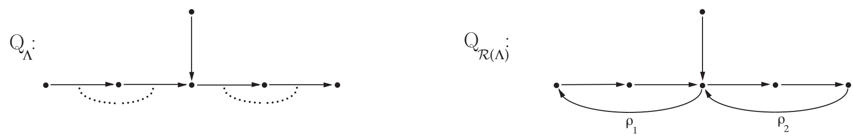
Proposition 2.4. [2, Theorem 2.6]. *Let $\Lambda = kQ_\Lambda/I$ be an algebra such that Q_Λ has no oriented cycles and* $\operatorname{gldim}\Lambda \leq 2$. *Let $\mathcal{R}$ be a system of relations for Λ. Then the quiver of $\mathcal{R}(\Lambda)$ is given by:*

(a) $(Q_{\mathcal{R}(\Lambda)})_0 = (Q_\Lambda)_0$

(b) *For $i, j \in Q_\Lambda$, the arrows from i to j in $(Q_{\mathcal{R}(\Lambda)})_1$ are the arrows from i to j in Q_Λ plus Card$(e_i.\mathcal{R}.e_j)$ additional arrows.*

This is, when Q_Λ has no oriented cycles, the vertices of the relation extension $\mathcal{R}(\Lambda)$ are the vertices of Λ, and the arrows of $\mathcal{R}(\Lambda)$ are obtained by adding to the arrows of Λ one arrow from i to j for each relation from j to i in a system of relations for Λ, for any i, j.

Example 2.5. Consider again the algebra Λ of Example 2.3



Here, the arrows ρ_1 and ρ_2 correspond respectively to the relations $\alpha_2\alpha_1 = 0$ and $\alpha_5\alpha_4 = 0$.

Now we turn our attention to the relations for these algebras, looking first at the trivial extension of Λ. The relations for $T(\Lambda)$ are described in [25] for algebras Λ whose oriented cycles are zero in Λ. In the particular case when $\operatorname{rad}^2(\Lambda) = 0$ they were described by Yamagata in [47]. When Λ is schurian and has no oriented cycles the relations for $T(\Lambda)$ were studied by Fernández in [23], and can also be obtained from the relations for the repetitive algebra, given by Asashiba in [6].

Recall that the algebra Λ is called *triangular* if the quiver of Λ has no oriented cycles.

For simplicity, we will assume that the algebra Λ is schurian, triangular, and such that parallel paths in Q_Λ are equal in Λ. Recall that every path γ in $T(\Lambda)$ is contained in an oriented chordless cycle $\mathcal{C}$. Following [25], we will say that the *supplement of γ in $\mathcal{C}$* is $e_{s(\gamma)}$ if all the arrows of $\mathcal{C}$ are arrows of γ, otherwise it is the path consisting of the remaining arrows of $\mathcal{C}$. We can describe now the relations for $T(\Lambda)$.

Proposition 2.6. [25, Corollary 3.2] *Let $\Lambda = kQ_\Lambda/I$ be a schurian triangular algebra such that parallel paths in Q_Λ are equal in Λ. Then $\Lambda = kQ_\Lambda/I_{T(\Lambda)}$ where the admisssible ideal $I_{T(\Lambda)}$ is generated by:*

(i) *the paths consisting of $n+1$ arrows in an oriented chordless cycle of length n,*
(ii) *the paths whose arrows do not belong to a single oriented chordless cycle, and*
(iii) *the difference $\gamma - \gamma'$ of paths γ, γ' with the same origin and the same endpoint and having a common supplement in oriented chordless cycles of $Q_{T(\Lambda)}$.*

We observe that the stated relations are formulated in terms of the cycles of $T(\Lambda)$. Thus $T(\Lambda)$ is determined by its quiver, provided the hypothesis of the above proposition hold.

Example 2.7. Let Λ be as in Example 2.3. Then the relations for $T(\Lambda)$ are:

All compositions of three arrows in the two cycles of lenth two, and of four arrows in the two cycles of length three, are zero.

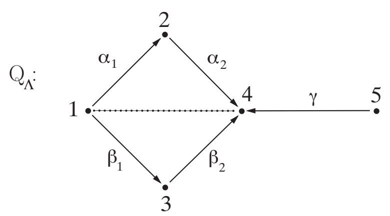
$\beta_{p_1}\beta_{p_2} = 0$, $\beta_{p_2}\beta_{p_4} = 0$, $\beta_{p_3}\beta_{p_4} = 0$, $\alpha_2\alpha_1 = 0$, $\alpha_5\alpha_4 = 0$

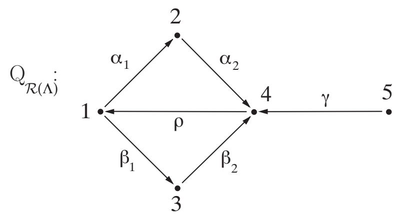
Finally, since the paths $\alpha_3\beta_{p_3}$, $\alpha_2\beta_{p_2}$ have the same suplement α_4 in the cycles $\alpha_4\alpha_3\beta_{p_3}$ and $\alpha_4\alpha_2\beta_{p_2}$ respectively, we get the commutativity relation:

$\alpha_3\beta_{p_3} = \alpha_2\beta_{p_2}$.

We now turn again our attention to the relation extension. The relations for $\mathcal{R}(\Lambda)$ are not known in general. However, they can be computed in many particular cases, as the following example illustrates.

Example 2.8.





We know $\mathcal{R}(\Lambda)$ is a split extension of Λ, thus the relations for Λ are also relations for $\mathcal{R}(\Lambda)$, as observed at the beginning of 2.2 . So we only have to determine if the paths involving ρ are zero. We first consider $\rho\gamma$. The indecomposable projective Λ-modules are:

$$P_1 = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}\ ,\ P_2 = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\ ,\ P_3 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\ ,\ P_4 = S_4\ ,\ P_5 = \begin{smallmatrix} 5 \\ 4 \end{smallmatrix}\ .$$

Then the exact sequence $0 \to P_4 \xrightarrow{\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right]} P_2 \oplus P_3 \xrightarrow{[\alpha_1, -\beta_1]} P_1 \to S_1 \to 0$ determines an element in $\mathrm{Ext}^2_\Lambda(S_1, S_4) = \mathrm{Ext}^2_\Lambda(I_1, P_4)$ corresponding to the relation $\rho = \alpha_2\alpha_1 - \beta_2\beta_1 = \left[\begin{smallmatrix}\alpha_1 & -\beta_1\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right] = 0$. Let V be the image of $\left[\begin{smallmatrix}\alpha_1 & -\beta_1\end{smallmatrix}\right]$. If we identify $\mathrm{Ext}^2_\Lambda(S_1, P_4)$ with $\mathrm{Ext}^1_\Lambda(V, P_4)$, we get that the relation $\rho\gamma$ corresponds to the pushout of the sequence $0 \to P_4 \xrightarrow{\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right]} P_2 \oplus P_3 \to V \to 0$ and the morphism $\gamma : P_4 \to P_5$. This pushout sequence splits if and only if the map γ factors through $\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right]$. This is not the case, because the only map from $P_2 \oplus P_3$ to P_5 is the zero map. Thus $\rho\gamma \neq 0$.

