

## Local regularity in non-linear generalized functions

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**Abstract.** In this review article we present regularity properties of generalized functions which are useful in the analysis of non-linear problems. It is shown that Schwartz distributions embedded into our new spaces of generalized functions, with additional properties described through the association, belong to various classical spaces with finite or infinite type of regularities.

### 1. Introduction

Generalized function algebras of Colombeau type contain Schwartz's distribution spaces and the embeddings preserve all the linear operations for distributions. A great advantage of the generalized function algebra approach is that various classes of nonlinear problems can be studied in these frames as well as linear problems with different kinds of singularities. We refer to [5], [7], [8], [16] and [26] for the theory of generalized function algebras and their use in the study of various classes of equations. For

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the purpose of local and microlocal analysis, one is lead to study classical function spaces within these algebras.

In this survey paper we will present our investigations related to local analysis. We mention that in number of papers we have also studied wave front sets of  $\mathcal{G}^\infty$  and analytic types. The attention in article will be focused on regularity theory in generalized function algebras. This regularity theory is parallel to the corresponding one within distribution spaces related to analytic, real analytic, harmonic and Besov type spaces, especially to Zygmund type spaces.

Elements of algebras of generalized functions are represented by nets  $(f_\varepsilon)_\varepsilon$  of smooth functions, with appropriate growth as  $\varepsilon \rightarrow 0$ , so that the spaces of Schwartz's distributions are embedded into the corresponding algebras, and that for the space of smooth functions the corresponding algebra of smooth generalized function is  $\mathcal{G}^\infty$  (see [26], [44]). Elements of these algebras are obtained through the regularization of distributions (convolving them with delta nets) and the construction of appropriate algebras of moderate nets and null nets of smooth functions and their quotients, as Colombeau did ([8]) with his algebra  $\mathcal{G}(\mathbb{R}^d)$ , in such a way that distributions are included as well as their natural linear operations.

The main goal of our investigations has been to find out conditions with respect to the growth order in  $\varepsilon$  or integrability conditions with respect to  $\varepsilon$  which characterize generalized function spaces and algebras with finite type regularities. Our definitions for such generalized function spaces enable us to obtain information on the regularity properties of Schwartz distributions embedded into the corresponding space of generalized functions.

One can find many articles in the literature where local and microlocal properties of generalized functions in generalized function algebras have been considered. Besides the quoted monographs we refer to the papers [3], [4], [9], [10] [12]–[14], [17], [20]–[25], [27]–[37], [40], [43]. We also mention that some Tauberian theorems for regularizing transforms, [33], [38], [39], [11], are also valuable tools for the study of regularity properties of generalized functions.

The paper is organized as follows. In Section 2 we recall basic definitions of Colombeau type algebras, as well as basic notions related to Littlewood-Paley decompositions, Besov and Zygmund type spaces. Holomorphic generalized functions are described in Section 3, whereas the real analytic functions are presented in Section 4. We note that a fine contribution to the understanding of these classes has been given by Aragona and his collaborators, as well as Colombeau as a founder of the theory. Differences between classical and generalized harmonic functions are presented as well as the fact that for the analytic and real analytic generalized functions standard points are sufficient for their description. This is not the case

for the harmonic generalized functions, whose treatment needs additional preparation. Harmonic generalized functions with specific notions of  $H$ -boundedness and generalized removable singularities are reviewed in Section 5. Following [6], we discuss various properties of generalized functions with dependence on the variable  $\varepsilon$ . Continuity, smoothness and the measurability condition with respect to  $\varepsilon$  are discussed in Section 6. Actually in this section we introduce for the first time in the literature the analysis of generalized functions through the integration with respect to  $\varepsilon$ . More precisely, we introduce in Section 6 the spaces  $\mathcal{G}_q(\Omega)$ , where the growth order is measure in an  $L^q$  space with respect to the variable  $\varepsilon$ . In Sections 7 and 8 we present Besov type spaces of generalized functions through the classes  $G_q(\Omega)$ . The regularity type results from Section 8 are rather recent. The last section, Sections 9, is devoted to Zygmund type spaces. These spaces are essentially related to the process of regularizations of Schwartz distributions which enable us to give precise characterizations of regularity properties of Schwartz distributions embedded into the corresponding space of Zygmund type generalized functions.

