São Paulo Journal of Mathematical Sciences **6**, 2 (2012), 365–374

Orbit Equivalence of Global Attractors for S^1-Equivariant Parabolic Equations

Carlos Rocha

Center for Mathematical Analysis, Geometry and Dynamical Systems
Departamento de Matemática, Instituto Superior Técnico
Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail address: crocha@math.ist.utl.pt

Abstract. We consider the global attractor $\mathcal{A}_f$ for the semiflow generated by a scalar semilinear parabolic equation of the form $u_t = u_{xx} + f(u, u_x)$, defined on the circle, $x \in S^1$. Using a characterization of the period maps for planar Hamiltonian systems of the form $u'' + g(u) = 0$ we discuss questions related to the topological equivalence between global attractors.

1. Introduction

In the following we consider the global attractors of semiflows generated by scalar semilinear parabolic equations of the form

$$u_t = u_{xx} + f(u, u_x) \ , \tag{1}$$

defined on the circle, $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, with appropriate smooth and dissipative nonlinearities $f = f(u, u_x)$. Our objective is to discuss questions related to the usual notion of topological equivalence between attractors.

It is well known that all the stable and unstable manifolds of equilibria and periodic orbits of the semiflow generated by (1) are automatically transversal (see [3] and [14] or citations there in). Hence, if all the equilibria and periodic orbits are hyperbolic, the semiflow has the Morse–Smale property [12]. Therefore, existence of a smooth homotopy between two nonlinearities f_0 and f_1 preserving hyperbolicity of all the equilibria and periodic orbits of (1) entails the topological equivalence of the corresponding attractors (see for example [12]). This is the main tool to obtain a

2000 *Mathematics Subject Classification.* Primary: 34B15, 35B41, 37G35.
Key words: Global attractors, nonlinear boundary value problems, Morse-Smale systems.

classification of the Morse–Smale global attractors of (1). However, the explicit construction of such homotopies is, in general, very difficult and depends on the class of nonlinearities considered. In Section 2 we review the characterization of the global attractor for the dynamical system generated by (1) and the notion of orbit equivalence of global attractors in this setting. In Section 3 we apply some previous results on the realization of period maps by planar ODE systems of the form $u'' + g(u) = 0$ used to address the existence of homotopies between different nonlinearities in (1), in the class of nonlinearities $f = f(u)$. Then, in Sections 4 and 5, we discuss some partial results in the general class of nonlinearities $f = f(u, u_x)$.

2. **Orbit equivalence of attractors**

We consider nonlinearities $f \in C^2(\mathbb{R}^2)$ satisfying adequate dissipative conditions (for example, $uf(u,0) < 0$ for all $|u| > M$ and $|f(u,p)| < C(u)(1 + |p|^\gamma)$ for $\gamma < 2$). Then, (1) generates a dynamical system in the Hilbert space $X = H^s(S^1)$ which, for $s > 3/2$, embeds into the set of continuously differentiable functions [13], $X \subset C^1(S^1)$. The semigroup $\varphi_t : X \to X, t \geq 0$, defined by $u_0 \mapsto \varphi_t(u_0) = u(t,\cdot)$, with $u(0,\cdot) = u_0$, has a nonempty global attractor $\mathcal{A} = \mathcal{A}_f$ [2, 11, 12], which is the maximal compact invariant subset of X.

The characterization of the global attractor $\mathcal{A}$ has been considered in the mathematical literature by many authors. See for example [15, 7, 8, 1, 5]. In general, $\mathcal{A}$ is composed of equilibria, periodic orbits and their heteroclinic orbit connections, that is, solutions for which the α- and the ω-limits are two different equilibria or periodic solutions. The set of equilibria $\mathcal{E} = \mathcal{E}_f$ of (1) is given by the 2π-periodic solutions of the ODE

$$u_{xx} + f(u, u_x) = 0 \ . \tag{2}$$

The periodic solutions of (1) are rotating waves. These are solutions of the form $u(t,x) = v(x - ct)$ rotating around S^1 with speed $c \neq 0$, and the set $\mathcal{R} = \mathcal{R}_f$ of rotating waves of (1) is given by the 2π-periodic solutions of the one-parameter family of ODEs parameterized in $c \neq 0$

$$v_{xx} + f(v, v_x) + cv_x = 0 \ . \tag{3}$$

Then, denoting by $\mathcal{H} = \mathcal{H}_f$ the set of heteroclinic orbit connections between two elements of $\mathcal{E} \cup \mathcal{R}$, we have that

$$\mathcal{A} = \mathcal{E} \cup \mathcal{R} \cup \mathcal{H} \ . \tag{4}$$

Here we recall the usual notion of topological equivalence between attractors. Two attractors are *orbit equivalent*, $\mathcal{A}_0 \cong \mathcal{A}_1$, if there is a homeomorphism $h : \mathcal{A}_0 \to \mathcal{A}_1$ taking orbits of one into orbits of the other and preserving the time direction.