In a similar way, we get that $\rho\alpha_2 = 0$, because $\alpha_2 : P_4 \to P_2$ coincides with the composition $\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right]$. Also, $\rho\beta_2 = 0$, because $\beta_2 = \left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}\alpha_2\\ \beta_2\end{smallmatrix}\right]$.

We also get that $\alpha_1\rho = 0$ and $\beta_1\rho = 0$, using an injective copresentation of S_4.

Therefore a system of relations for $\mathcal{R}(\Lambda)$ is $\alpha_2\alpha_1 - \beta_2\beta_1 = 0$, $\rho\alpha_2 = 0$, $\rho\beta_2 = 0$, $\alpha_1\rho = 0$ and $\beta_1\rho = 0$.

An example is given in [2, Example 2.8] where the relations for the relation extension are calculated with a different method.

3. Iterated tilted algebras and trivial extensions

We recall that a module $M \in \operatorname{mod}\Lambda$ is a *tilting module* if M has projective dimension at most one, $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and M decomposes into precisely n pairwise non-isomorphic direct summands, where n is the number of pairwise non-isomorphic simple Λ-modules, or equivalently the number of vertices of the quiver of Λ.

If H is a hereditary algebra and M a tilting H-module then $\mathrm{End}^{\mathrm{op}}_H(M)$ is called a *tilted algebra*. Since the opposite of a tilted algebra is again a tilted algebra we will consider the endomorphism algebras themselves instead of their opposites. An algebra B is called an *iterated tilted algebra of type* Q if there exist algebras $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$ such that Λ_1 is hereditary with quiver Q, $\Lambda_t = B$ and for each $i = 1, \ldots, t-1$ we have $\Lambda_{i+1} \simeq \mathrm{End}_{\Lambda_i}(M_i)$ for some tilting Λ_i-module M_i.

The connection of tilted and iterated tilted algebras with trivial extensions is very interesting and was studied by many mathematicians. Results in this direction were first proven by Tachikawa and Wakamatsu in the hereditary case. We will be particularly interested in two theorems, one of them relates trivial extensions and tilted algebras and is due to Hughes and Waschbüsh. The other, due to Assem, Happel and Roldán, studies the relation of trivial extensions with iterated tilted algebras.

In order to state these results we recall the notion of *Cartan class* of a selfinjective algebra of finite representation type, introduced by Riedtmann in [39]. The stable Auslander-Reiten quiver of such an algebra Γ is isomorphic to $\mathbb{Z}Q/G$, where Q is a Dynkin diagram and G is a group of automorphisms of $\mathbb{Z}Q$, such that the action of G on $\mathbb{Z}Q$ is admissible. The Dynkin type of Q is uniquely determined and is called the Cartan class of Γ. (See [39])

We can state now the mentioned theorems, which will be important in the sequel.

Theorem 3.1. [32] *The following conditions are equivalent for a finite dimensional algebra* Λ*:*

(a) $T(\Lambda)$ *is of finite representation type and Cartan class* Q*.*
(b) *There exists a tilted algebra* Λ' *of Dynkin type* Q *such that* $T(\Lambda) \simeq T(\Lambda')$*.*

Theorem 3.2. [5] *The following conditions are equivalent for a finite dimensional algebra* Λ*:*

(a) $T(\Lambda)$ *is of finite representation type and Cartan class* Q*.*
(b) Λ *is iterated tilted of Dynkin type* Q*.*

Next we will show how these two theorems can be combined with the description of the quiver and the relations for $T(\Lambda)$, to decide both if Λ is iterated tilted of Dynkin type Q and if $T(\Lambda)$ is of finite representation type of Cartan class Q. In order to apply the first theorem it is important to know when $T(\Lambda) \simeq T(\Lambda')$, for two given algebras Λ and Λ'.

We observe first that the quiver of $T(\Lambda)$ is constructed from the quiver of Λ by adding some arrows, and there is one such arrow in each oriented chordless cycle of $T(\Lambda)$. Clearly, by deleting these arrows in $T(\Lambda)$ we obtain Λ. We would like to know what subsets of arrows of $T(\Lambda)$ can be deleted to obtain an algebra whose trivial extension is isomorphic to $T(\Lambda)$.

Definition 3.3. A subset of the set of arrows Δ of a quiver Q is called *admissible cut* of Q if it contains exactly one arrow of each oriented chordless cycle in Q.

Definition 3.4. Let $\Lambda = kQ_\Lambda/I$ be an algebra given by a quiver Q_Λ and an admissible ideal I. A *quotient of* Λ *by an admissible cut of* Λ is an algebra of the form $kQ_\Lambda/\langle I \cup \Delta\rangle$ where Δ is an admissible cut of Q_Λ.

The following theorem gives necessary and sufficient conditions for two trivial extensions to be isomorphic, under the assumption that one of them is schurian triangular.

Theorem 3.5. [23, Corolario 1.3.19] *Let* Λ *be a schurian triangular algebra. Then an algebra* Λ' *is an an admissible cut of* $T(\Lambda)$ *if and only if* $T(\Lambda) \simeq T(\Lambda')$*.*

The proof uses elementary arguments. But the situation turns more complicated in the general case, without the assumption that the algebra Λ is schurian and triangular. A generalization of the above theorem has been obtained in [24] for algebras whose oriented cycles are zero. However, the arguments used in the schurian triangular case do not apply in this case, and it is necessary to use the following result, due to Wakamatsu:

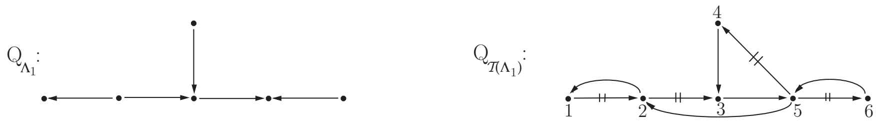
Theorem 3.6. [46]*. Let* Λ *and* Λ' *be artin algebras. Then* $T(\Lambda) \simeq T(\Lambda')$ *if and only if there exist an artin algebra* S *and an* $S-S$*-bimodule* M *such that* $\Lambda \simeq S \ltimes M$ *and* $\Lambda' \simeq S \ltimes DM$*.*

Given an admissible cut Δ of an algebra without non-zero oriented cycles the algebra S and the bimodule M can be described in terms of Δ, and this is the tool used in [24] to prove one of the implications of the theorem.