Let us note that many other authors have made great contributions to developing the local and microlocal analysis of generalized functions of Colombeau type. Our collaborators Oberguggenberger, Vernaev and Valmorin contributed very much to the results presented in this paper. Among the main contributors to the local and micro local theory of Colombeau type generalized functions, we have to underline the role of Colombeau, Oberguggenberger, Aragona, Hörmann, Kunzinger, Vernaev, Gareto, Marti, Valmorin, as well as their (and our ) coauthors. In this sense, our list of references is rich enough but not complete.

## 2. Colombeau algebras and spaces

Let  $\Omega$  be an open subset of  $\mathbb{R}^d$ . We consider the families of local Sobolev seminorms  $\|\phi\|_{W^{m,p}(\omega)} = \sup\{\|\phi^{(\alpha)}\|_{L^p(\omega)}; |\alpha| \leq m\}$ , where  $m \in \mathbb{N}_0$ ,  $p \in [1, \infty]$ , for  $\omega \subset\subset \Omega$  (which means that  $\bar{\omega}$  is compact in  $\Omega$ ).

The spaces of moderate nets and negligible nets  $\mathcal{E}_{L_{loc}^p, M}(\Omega)$  and  $\mathcal{N}_{L_{loc}^p}(\Omega)$ , resp.,  $\mathcal{E}_{L^p, M}(\Omega)$  and  $\mathcal{N}_{L^p}(\Omega)$ , consist of nets  $(f_\varepsilon)_{\varepsilon \in (0,1)} = (f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1)}$  with the properties (given by Landau's big  $O$  and small  $o$ )

$$(\forall m \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\exists a \in \mathbb{R})(\|f_\varepsilon\|_{W^{m,p}(\omega)} = O(\varepsilon^a)) \quad (2.1)$$

$$\text{and } (\forall m \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\forall b \in \mathbb{R})(\|f_\varepsilon\|_{W^{m,p}(\omega)} = O(\varepsilon^b)),$$

The Sobolev lemma implies  $\mathcal{E}_{L_{loc}^p}(\Omega)$  and  $\mathcal{N}_{L_{loc}^p}(\Omega)$  are algebras, but also that

$$E_M(\Omega) = E_{L_{loc}^\infty}(\Omega) = \mathcal{E}_{L_{loc}^p}(\Omega), \quad \mathcal{N}(\Omega) = \mathcal{N}_{L_{loc}^\infty}(\Omega) = \mathcal{N}_{L_{loc}^p}(\Omega), \quad p \geq 1.$$

Thus the Colombeau algebra of generalized functions can be defined as

$$\mathcal{G}(\Omega) = \mathcal{E}_{L^p_{loc}}(\Omega) / \mathcal{N}_{L^p_{loc}}(\Omega), \quad p \geq 1.$$

Recall [26], that the algebra of regular generalized functions as defined as  $\mathcal{G}^\infty(\Omega) = \mathcal{E}_M^\infty(\Omega) / \mathcal{N}(\Omega)$ , where  $\mathcal{E}_M^\infty(\Omega)$  consists of nets  $(f_\varepsilon)_{\varepsilon \in (0,1)} \in \mathcal{E}(\Omega)^{(0,1)}$  with the property

$$(\forall K \subset\subset \Omega)(\exists a \in \mathbb{R})(\forall n \in \mathbb{N})(|\sup_{x \in K} f_\varepsilon^{(n)}(x)| = O(\varepsilon^a)).$$