When the global attractor $\mathcal{A}$ is Morse–Smale there exists a smooth homotopy $f^\tau, \tau \in [0,1]$, which preserves the hyperbolicity of all the equilibria and periodic orbits and reduces (1) to a problem for which f^1 is an even function of the second variable. In this case equation (2) is reversible with respect to the reflection $x \mapsto -x$. In addition, equation (2) for $f = f^1$ is integrable (see [16]) with a first integral $H = H(u, u_x)$ which is also an even function of the second variable. This homotopy was constructed in [7] and was used to obtain the heteroclinic orbit connections in $\mathcal{A}$, see also [9]. The resulting flow for f^1 has all the rotating waves "frozen" to speed $c = 0$ and possesses an embedded flow that satisfies Neumann boundary conditions in the half interval $x \in [0, \pi]$. For this reason the homotopy $f^\tau, \tau \in [0,1]$, is called the *freezing and symmetrizing homotopy*. The mentioned frozen waves are nonhomogeneous stationary solutions of (1), i.e. nonconstant 2π-periodic solutions of (2), which cannot be strictly hyperbolic. In fact, due to S^1-equivariance these stationary solutions occur in one parameter families of S^1 shifted copies, and are nonisolated. Therefore, in the case of nonhomogeneous stationary solutions, the above hyperbolicity is understood as normal hyperbolicity (admitting one simple zero eigenvalue). Then, the description of the orbit connections in $\mathcal{A}_{f^0} \cong \mathcal{A}_{f^1}$ follows from the previously established characterization of the global attractor for the embedded Neumann flow [10, 4, 6, 19].

3. The class of nonlinearities $f = f(u)$

For nonlinearities of the form $f(u, u_x) = g(u)$ the global attractor $\mathcal{A}_g$ has only equilibria and heteroclinic orbits since all rotating waves are frozen. The set of equilibria $\mathcal{E}_g$ is then given by the 2π-periodic solutions of the ODE

$$v_{xx} + g(v) = 0 \ . \tag{5}$$

This set is composed of two subsets: the spatially homogeneous stationary solutions, i.e. the subset $\mathcal{Z}_g = \{u(x,t) = e : g(e) = 0\}$ corresponding to the zeros of g; and the spatially nonhomogeneous stationary solutions, i.e. the subset $\mathcal{F}_g$ of frozen waves,

$$\mathcal{E}_g = \mathcal{Z}_g \cup \mathcal{F}_g \ . \tag{6}$$

Furthermore, hyperbolicity of an equilibrium $e \in \mathcal{Z}_g$ occurs if and only if $g'(e) \neq 0$.

In association with (5) we have the *period map* $T = T(a) : D \subset \mathbb{R} \to \mathbb{R}_+$ which is given by the minimal period of the solution $v = v(x, a)$ that satisfies Neumann initial conditions:

$$v(0,a) = v(T(a), a) = a \quad , \quad v_x(0,a) = v_x(T(a), a) = 0 \ . \tag{7}$$

The nonhomogeneous stationary solutions $u \in \mathcal{F}_g$ of $(1)_g$ correspond to the solutions $v = v(\cdot, a)$ of (5) for which $T(a) = 2\pi/\ell$, $\ell \in \mathbb{N}$. Moreover, they are hyperbolic if and only if $T'(a) \neq 0$. The integers ℓ here are called the *periodic lap numbers* of the stationary solutions v. See [8] and [9] for details.

Reference [17] gives a complete characterization of the period maps $T = T_g$ in terms of the Morse type of the potential function $G(u) = \int_0^u g(s)ds$ and the sequences of positive integers corresponding to the periodic lap numbers of the 2π-periodic solutions of (5). See also [8].