The following example illustrates how these techniques can be combined to decide if an algebra is iterated tilted of Dynkin type.

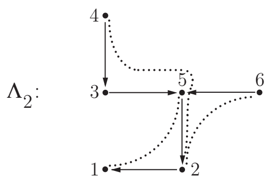
Example 3.7. Consider the algebra Λ of Example 2.3, and its trivial extension.

By 3.2, to know if Λ is iterated tilted of Dynkin type we only need to decide if $T(\Lambda)$ is of finite representation type. To decide this we look for an algebra Λ_1 such that $T(\Lambda) \simeq T(\Lambda_1)$ and such that we are able to determine easily if it is iterated tilted of Dynkin type or not. To do so we choose the admissible cut of $T(\Lambda)$ whose arrows are indicated above with two short parallel segments $||$, and consider the corresponding quotient Λ_1.



Here, Λ_1 is a hereditary algebra of Dynkin type E_6. It follows then that $T(\Lambda_1)$ is of finite representation type of Cartan class E_6 (by 3.1 or 3.2). Since Λ_1 is a quotient of $T(\Lambda)$ by an admissible cut we can apply 3.5 to conclude that $T(\Lambda) \simeq T(\Lambda_1)$. Thus $T(\Lambda)$ is of finite representation type and Cartan class E_6 and using again 3.2 we conclude that Λ is iterated tilted of type E_6.

Now we can find other iterated tilted algebras of the same Dynkin type by choosing other admissible cuts of $T(\Lambda)$, and considering the corresponding quotients. For example, the algebra Λ_2 given below is iterated tilted of type E_6, being a quotient of $T(\Lambda) \simeq T(\Lambda_1)$ by the admissible cut indicated in the last figure.



These techniques were used by Fernández in [23] to give a complete classification of trivial extensions of finite representation type under a unified approach, as well as of iterated tilted algebras of types A_n, D_n

and E_6. Moreover, she gave a method to decide easily if an algebra is iterated tilted of a given Dynkin type. Finding a quotient by an admissible cut Λ' of $T(\Lambda)$ whose representation type and Dynkin class can be determined is not always as simple as in the previous example. A very useful tool to do this was the list of minimal algebras of infinite representation type given by Bongartz [13] and by Happel and Vossieck [30]. Using this list, as well as different reduction techniques the complete classification was acchieved.

Using results of Assem, Nehring and Skowroński in [7] these ideas can be applied in many cases to decide if an algebra is iterated tilted of type $\tilde{D}_n$ or $\tilde{E}_p$.

Many of these classification results have been obtained by other mathematicians using different techniques. The first related results can be found in the early work of Riedtmann, where she classified selfinjective algebras of finite representation type [39, 40, 41]. The classification of iterated tilted algebras of Dynkin type A_n was done by Assem and Happel in [4]. The D_n case was studied by Assem and Skowroński in [10] and also, through the use of derived categories, by Keller ([33]). The classification for E_6 was done by Happel in his study of tiltig sets in cylinders [28], and Roggon completed the cases E_p, by means of invariants assigned to the algebras [43].

4. Cluster algebras and relation extensions of iterated tilted algebras

In this section we study the connection between cluster tilted algebras and iterated tilted algebras of global dimension at most two. The notion of cluster algebra was defined and studied by Fomin and Zelevinski in 2000. In connection with it, cluster categories were defined by Buan, Marsch, Reinecke, Reiten and Todorov, who developed a tilting theory for them [16]. We start by recalling definitions and known results which will be needed later.

Let H be a finite dimensional hereditary algebra over the algebraically closed field k. We denote by $\mathrm{D^b}(H)$ the bounded derived category of finitely generated H-modules [29]. Since H his hereditary each indecomposable object in $\mathrm{D^b}(H)$ is isomorphic to a complex concentrated in one degree (stalk complex). We will then identify modΛ with the full subcategory of $\mathrm{D^b}(H)$ of the complexes concentrated in degree 0.

Since H has finite global dimension, then $\mathrm{D^b}(H)$ is a triangulated category, and has Auslander-Reiten triangles. We will denote by $\tau_{\mathrm{D^b}(H)}$, or just by τ, the Auslander-Reiten translation in $\mathrm{D^b}(H)$ and by [1] the shift (or suspension) functor. Then $F = \tau^{-1}[1]$ is an autoequivalence.

The Auslander-Reiten quiver Γ of $\mathrm{D^b}(H)$ has been decribed by Happel in [29]. If $M \in \operatorname{mod} H$ is an indecomposable non-projective module, then $\tau_H(M) = \tau_{\mathrm{D^b}(H)}(M)$, so Γ contains a copy of Γ_H, for every $i \in \mathbb{Z}$. Moreover, $\tau_{\mathrm{D^b}(H)}(P_i[k]) = I_i[k-1]$. When the quiver Q of H is Dynkin then Γ consists of a single component isomorphic to the translation quiver $\mathbb{Z}D$ ([29, Ch.1, Cor. 5.6]).

If Q is not Dynkin then the structure of Γ is very different. Denote by $\mathcal{P}$, (resp. $\mathcal{I}$) the preprojective (resp. preinjective) component of the Auslander-Reiten quiver of H and by $\mathcal{R}$ the full subcategory of $\operatorname{mod} H$ given by the regular components. For each $r \in \mathbb{Z}$ the regular part $\mathcal{R}$ gives rise to $\mathcal{R}[r]$, given by the complexes $X \in \mathrm{D^b}(H)$ concentrated in degree r with $X_r \in \mathcal{R}$. Moreover, for each $r \in \mathbb{Z}$ there is a transjective component $\mathcal{I}[r-1] \vee \mathcal{P}[r]$ of Γ which we shall denote by $\mathcal{R}[r-\frac{1}{2}]$, and each component of Γ is contained in $\mathcal{R}[r]$ for some half-integer r.

The *cluster category of H* is the orbit category $\mathcal{C} = \mathrm{D^b}(H)/F^{\mathbb{Z}}$ (see [16]). Thus the objects of $\mathcal{C}$ are the objects of $\mathrm{D^b}(H)$ and the morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{D^b}(H)}(X, F^iY)$$

with the natural composition. Then $\mathcal{C}$ has a natural triangulated structure [34], and the Auslander-Reiten formula holds both in $\mathrm{D^b}(H)$ and in $\mathcal{C}$. That is,

$$\operatorname{Hom}_{\mathrm{D^b}(H)}(Y, \tau X) \simeq D\operatorname{Ext}^1_{\mathrm{D^b}(H)}(X,Y)$$

and

$$\operatorname{Hom}_{\mathcal{C}}(Y, \tau X) \simeq D\operatorname{Ext}^1_{\mathcal{C}}(X,Y).$$

The notion of tilting complex was defined by Happel in [29].