If the elements of the nets  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$  are constant functions in  $\Omega$  (i.e., seminorms reduce to the absolute value), then one obtains the corresponding algebras  $\mathcal{E}_0$  and  $\mathcal{N}_0$ ;  $\mathcal{N}_0$  is an ideal in  $\mathcal{E}_0$  and, as their quotient, one obtains the Colombeau algebra of generalized complex numbers:  $\bar{\mathbb{C}} = \mathcal{E}_0 / \mathcal{N}_0$  (or  $\bar{\mathbb{R}}$  if nets are real). It is a ring, not a field. For the analysis of harmonic generalized functions we will recall the definition of compactly supported generalized points, due to Oberguggenberger and Kunzinger in [27]. A net  $(x_\varepsilon)_\varepsilon$  in a general metric space  $(A, d)$  is called moderate, if

$$(\exists N \in \mathbb{N})(\exists x \in A)(d(x, x_\varepsilon) = O(\varepsilon^{-N})).$$

and an equivalence relation in  $A^{(0,1]}$  is introduced by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow (\forall p \geq 0)(d(x_\varepsilon, y_\varepsilon) = O(\varepsilon^p)).$$

$\tilde{A} = A / \sim$  is called the set of generalized points in  $A$ . If  $A = \Omega$  is an opens subset of  $\mathbb{R}^d$ , then  $\tilde{\Omega} = \Omega / \sim$  is the set of generalized points. Note  $\tilde{\mathbb{C}} = \bar{\mathbb{C}}$  ( $\tilde{\mathbb{R}} = \bar{\mathbb{R}}$ ). An element  $\tilde{x} \in \tilde{\Omega}$  is called compactly supported if  $x_\varepsilon$  lies in a compact set for  $\varepsilon < \varepsilon_0$  for some  $\varepsilon_0 \in (0, 1)$ . The set of compactly supported points  $\tilde{x} (\in \tilde{\Omega})$  is denoted by  $\tilde{\Omega}_c$ . Recall, nearly standard points are elements  $\tilde{x} \in \tilde{\Omega}_c$  with limit in  $\Omega$  that is, there exists  $x \in \Omega$  such that for a representative  $(x_\varepsilon)_\varepsilon$ ,  $x_\varepsilon \rightarrow x$ , as  $\varepsilon \rightarrow 0$ , holds. We denote by  $\tilde{\Omega}_{ns}$  the set of nearly standard points of  $\Omega$ .

The embedding of the Schwartz distribution space  $\mathcal{E}'(\Omega)$  into  $\mathcal{G}(\Omega)$  is realized through the sheaf homomorphism  $\mathcal{E}'(\Omega) \ni T \mapsto \iota(T) = [(T * \phi_\varepsilon | \Omega)_\varepsilon] \in \mathcal{G}(\Omega)$ , where the fixed net of mollifiers  $(\phi_\varepsilon)_\varepsilon$  is defined by  $\phi_\varepsilon = \varepsilon^{-d} \phi(\cdot/\varepsilon)$ ,  $\varepsilon < 1$ , and  $\phi \in \mathcal{S}(\mathbb{R}^d)$  satisfies

$$\int_{\mathbb{R}^d} \phi(t) dt = 1, \quad \int_{\mathbb{R}^d} t^m \phi(t) dt = 0, \quad |m| > 0.$$

$$(t^m = t_1^{m_1} \dots t_d^{m_d} \text{ and } |m| = m_1 + \dots + m_d).$$

This sheaf homomorphism [16], extended over  $\mathcal{D}'$ , gives the embedding of  $\mathcal{D}'(\Omega)$  into  $\mathcal{G}(\Omega)$ . We also use the notation  $\iota$  for the mapping from  $\mathcal{E}'(\Omega)$

into  $\mathcal{E}_M(\Omega)$ ,  $\iota(T) = (T * \phi_\varepsilon|_\Omega)_\varepsilon$ . Throughout this article,  $\phi$  is fixed and satisfies the above condition over its moments.