In the generic situation, the potential G is a Morse function, that is, all its critical points are nondegenerate (i.e. $g(e) = 0$ implies $g'(e) \neq 0$) and all its critical values are distinct. Moreover, all 2π-periodic orbits of (5) are hyperbolic. In this case, let $\mathcal{P}_n$ denote the set of all period maps $T = T_g$ corresponding to nonlinearities g with exactly n zeros. Then, the phase portrait of (5) contains $(n-1)/2$ centers, corresponding to the local minima of G, and a bounded open set of periodic orbits, which we call the *cyclicity set* $\mathcal{C}$, see [7].

The cyclicity set $\mathcal{C}$ decomposes into $(n-1)/2$ punctured disks (one for each center) and l annular regions, with $1 \leq l \leq (n-3)/2$. One easily verifies that the number l depends only of the Morse type of G determined by the ordering of the values assumed by G on the $(n+1)/2$ saddles corresponding to the local maxima. Let $K = (m-1)/2 + l$ denote the total number of connected regions in the cyclicity set, i.e. the punctured disks and annular regions $\mathcal{C}_r, 1 \leq r \leq K$. Then

$$\mathcal{C} = \bigcup_{1 \leq r \leq K} \mathcal{C}_r \ . \tag{8}$$

These regions are partially ordered by the nesting of the periodic orbits in the phase plane. In fact, the combinatorial structure given by the nesting of the periodic orbits establishes a total ordering on the set of connected regions. The idea is to notice that the nesting of the periodic orbits is a *regular bracket structure*, like the structure of the parentheses used in arithmetic expressions. Then, we use the total ordering of the left brackets in the regular bracket structure. We refer to [17] and [8] for details.

The domain D of $T_g \in \mathcal{P}_n$ consists of a finite number of open bounded intervals related to the punctured disks and annular regions that form the connected components $\mathcal{C}_r$ of the cyclicity set. To each region is associated a (possibly empty) sequence

$$S_r = (\ell_1^r, \ldots, \ell_{s_r}^r) \tag{9}$$

of s_r positive integers corresponding to the periodic lap numbers of the 2π-periodic solutions ordered by their minimum (initial) values $a \in D$. Then, we have a collection of $(n-1)/2+l$ sequences of positive integers,

$$S = (S_r)_{1\le r\le K} \ , \tag{10}$$

which is called the *lap signature* of the period map T_g, see [17], [8].

By explicitly computing a nonlinearity g for which $T = T_g$, we obtain:

Theorem 1: *(See* [17]*, Proposition 3.) A collection of sequences of positive integers $S = (S_r)_{1\le r\le K}$ is the lap signature of a period map $T \in \mathcal{P}_n$ corresponding to the regular bracket structure of a Morse type, if and only if:*

(i) *each sequence $S_r = (\ell_1, \dots, \ell_{s_r}\}$ corresponding to a punctured disk satisfies*

$$\ell_1 = 1 \quad , \quad |\ell_{j+1} - \ell_j| \le 1 \ , \ j = 1, \dots, s_r - 1 \ , \tag{11}$$

and, if $\ell_{i-1} \ne \ell_i = \cdots = \ell_j \ne \ell_{j+1}$ for $1 < i \le j < s_r$, then

$$(\ell_i - \ell_{i-1})(\ell_{j+1} - \ell_j) = (-1)^{i-j} \ ; \tag{12}$$

(ii) *each sequence $S_r = (\ell_1, \dots, \ell_{s_r})$ corresponding to an annular region satisfies Eqs. (11), (12) and*

$$\ell_{s_r} = 1 \text{ with } s_r \text{ even.} \tag{13}$$

We say that two nonlinearities g_0 and g_1 belong to the same *lap signature class* if the corresponding potential functions G_0 and G_1 have the same Morse type and the corresponding period maps, T^0 and T^1, have the same lap signature. In this case, using the results of [17] (see also [8]), we can construct a homotopy $T^\tau, \tau \in [0,1]$, between the period maps along which the lap signature of T^τ and the hyperbolicity of the equilibria are preserved. Then, from the explicit computation of a nonlinearity $g_\tau = g_\tau(u)$ for each period map T^τ, fixing the dissipative behavior outside a large interval $|u| > M$, we obtain a homotopy between the two nonlinearities that preserves hyperbolicity of all the equilibria. This shows that the lap signature class of $g = g(u)$ determines the global attractor $\mathcal{A}_g$ up to orbit equivalence and proves the following result:

Theorem 2: *If $g_0 = g_0(u)$ and $g_1 = g_1(u)$ belong to the same lap signature class then*

$$\mathcal{A}_{g_0} \cong \mathcal{A}_{g_1} \ . \tag{14}$$

4. The class of nonlinearities $f = f(u, u_x)$

Due to the existence of the freezing and symmetrizing homotopy mentioned in Section 2 the discussion of orbit equivalence only needs to be conducted in the class of reversible nonlinearities, that is the nonlinearities $f = f(u, u_x)$ which are even functions of the second variable

$$f(v, p) = f(v, -p) \ . \tag{15}$$

In the class of reversible nonlinearities the period map $T = T_f$ is defined using the first integral $H = H(u, u_x)$ also mentioned in Section 2. Moreover, the previous equivalence characterization result can easily be extended to this class when the period map T is realizable in the class of nonlinearities $f = f(u)$. In fact, let $f_0 = f_0(u, u_x)$ and $f_1 = f_1(u)$ denote two nonlinearities producing the same period map $T : D \to \mathbb{R}_+$, and let H_0 and H_1 denote the respective first integrals. In the following, let $(v, p_j(v))_{j=0,1}$ describe the orbit of $(2)_{f_j}$ through $(a, 0)$ corresponding to the periodic solution $v = v(\cdot, a)$ with minimum value $a \in D$. Also recall that the period map in both cases, $j = 0, 1$, has the form

$$T_{f_j}(a) = 2 \int_a^{b(a)} \frac{dv}{p_j(v)} = T(a) \ . \tag{16}$$

The maps $Q_j : (x, a) \mapsto (v, p_j(v))$ are local diffeomorphisms onto the cyclicity sets of each phase portrait. Then, following Lemma 5.1 of [7] we have that

$$\frac{1}{p_\tau} = \tau \frac{1}{p_1} + (1 - \tau) \frac{1}{p_0} \tag{17}$$

provides a homotopy between the phase portraits of both Eqs. $(2)_{f_j}$ along which the period map T is preserved. Outside the $(v, 0)$-axis the argument essentially follows from the computation

$$JQ_\tau = \tau \left(\frac{p_\tau}{p_1} \right)^2 JQ_1 + (1 - \tau) \left(\frac{p_\tau}{p_0} \right)^2 JQ_0 \ , \tag{18}$$

where $JQ = \det(Q_x, Q_a)$ denotes the Jacobi determinant. From (17) we conclude that $Q_\tau : (x, a) \mapsto (v, p_\tau(v))$ is a local diffeomorphism between the cyclicity sets $\mathcal{C} = \mathcal{C}^0$ and $\mathcal{C}^\tau$ for each $\tau = [0, 1]$. Therefore, we again conclude that the global attractors, $\mathcal{A}_{f_0} \cong \mathcal{A}_{f_1}$, are orbit equivalent.

5. The class of nonlinearities $f = f(u, u_x)$ of simple type

The general discussion of orbit equivalence between attractors in the class of nonlinearities $f = f(u, u_x)$ requires additional arguments. In fact, given a function T the existence of a nonlinearity $g = g(u)$ for which $T_g = T$ is not assured due to the existence of constraints on the realization of period maps by nonlinearities $g = g(u)$. These constraints assume the form of a limitation on the negative values of the derivative of an appropriate rescaled version of T. See [17] and [18] for details.

A small contribution to this discussion is obtained by considering very simple phase plane diffeomorphisms which preserve the period map T_f.

Consider a reversible nonlinearity $g = g(v, p)$ and let

$$(v, p(v)) = (v(\cdot, a), p(v(\cdot, a))) \tag{19}$$

describe the periodic orbits of $(2)_g$ on the phase plane, where $v = v(\cdot, a)$ denotes the solution with $v(0, a) = a, v_x(0, a) > 0$. Let $\mathcal{C}_g$ denote the cyclicity set on the phase plane of $(2)_g$ and let $\mathcal{C}_0 \subset \mathcal{C}_g$ denote a region corresponding to a punctured disk. Moreover, let $\Phi : \mathcal{C}_0 \to \mathbb{R}^2$ denote the scaling map

$$\Phi(v, p) = (\Omega(v, p)v, \Omega(v, p)p) \tag{20}$$

where $\Omega : \mathcal{C}_0 \to \mathbb{R}$ is assumed to be constant on the periodic orbits of $(2)_g$ in $\mathcal{C}_0$, i.e.