Definition 4.1. A complex T in $\mathrm{D^b}(H)$ is a *tilting complex* if $\operatorname{Hom}_{\mathrm{D^b}(H)}(T, T[i]) = 0$ for all $i \neq 0$, and the only complex X such that $\operatorname{Hom}_{\mathrm{D^b}(H)}(T, X[i]) = 0$ for all i is $X = 0$.

It was proven by Reiten and Van den Berg ([37, Cor. 3.3 and Lemma 3.5] that the second condition can be replaced by

(T_2) The number of nonisomorphic summands of T coincides with the number of nonisomorphic simple H-modules.

Definition 4.2. An object T in $\mathcal{C}_H$ is called a *cluster tilting object* if $\mathrm{Hom}_{\mathcal{C}}(T, T[1]) = 0$ and condition (T_2) holds.

Tilted algebras, that is, endomorphism rings of tilting modules, have played an important role in the representation theory of algebras. It was proven in [29, Cor. 5.5 of Chap. 4] and [38], that the endomorphism ring of a tilting complex in the derived category is an iterated tilted algebra, and each iterated tilted algebra is of this form. *Cluster tilted algebras* are defined as endomorphism rings of cluster tilting objects in $\mathcal{C}$. When $\mathcal{A}$ is the cluster algebra associated to H, then the quivers of the cluster tilted algebras arising from $\mathcal{C}$ coincide with the quivers of the exchange matrices associated to $\mathcal{A}$.

Let T be an H-module. Then T can be considered both as an object in $\mathrm{D^b}(H)$ and as an object in $\mathcal{C}$. We observe that T is tilting in $\mathrm{mod}H$ if and only if T is a tilting complex in $\mathrm{D^b}(H)$, if and only if T is a cluster tilting object in $\mathcal{C}$.

On the other hand, it is proven in [16] that any cluster tilting object in $\mathcal{C}$ can be represented by a tilting module over a hereditary algebra H' derived equivalent to H. Therefore, if C is a cluster tilted algebra there exists a hereditary algebra H' such that C is isomorphic to the endomorphism ring of a tilting H'-module in $\mathrm{D^b}(H')$ [16, 3.3].

If T is a tilting complex in $\mathrm{D^b}(H)$, it is not always true that T defines a cluster tilting object in $\mathcal{C}$. However, if we further assume that the iterated tilted algebra has global dimension at most two, then T is also tilting in $\mathcal{C}$. This fact was proven by Barot, Fernández. Pratti, Platzeck and Trepode in [11, Cor. 3.15], and independently by Osamu Iyama in [31, Thm. 1.22] and by Claire Amiot in [1, 4.10] using different techniques. Our next aim is to discuss the central ideas of the proof of this result, following the approach in [11].

We start by defining a procedure, the "rolling of tilting complexes", which associates to each tilting complex T a new complex $\rho(T)$ such that T and $\rho(T)$ define the same object in the cluster category $\mathcal{C}$. The construction depends on wether the quiver is Dynkin or not.

A first approach to define ρ is to choose some summands T_i of T and replace them by $F^{-1}(T_i)$. However, this does not always lead to a tilting complex. The following lemma gives sufficient conditions for the new module to be a tilting complex.

Lemma 4.3. [11, Lemma 3.7] *Let $T = T' \oplus X$ be a tilting complex in $\mathrm{D^b}(H)$ such that $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$ and let $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. If $\mathrm{gldim}\, B \leq 2$, then $\widetilde{T} = T' \oplus F^{-1}X$ is a tilting complex in $\mathrm{D^b}(H)$ if and only if $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[k]) = 0$ for $k = 0, -1$.*

So we consider a tilting complex T in $\mathrm{D^b}(kQ)$ and look for a summand X of T satisfying the conditions in the lemma.

We assume first that Q is a Dynkin quiver. We denote by $\leq$ the partial order induced in $\Gamma = \mathbb{Z}Q$ by the arrows. Since $T = \bigoplus_{i=1}^n T_i$ has only finitely many summands we can easily find a section $\Sigma = \{\Sigma_1, \ldots, \Sigma_n\}$ such that $T \leq \Sigma$, that is, $T_i \leq \Sigma_j$ for all i and j. Here, by a *section* we mean a set of representatives $\Sigma_1, \ldots, \Sigma_n$ of the τ-orbits of Γ such that $\Sigma_1, \ldots, \Sigma_n$ induce a connected subquiver of Γ, and n is the the number of vertices in the quiver Q. If Σ_j is maximal in Σ and $\Sigma_j \notin \{T_1, \ldots, T_n\}$ then $\Sigma' = \Sigma \setminus \{\Sigma_j\} \cup \{\tau\Sigma_j\}$ is also a section satisfying $T \leq \Sigma'$. After finitely many steps we get a section $\Sigma(T)$ such that $T \leq \Sigma(T)$ and all maximal elements in $\Sigma(T)$ belong to $\mathrm{add}T$. Then $\Sigma(T)$ is uniquely defined by T.

Definition 4.4 (Rolling of tilting complex, the Dynkin case)**.** With the previous notations, let X be the sum of those summands of T which belong to $\Sigma(T)$ and T' a complement of X in T. Then define the *rolling* of T to be $\rho(T) = T' \oplus F^{-1}X$.

Now consider the case where Q is not Dynkin. As we mentioned at the beginning of this section, $\mathrm{D^b}(kQ)$ is composed by the parts $\mathcal{R}[r]$ for $r \in \mathbb{Z}/2$. Now, write $T = \bigoplus_{a \in \mathbb{Z}/2} T_{\mathcal{R}[a]}$, where $T_{\mathcal{R}[a]} \in \mathcal{R}[a]$.

Definition 4.5 (Rolling of tilting complex, the non-Dynkin case)**.** With the previous notation let m be the largest half-integer such that $T_{\mathcal{R}[m]}$ is non-zero. Then define $X = T_{\mathcal{R}[m]}$ and T' to be the complement of X in T. Define the *rolling* of T to be $\rho(T) = T' \oplus F^{-1}X$.

Proposition 4.6. *Let T be a tilting complex in $\mathrm{D^b}(H)$ such that $\mathrm{gldim}\, \mathrm{End}_{\mathrm{D^b}(H)}(T) \leq 2$. Then $\rho(T)$ is again a tilting complex.*

Proof. Assume that $H = kQ$, and consider first the case when Q is a Dynkin quiver. It follows from the definition of $\Sigma(T)$ that $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$. Let now V be an indecomposable module in the section $\tau\Sigma$. Then $\tau^{-1}V \in \Sigma$, so there is a maximal element T_1 in Σ such that $\tau^{-1}V \leq T_1$. So $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau^{-1}V, T_1) \neq 0$, since both $\tau^{-1}V$ and T_1 belong to Σ. Since all maximal elements in Σ belong to $\mathrm{add}\, T$ it follows that $T_1 \in \mathrm{add}\, T$. Then, using the Auslander-Reiten formula, we get

$$\mathrm{Ext}^1_{\mathrm{D^b}(H)}(T_1, V) \simeq D(\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau^{-1}V, T_1)) \neq 0.$$

This proves that V is not in $\mathrm{add}\, T$, because T is a tilting complex and therefore $\mathrm{Ext}^1_{\mathrm{D^b}(H)}(T, T) = 0$. Thus τX has no summands in $\mathrm{add}\, T$, because $X \in \mathrm{add}\, \Sigma$. It follows from the definition of T' that $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T') = 0$ and $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[-1]) = 0$. Then we get from the preceding lemma that $\rho(T)$ is a tilting complex in $\mathrm{D^b}(H)$.

When the quiver Q is not Dynkin the proof is easier, since it is not difficult to prove that the conditions for $\rho(T)$ to be tilted given in Lemma 4.3 are satisfied.

□

Given an iterated tilted algebra B, there are a hereditary algebra H and a tilting complex T in $\mathrm{D^b}(H)$ such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. It can be proven that $\mathrm{End}_{\mathrm{D^b}(H)}(\rho(T))$ does not depend on the choice of H or T ([11]). Then we can give the following definition.

Definition 4.7. The *rollling of the iterated tilted algebra B* is the endomorphism algebra $\rho(B) = \mathrm{End}_{\mathrm{D^b}(H)}(\rho(T))$, where H is a hereditary algebra such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ and T a tilting complex in $\mathrm{D^b}(H)$.

4.1. **Iterated rolling.** Let T be a tilting complex in $\mathrm{D^b}\, H$ such that the global dimension of $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ is at most two. We proved that the rolling of T is again a tilting complex defining the same object as T in the cluster category $\mathcal{C}$. We will see next that it is possible to iterate the rolling procedure, and find a natural number n such that $\rho^n(T)$ is a tilting module over an algebra H' derived equivalent to H. This will prove that $\rho^n(T)$ defines a cluster tilting object in $\mathcal{C}$, so T also does, because rolling preserves elements of $\mathcal{C}$.

To be able to apply the previous proposition to $\rho(T)$ we need the global dimension of $\rho(B) = \mathrm{End}_{\mathrm{D^b}(H)}(\rho(T))$ to be at most 2. This is true, and we state it in the following proposition.

Proposition 4.8. [11, Prop. 3.11] *Let B be an iterated tilted algebra. If* $\operatorname{gldim} B \leq 2$ *then* $\operatorname{gldim} \rho(B) \leq 2$.

This shows that the iteration procedure can be carried on. Assume next that $H = kQ$, with Q Dynkin. Then, for a section Σ in $\mathrm{D^b}(H)$ we consider the hereditary algebra $H(\Sigma)$ whose indecomposable injective modules (concentrated in degree zero) are the objects in Σ. This algebra is derived equivalent to H. Next, for each tilting complex T in $\mathrm{D^b}(H)$ we define the natural number $n_\Sigma(T)$ as the sum of the lengths all the paths from the indecomposable summands of T which are not $H(\Sigma)$-modules to the other indecomposable summands of T. Then $n_\Sigma(T) = 0$ if and only if T is an $H(\Sigma)$-module.

We denote by $\Sigma(T)$ the section used in the definition of $\rho(T)$. For for each $h \geq 0$, let $n_h(T) = n_{\Sigma(\rho^h(T))}(\rho^h(T))$. Then $n_h(T) > 0$ implies that $n_{h+1}(T) < n_h(T)$, and $n_h(T) = 0$ implies $n_{h+1}(T) = 0$ (see [11, Lemma 3.3]).

The following proposition will be important in the sequel.

Proposition 4.9. *Let H be a hereditary algebra and T be a tilting complex in $\mathrm{D^b}(H)$ such that* $\operatorname{gldim} \mathrm{End}_{\mathrm{D^b}(H)}(T) \leq 2$. *Then there exists a natural number n such that $\rho^n(T)$ is a tilting module over an algebra derived equivalent to H. Moreover, if $\rho^k(T)$ is such a tilting module for some k, then $\rho^{k+1}(T)$ has the same property.*

Proof. First consider the case when the quiver Q is Dynkin. With the notations of the preceding paragraph, there is a natural number n such that $n_k(T) = 0$ for all $k \geq n$. Then $\rho^k(T)$ is a module over the hereditary algebra $H(\Sigma(\rho^k(T))$.

The proof in the non Dynkin case is based in the fact that, for sufficiently large h, $\rho^h(T)$ belongs to $\mathcal{R}[p] \cup \mathcal{R}[p + \frac{1}{2}]$ for some half integer p (see [11, Theorem 1.2]). $\square$

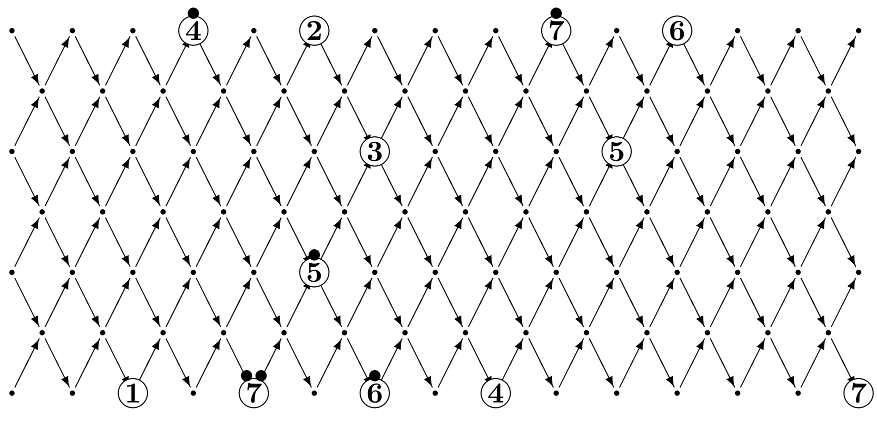
Using the fact that tilting modules define cluster tilting objects in $\mathcal{C}$, and since T and $\rho^h(T)$ define isomorphic objects in $\mathcal{C}$, we obtain the following corollary.

Corollary 4.10. ([11, Cor. 3.15], [31, Thm. 1.22], [1, 4.10]) *Let T be a tilting complex in $\mathrm{D^b}(H)$ such that* $\operatorname{gldim}\operatorname{End}_{\mathrm{D^b}(H)}(T) \leq 2$. *Then T defines a cluster tilting object in the cluster category $\mathcal{C}$ of H and $\operatorname{End}_{\mathcal{C}}(T)$ is a cluster tilted algebra.*

Let B be an iterated tilted algebra with $\operatorname{gldim} B \leq 2$. Let H be a hereditary algebra H and T a tilting complex in $\mathrm{D^b}(H)$ such that $B \simeq \operatorname{End}_{\mathrm{D^b}(H)}(T)$. Then there is a natural number h so that $\operatorname{End}_{\mathcal{C}}(T) \simeq \mathcal{R}(\rho^h(B))$. As we observed before, $\rho(B)$ doesn't depend on the choice of H and T, so $\operatorname{End}_{\mathcal{C}}(T)$ is also independent on such choice, and we can define the *cluster tilted algebra $C(B)$ associated to B* to be the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T)$.