$$\Omega(v(\cdot, a), p(v(\cdot, a))) = \Omega(a, 0)\ . \tag{21}$$

On the appropriate interval $(a_-, a_+) \subset \mathbb{R}$ let $\omega : (a_-, a_+) \to \mathbb{R}$ denote the function given by $\omega(a) = \Omega(a, 0)$. Furthermore let ω be continuously differentiable and assume only positive values, $\omega(a) > 0$. Then, the scaling map Φ preserves the periodic orbits up to the scale change $\omega(a) = \Omega(a, 0)$ which depends smoothly on the orbits. Notice that for constant $\omega = 1$ the scaling map Φ is the identity in $\mathcal{C}_0$.

In general Φ is not injective, but since the Jacobi determinant $J\Phi(v, p)$ satisfies

$$J\Phi(v, p) = \Omega(v, p)\ (\Omega(v, p) + \nabla\Omega(v, p) \cdot (v, p))\ , \tag{22}$$

we conclude that Φ can be extended to a global plane diffeomorphism if Ω satisfies the condition

$$\Omega(v, p) + \nabla\Omega(v, p) \cdot (v, p) > 0\ . \tag{23}$$

We then construct a homotopy $\Phi^\tau, \tau \in [0, 1]$, between the identity $\Phi^0 =$ id in $\mathcal{C}_0$ and the scaling map $\Phi^1 = \Phi$ of the form

$$\Phi^\tau(v, p) = (\Omega^\tau(v, p)v, \Omega^\tau(v, p)p) \tag{24}$$

by taking $\omega^\tau(a) = \Omega^\tau(a, 0)$ of the form

$$\omega^\tau = (1 - \tau) + \tau\omega \quad , \quad \tau \in [0, 1]\ . \tag{25}$$

Then Ω^τ is a linear convex combination of Ω and $\Omega^0 \equiv 1$. Hence Ω^τ also satisfies condition (23) for all $\tau \in [0, 1]$.

On each cyclicity set $\Phi^\tau(\mathcal{C}_0)$ resulting from the (nonlinear) scaling Φ^τ we obtain a vector field corresponding to the scaling of $(2)_g$. The resulting nonlinearity has the form

$$f_\tau(v,p) = \Omega^\tau(v,p) g(v/\Omega^\tau(v,p), p/\Omega^\tau(v,p)) \quad , \quad \tau \in [0,1] \tag{26}$$

and, after an appropriate extension to all $(v,p) \in \mathbb{R}^2$, constitutes a homotopy in the class of reversible nonlinearities $f = f(u, u_x)$.

The remarkable property of the scaling (20) is the invariance of the period map up to the scale change. In fact, the periods of the orbits $(v(\cdot,a), p(v(\cdot,a)))$ and $(\omega v(\cdot,a), \omega p(v(\cdot,a)))$ are equal. Computing the period map T_{f_τ} for $\tau \in [0,1]$, using that Ω^τ is constant along the periodic orbits

$$\Omega^\tau(v(\cdot,a), p(v(\cdot,a))) = \omega^\tau(a) \ , \tag{27}$$

we obtain

$$T_{f_\tau}(\omega^\tau(a)a) = 2\int_{\omega^\tau(a)a}^{\omega^\tau(a)b(a)} \frac{d\tilde{v}}{\omega^\tau(a)p(\tilde{v}/\omega^\tau(a))} = 2\int_a^{b(a)} \frac{dv}{p(v)} = T_g(a) \ . \tag{28}$$

Therefore, the period map T_{f_τ} is preserved up to the (nonlinear) scale $\omega^\tau(\cdot)$.

We then attempt to choose $\omega = \omega(a)$ such that the period map T_{f_1} satisfies the constraints on the realization of period maps by nonlinearities $g = g(u)$ mentioned in the beginning of this Section. Notice that condition (23) can always be ensured in $\mathcal{C}_0$. In fact, by adding a constant to $\omega(a)$, the term Ω can always be made larger than the term with $\nabla\Omega$.

Therefore, in the restricted region of a punctured disk, there exists a homotopy in the class of reversible $f = f(u, u_x)$ from $g = g(u, u_x)$ to a nonlinearity $f_0 = f_0(u, u_x)$ such that T_{f_0} is realizable in the class of nonlinearities $f = f(u)$. Moreover, this operation can be performed sequentially on all the punctured disk regions of $g = g(u, u_x)$ without interference between regions.