We illustrate the above results with the following example.

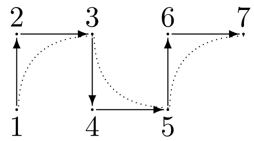
Example 4.11. Let Q be a quiver of type $\mathbb{A}_7$ with some orientation and $H = kQ$. We depict below the Auslander-Reiten quiver Γ of $\mathrm{D^b}(H)$, where the symbol ⓘ indicates the indecomposable summand T_i of the tilting complex $T = \bigoplus_{i=1}^{7} T_i$. Moreover, $F^{-1}T_i$ and $F^{-2}T_i$ are indicated by the symbols ⓘ, ⓘ respectively.

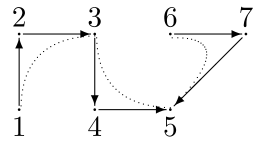


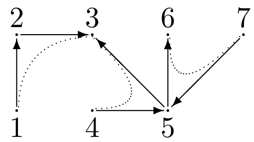
We then have the tilting complexes:

$$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7$$
$$\rho(T) = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus F^{-1}T_6 \oplus F^{-1}T_7$$
$$\rho^2(T) = T_1 \oplus T_2 \oplus T_3 \oplus F^{-1}T_4 \oplus F^{-1}T_5 \oplus F^{-1}T_6 \oplus F^{-2}T_7$$

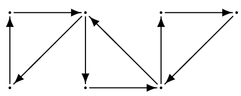
The corresponding iterated tilted algebras $B_h = \mathrm{End}_{\mathrm{D^b}(H)}(\rho^h(T))$, $i = 0, 1, 2$, are given by the following quivers with relations.

B_0:


B_1:


B_2:


The algebra B_2 is tilted, and by 4.9 we know that all algebras B_i for $i > 2$ are also tilted. Observe that the relation extensions of all the algebras $\rho^h(B)$ have the same quiver, which coincides with the quiver of the cluster tilted algebra $C(B) = \mathrm{End}_{\mathcal{C}}(T)$ and is shown in the following picture.



4.2. **Relation extensions of iterated tilted algebras and cluster tilted algebras.** If T is a tilting complex in $\mathrm{D^b}(H)$ and $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ then we have an equivalence of categories $G\colon \mathrm{D^b}(H) \to \mathrm{D^b}(B)$ derived from $\mathrm{Hom}(T, -)$ such that $G(T) = B$ and $G(\tau T[1]) = \mathrm{D}B$ (see [29]) . Moreover, if X and Y are objects of of $\mathrm{D^b}(H)$ such that GX and GY are B-modules, then $\mathrm{Ext}^i_B(GX, GY) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(X, Y[i])$

for all $i \in \mathbb{Z}$. Using the functoriality of these isomorphisms and Serre duality it can be proven that there is an isomorphism of B-B-bimodules $\mathrm{Ext}^2_B(DB, B) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(T, FT)$ (see [2, 3.3]).

Assume moreover that gldim$B \leq 2$ and let $C(B) = \mathrm{End}_{\mathcal{C}}(T)$ be the cluster tilted algebra associated to B. Then

$$C(B) = \mathrm{End}_{\mathcal{C}}(T) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^iT)$$

The summand corresponding to $i = 0$ is $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$, which is a subalgebra of $C(B)$. The next is the B-B-bimodule $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, FT)$ $\simeq \mathrm{Ext}^2_B(DB, B)$. Therefore we get a projection map

$$\pi : C(B) \to \mathcal{R}(B)$$

which is a homomorphism of B-B-bimodules.

When T is a tilting module, then B is a tilted algebra and $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^iT) = 0$ for all $i \geq 2$. So the map $\pi : C(B) \to \mathcal{R}(B)$ is bijective, and a straightforward verification shows that it is in fact an algebra isomorphism. Combining these facts the following result due to Assem, Brüstle and Schiffler can be proven:

Theorem 4.12. [2, Theorem 3.4] *An algebra C is cluster tilted if and only if there exists a tilted algebra B such that $C \simeq \mathcal{R}(B)$.*

We turn now our attention to iterated tilted algebras of global dimension at most two, and study the relation between their relation extensions and cluster tilted algebras.

Let B be an iterated tilted algebra. Then there are a hereditary algebra H and a tilting complex T in $\mathrm{D^b}(H)$ such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. We have seen that there is homomorphism of B-B-bimodules π : $C(B) \to \mathcal{R}(B)$, which in general is not an isomorphism, not even an algebra homomorphism. However, if we assume moreover that the global dimension of B is at most two then π is an algebra homomorphism whose kernel is contained in $\mathrm{rad}^2 C(B)$, and therefore the algebras $C(B)$ and $\mathcal{R}(B)$ have isomorphic quivers. The proof of this result is done in several steps, which we will outline next.

First, we need to study the behaviour of relation extensions under rolling. We assume throughout the rest of this section that gl.dim $B \leq 2$. We denote by $\Psi : C(\rho(B)) = \mathrm{End}_{\mathcal{C}}(\rho(T)) \to \mathrm{End}_{\mathcal{C}}(T) = C(B)$

the canonical isomorphism, given by the direct sum of the following bijective maps

$$\mathrm{id}\colon \mathrm{End}_{\mathcal{C}}(T') \to \mathrm{End}_{\mathcal{C}}(T'), \quad \sigma^{-1}\colon \mathrm{Hom}_{\mathcal{C}}(T', F^{-1}X) \to \mathrm{Hom}_{\mathcal{C}}(T', X),$$
$$\sigma F\colon \mathrm{Hom}_{\mathcal{C}}(F^{-1}X, T') \to \mathrm{Hom}_{\mathcal{C}}(X, T'),\ F\colon \mathrm{End}_{\mathcal{C}}(F^{-1}X) \to \mathrm{End}_{\mathcal{C}}(X).$$

Here σ denotes the shift in the $\mathbb{Z}$-graduation, that is

$$\sigma\colon \bigoplus_{i\in\mathbb{Z}}(Y, F^iZ) \to \bigoplus_{i\in\mathbb{Z}}(Y, F^{i+1}Z), (f_i)_{i\in\mathbb{Z}} \mapsto (f_{i+1})_{i\in\mathbb{Z}},$$

and we abbreviated $(Y,Z) = \mathrm{Hom}_{\mathrm{D^b}(H)}(Y,Z)$.

We recall that the underlying vector spaces of $\mathcal{R}(\rho(B))$ and $\mathcal{R}(B)$ can be identified respectively with

$$\mathrm{End}_{\mathrm{D^b}(H)}(\rho(T)) \oplus \mathrm{Hom}_{\mathrm{D^b}(H)}(\rho(T), F\rho(T))$$

and

$$\mathrm{End}_{\mathrm{D^b}(H)}(T) \oplus \mathrm{Hom}_{\mathrm{D^b}(H)}(T, FT),$$

which are also summands of $\mathrm{End}_{\mathcal{C}}(\rho(T))$ and $\mathrm{End}_{\mathcal{C}}(T)$. Then a k-linear transformation $\Theta : \mathcal{R}(\rho(B)) \to \mathcal{R}(B)$ can be defined so that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{End}_{\mathcal{C}}(\rho(T)) \arrow[r, "\overset{\Psi}{\simeq}"] \arrow[d, "\pi(\rho(B))"'] & \mathrm{End}_{\mathcal{C}}(T) \arrow[d, "\pi(B)"] \\
\mathcal{R}(\rho(B)) \arrow[r, "\Theta"] & \mathcal{R}(B)
\end{tikzcd}$$

It can be proven that Θ is a surjective algebra homomorphism and $\mathrm{Ker}\,\Theta \subseteq \mathrm{rad}^2\,\rho(B)$. Since Ψ is an isomorphism, the commutativity of this diagram implies that if $\pi(\rho(B))$ is an algebra homomorphism, then $\pi(B)$ is also an algebra homomorphism. These results are proven in [11, Prop. 3.17], and their application to iterated rolling leads to the following result.

Theorem 4.13. *If B is an iterated tilted algebra of* $\mathrm{gldim}\,B \leq 2$*. Let H be a hereditary algebra and T a tilting complex in $\mathrm{D^b}(H)$ such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ and let $C(B)$ be the cluster tilted algebra $\mathrm{End}_{\mathcal{C}(H)}(T)$. Then there exists a sequence of algebra homomorphisms*

$$B \to C(B) \xrightarrow{\pi} \mathcal{R}(B) \to B$$

whose composition is the identity map. Moreover, π is an epimorphism whose kernel is contained in $\operatorname{rad}^2 C(B)$*. In particular $C(B)$ and $\mathcal{R}(B)$ have the same quivers and are both split extensions of B.*

Proof. Consider h such that the algebra $\rho^h(B)$ is tilted of type Q and $C(B) = \operatorname{End}_{\mathcal{C}}(\rho^h(T))$ is cluster-tilted (Prop. 4.9). Then $\pi(\rho^h(B))\colon C(B) \to \mathcal{R}(\rho^h(B))$ is an isomorphism, by 4.12. Hence by the observation preceding this theorem we get inductively for $i = h-1, h-2, \ldots, 1, 0$ that $\pi(\rho^i(B))$ is an algebra homomorphism. Let $\Theta_i\colon \mathcal{R}(\rho^i(B)) \to \mathcal{R}(\rho^{i-1}(B))$ be the surjective morphism above defined, with kernel contained in $\operatorname{rad}^2 \mathcal{R}(\rho^i(B))$. Then the composition $\Theta_1\Theta_2\cdots\Theta_h : \mathcal{R}(\rho^h(B)) \to \mathcal{R}(\rho^0(B)) = \mathcal{R}(B)$ has the same properties, and so does $\pi(B) : C(B) \to \mathcal{R}(B)$, as follows using the commutativity of the above diagram.

Let $B \to \operatorname{End}_{\mathcal{C}}(T)$ and $\mathcal{R}(B) \to B$ be the canonical inclusion and projection, respectively. Then it follows from the definition of $\pi = \pi(B)$ that the composition $B \to \operatorname{End}_{\mathcal{C}}(T) \xrightarrow{\pi(B)} \mathcal{R}(B) \to B$ is the identity map and consequently both algebras C and $\mathcal{R}(B)$ are split extensions of B. $\square$

A different proof of the last assertion of this theorem, relating the quivers of $\operatorname{End}_{\mathcal{C}}(T)$ and $\mathcal{R}(B)$, is given in [1, 4.17].

An interesting consequence of this theorem is the following result.

Corollary 4.14. [11, 3.21] *If B is an iterated tilted algebra of Dynkin type and* $\operatorname{gldim} B \leq 2$ *then $\mathcal{R}(B)$ is of finite representation type.*

Proof. Let H be a hereditary algebra and T a tilting complex in $\mathrm{D^b}(H)$ such that $B = \operatorname{End}_{\mathrm{D^b}(H)}(T)$, and let C be the cluster tilted algebra $C = \operatorname{End}_{\mathcal{C}(H)}(T)$. Since H is hereditary we know by [19, Cor. 2.4] that C is of finite representation type, thus so is the quotient $\mathcal{R}(B)$ of C. $\square$

5. **The Dynkin case**

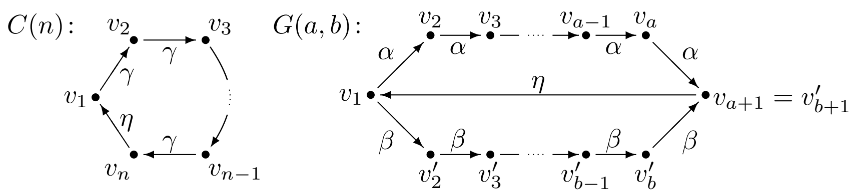
The notion of admissible cut played an important role in the study of trivial extensions of finite dimensional algebras. As seen in 3.5, for a schurian triangular algebra Λ, an algebra Λ' is an admissible cut of $T(\Lambda)$ if and only if $T(\Lambda) \simeq T(\Lambda')$. Our next purpose is to show how admissible cuts can be used to study cluster tilted algebras of Dynkin type.

To do this we need the description of the relations of cluster tilted algebras of Dynkin type given by Buan, Marsch and Reiten in [17]. We will say that an arrow α is *parallel*, (resp. *antiparallel*) to a relation (or a path or an arrow) ρ if $s(\alpha) = s(\rho)$ and $t(\alpha) = t(\rho)$ (resp. $s(\alpha) = t(\rho)$ and $t(\alpha) = s(\rho)$). We recall that a relation ρ is called *minimal* if $\rho = \sum_i \beta_i \rho_i \gamma_i$ where ρ_i is a relation for every i, then β_i and γ_i are scalars for some index i (see [17]).

The following description follows immediately from [17, Thm. 4.1].