Consequently, we obtain a restricted class of reversible nonlinearities $f = f(u, u_x)$ for which the previous homotopy can be extended to all the phase space (v,p). We say that a reversible $f = f(u, u_x)$ is of *simple type* if each 2π-periodic orbit of (2) encircles exactly one center in the phase plane (v,p). Hence, for the lap signature class of T_f all the sequences S_r corresponding to annular regions of the cyclicity set $\mathcal{C}_f$ are empty.

Therefore, in the restricted class of reversible nonlinearities $f = f(u, u_x)$ of simple type the constraints on the realization of period maps by nonlinearities $g = g(u)$ can be circumvented, and we obtain

Theorem 3: *If $f = f(u, u_x)$ and $g = g(u, u_x)$ belong to the same lap signature class of simple type, then*

$$\mathcal{A}_f \cong \mathcal{A}_g \ . \tag{29}$$

In general, however, the possible constraints on the realization of period maps by nonlinearities $g = g(u)$ remains a serious drawback on the discussion of orbit equivalence of global attractors. In fact we miss an adequate extension of the scaling map Φ to the annular regions $\mathcal{C}_r$ of the cyclicity set. This is due to the interference with the previous construction of the scaling map on the (enclosed) punctured disk regions. Therefore, the general construction remains an open problem.

Acknowledgments: This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT/Portugal).

References

[1] S. Angenent and B. Fiedler. The dynamics of rotating waves in scalar reaction diffusion equations. *Trans. Amer. Math. Soc.*, **307**, 545–568 (1988).

[2] A. V. Babin and M. I. Vishik. *Attractors of Evolution Equations.* North Holland, Amsterdam, 1992.

[3] R. Czaja and C. Rocha. Transversality in scalar reaction-diffusion equations on a circle. *J. Differential Equations*, **245**, 692–721 (2008).

[4] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations, *J. Differential Equations*, **125**, 239–281 (1996).

[5] B. Fiedler and J. Mallet-Paret. The Poincaré-Bendixson theorem for scalar reaction diffusion equations. *Arch. Rational Mech. Anal.*, **107**, 325–345 (1989).

[6] B. Fiedler and C. Rocha. Orbit equivalence of global attractors of semilinear parabolic differential equations. *Trans. Amer. Math. Soc.*, **352**, 257–284 (2000).

[7] B. Fiedler, C. Rocha and M. Wolfrum. Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle, *J. Differential Equations*, **201**, 99–138 (2004).

[8] B. Fiedler, C. Rocha and M. Wolfrum. A permutation characterization of Sturm global attractors of Hamiltonian type. *J. Differential Equations*, **252**, 588–623 (2012).

[9] B. Fiedler, C. Rocha and M. Wolfrum. Sturm global attractors for S^1-equivariant parabolic equations. *Netw. Heterog. Media*, **7**, 617–659 (2012).

[10] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Differential Equations*, **91**, 75–94 (1991).

[11] J. K. Hale. *Asymptotic Behavior of Dissipative Systems.* Math. Surv. **25**. AMS Publications, Providence, 1988.

[12] J. K. Hale, L. T. Magalhães and W. M. Oliva. *Dynamics in infinite dimensions. Second edition.* Applied Mathematical Sciences, 47. Springer–Verlag, New York, 2002.

[13] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Lect. Notes in Math., Vol. 840. Springer, New York, 1981.

[14] R. Joly and G. Raugel. Generic Morse-Smale property for the parabolic equation on the circle. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **27**, 1397–1440, (2010).

[15] H. Matano and K.–I. Nakamura. The global attractor of semilinear parabolic equations on S^1. *Discrete Contin. Dyn. Syst.*, **3**, 1–24 (1997).

[16] C. Ragazzo. Scalar autonomous second order ordinary differential equations. *Qual. Theory Dyn. Syst.*, **11**, 277-415 (2012).

[17] C. Rocha. Realization of period maps of planar Hamiltonian systems. *J. Dynam. Differential Equations*, **19**, 571–591 (2007).

[18] R. Schaaf. *Global solution branches of two point boundary value problems.* Lect. Notes in Math, Vol. 1458. Springer–Verlag, New York, 1990.

[19] M. Wolfrum. A sequence of order relations, encoding heteroclinic connections in scalar parabolic PDE. *J. Differential Equations*, **183**, 56–78 (2002).