Theorem 5.1. *Let C be a cluster-tilted algebra of Dynkin type. Then $C \simeq kQ_C/I_C$, where:*

a) *For each arrow in Q_C there exist at most two shortest antiparallel paths to η. If there is at least one and Σ_η denotes the full subquiver of Q_C given by the vertices of η and the antiparallel paths, then the quiver Σ_η is isomorphic to $C(n)$ (for some n) or to $G(a, b)$ (for some a, b), as shown in the following picture.*



(b) *The ideal I_C is generated by minimal zero relations and minimal commutativity relations, and each of them is antiparallel to exactly one arrow. If an arrow η is antiparallel to the minimal zero relation ρ, then $\Sigma_\eta \simeq C(n)$ and $\rho = \gamma^{n-1}$. If η is antiparallel to the minimal commutativity relation $\rho_1 = \rho_2$, then $\Sigma_\eta \simeq G(a, b)$ and $\rho_1 = \alpha^a \neq 0, \rho_2 = \beta^b \neq 0$.*

Thus, in the Dynkin case cluster algebras are determined by their quivers. It has been proven in [21] that this result extends to the non Dynkin case. We find here an analogy with trivial extensions of schurian triangular algebras, since they are also determined by their quivers.

From the above description we get that, given a minimal set of minimal relations $\mathcal{R}$ in C, each arrow in an oriented cycle is antiparallel to a unique minimal relation in $\mathcal{R}$, and all minimal relations are obtained in this way. So, if $\Delta = \{\alpha_1 \ldots, \alpha_t\}$ is an admissible cut of C, then each α_i is antiparallel to a relation ρ_i, and ρ_i is a relation in the quotient algebra C/Δ. More precisely:

Proposition 5.2. [11, Prop. 4.16] *Let B be a quotient by an admissible cut of a cluster-tilted algebra C of Dynkin type. Write $B = kQ_B/I_B$ where Q_B is the quiver of B and I_B is an admissible ideal generated by the minimal set of minimal relations $\{\rho_i \mid i = 1, \ldots t\}$. Then C is a split extension of B by an ideal $M = \langle \alpha_1, \alpha_2, \ldots, \alpha_t \rangle$, generated by arrows such that α_i is antiparallel to ρ_i for each $i = 1, \ldots, t$.*

Proof. The first part of the proposition follows from the preceding observation. The fact that C is split extension of B by M can be proven using Theorem 5.1 and the characterization of split extensions in terms of their relations given in [3, Thm. 2.5]. □

Using the description of the quiver of the relation extension given in 2.4 we can state the following Corollary:

Corollary 5.3. *Let B be a quotient by an admissible cut of a cluster-tilted algebra C of Dynkin type such* $\mathrm{gldim} B \leq 2$. *Then* $Q_{\mathcal{R}(B)} = Q_C$.

Example 5.4. One can easily see that the algebras B_i, $i = 0, 1, 2$, in Example 4.11 are all admissible cuts of $C(B_0)$. By the preceding corollary we know that all their relation extensions have the same quiver, which coincides with the quiver of $C(B_0)$. This fact follows also from Theorem 4.13, which implies that $\mathcal{R}(B_i)$ is the quotient of $C(B_i) = C(B_0)$ by an ideal contained in $\mathrm{rad}^2 C(B_i)$. We also know that $\mathcal{R}(B_2) \simeq C(B_2)$.

However, the relation extension of an iterated tilted algebra B of Dynkin type and global dimension at most two is in general not isomorphic to the cluster algebra $C(B)$. For example, the relation extension $\mathcal{R}(B_0)$ of 4.11 is not isomorphic to $C(B_0)$. In fact, it follows from the definition of the product in the relation extension, that the product of arrows corresponding to relations is zero in the relation extension. So the path $5 \to 3 \to 1$ is zero in $\mathcal{R}(B_0)$. Since this path is not contained in any cycle, it is nonzero in $C(B_0)$, as follows from the description of the relations of cluster tilted algebras given in 5.1.

We have seen in section 4.2 that when B is an iterated tilted algebra and $\operatorname{gldim} B \leq 2$, then there exists a sequence of algebra homomorphisms

$$B \to C(B) \xrightarrow{\pi} \mathcal{R}(B) \to B$$

whose composition is the identity map. Moreover, π is an epimorphism whose kernel is contained in $\operatorname{rad}^2 C(B)$. In particular $C(B)$ and $\mathcal{R}(B)$ have the same quivers and are both split extensions of B. It follows from this result that B is an admissible cut of $\mathcal{R}(B)$ if and only if B is an admissible cut of $C(B)$.

In general, it is not necessarily true that an algebra of global dimension 2 is an admissible cut of its relation extension, as the example in [11, 4.14] shows. It is not known wether this holds if we further assume that the algebra is iterated tilted. However, in the Dynkin case the situation is particularly nice, as the following result shows.

Theorem 5.5. [11, Theorem 4.20] *Let Q be a Dynkin quiver. An algebra B with* $\operatorname{gldim} B \leq 2$ *is iterated tilted of type Q if and only if it is the quotient of a cluster-tilted algebra of type Q by an admissible cut.*

The proof of this theorem uses several techniques and results. To prove the sufficiency, a result of Assem and Skowroński in [8] is used, stating that if the Tits form q_A of an algebra A is positive definite and A is strongly simply connected, then A is iterated tilted of Dynkin type.

Let B be the quotient by an admissible cut of a cluster tilted algebra C of Dynkin type Q. The proof of the positiveness of q_B is based on a result on quasi-Cartan companion matrices due to Barot, Geiss and Zelevinski, which states that the quiver Q_C admits a positive definite quasi-Cartan companion (see [12]). On the other hand, a careful study of the cycles in B, and known results about strong and simple connectedness are used to prove that B is strongly simply connected.

To prove the necessity, let B an iterated tilted algebra of Dynkin type Q. By Theorem 4.13 we kow that $C(B)$ is a split extension of B , thus B is the quotient of the cluster tilted algebra $C(B)$ by the ideal J generated by the arrows of $Q_{C(B)}$ which are not in Q_B. The fact that J is an admissible cut of C is proven using that $C(B)$ and $\mathcal{R}(B)$ have the same quiver, and the description of the relations for the cluster tilted algebra given in [17] (see 5.1).

Therefore, in the Dynkin case admissible cuts of cluster tilted algebras and iterated tilted algebras of global dimension at most 2 coincide (up to isomorphism). This result can not be extended to arbitrary iterated tilted algebras of global dimension at most two, not even in the euclidean case, as shown in [11, 4.21].

5.1. **Applications.** These results can be applied in different ways. For example, in many cases they can be combined with known results about cluster tilted algebras, to determine that a given algebra of global dimension two is not iterated tilted, as the example in [11, 4.14] shows.

These ideas can also be applied to classify cluster-tilted algebras of Dynkin type. In fact, Bordino, Fernández and Trepode classified those of type E_p using the results in the last two sections, and the fact that the classification of iterated tilted algebras is known.

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