We will use a continuous Littlewood-Paley decomposition of the unity (see [42] or [19, Sect. 8.4], for instance). Let  $\varphi \in \mathcal{S}(\mathbb{R}^d)$  such that its Fourier transformation  $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^d)$  is a real valued radial (independent of rotations) function with support contained in the unit ball such that  $\hat{\varphi}(y) = 1$  if  $|y| \leq 1/2$ . Set  $\hat{\psi}(y) = -\frac{d}{d\varepsilon}\hat{\varphi}(\varepsilon y)|_{\varepsilon=1} = -y \cdot \nabla \hat{\varphi}(y)$ . The support of  $\hat{\psi}$  is contained in the set  $1/2 \leq |y| \leq 1$ . Then,  $\varphi$  has the same moment properties as  $\phi$  given above but in the sequel we will use term mollifier only for  $\phi$  since it will be used for the embedding of distributions into various spaces and algebras of generalized functions. The function  $\psi$  is a wavelet (all the moments of  $\psi$  are equal to zero).

One has [19, Sect. 8.6]: for any  $u \in \mathcal{S}'(\mathbb{R}^d)$ ,

$$u = u * \varphi + \int_0^1 u * \psi_\eta \frac{d\eta}{\eta} = u * \varphi_\varepsilon + \int_0^\varepsilon u * \psi_\eta \frac{d\eta}{\eta}, \quad 0 < \varepsilon \leq 1, \quad (2.2)$$

and hence,

$$u * \varphi_\varepsilon = u * \varphi + \int_\varepsilon^1 u * \psi_\eta \frac{d\eta}{\eta}, \quad 0 < \varepsilon \leq 1. \quad (2.3)$$

We recall that the Besov spaces  $B_{q,p}^s(\mathbb{R}^d)$ ,  $p, q \in (0, \infty]$ ,  $s \in \mathbb{R}$ , are defined as

$$B_{q,p}^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{q,p}^s < \infty\}, \text{ where}$$

$$\|u\|_{q,p}^s := \|f * \phi\|_{L^p(\mathbb{R}^d)} + \left( \int_0^1 y^{-sq} \|f * \psi_y\|_{L^p(\mathbb{R}^d)}^q dy/y \right)^{1/q}. \quad (2.4)$$

The definition is independent of the choice of the pair  $\varphi$ . If we discretize  $y = 2^{-j}$ ,  $j \in \mathbb{N}$ , then one can replace [42] the second term in (2.4) by

$$\left( \sum_{j=1}^{\infty} 2^{-jq} \|f * \psi_{2^{-j}}\|_{L^p(\mathbb{R}^d)}^q \right)^{1/q}.$$

They are quasi-Banach spaces if  $\min\{p, q\} \leq 1$ , and Banach spaces if  $\min\{p, q\} \geq 1$ . In the sequel we consider the second case i.e.  $\min\{p, q\} \geq 1$ .

In particular, the Zygmund spaces are defined by

$$C_*^s(\mathbb{R}^d) := \{u \in \mathcal{S}' : |u|_*^s := \|\phi * u\|_{L^\infty} + \sup_{y \in (0,1)} (y^{-s} \|\psi_y * u\|_{L^\infty}) < \infty\},$$

so that  $C_*^s(\mathbb{R}^d) = B_{\infty,\infty}^s(\mathbb{R}^d)$ . We will also consider spaces  $B_{q,L_{loc}^p}^s(\Omega)$  consisting of tempered distributions with the property:  $f \in B_{q,L_{loc}^p}^s(\Omega)$  if  $\theta f \in B_{q,p}^s(\mathbb{R}^d)$  for every  $\theta \in \mathcal{D}(\Omega)$ .

### 3. Holomorphic generalized functions [29]

We denote by  $\mathcal{O}(\Omega)$  the space of holomorphic functions on  $\Omega$ , where  $\Omega$  is an open subset of  $\mathbb{R}^2 = \mathbb{C}$ ;  $D(z_0, r)$  denotes a disc with the center  $z_0$  and radius  $r > 0$ .

**Definition 3.1.** A generalized function  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$  is said to be holomorphic if  $\frac{\partial f}{\partial \bar{z}} = 0$  in  $\mathcal{G}(\Omega)$ .

The set of holomorphic generalized functions is denoted by  $\mathcal{G}_H(\Omega)$ .

Due to Colombeau-Galé [9] we know that  $f \in \mathcal{G}_H(\Omega)$  if and only if for every relatively compact open set  $\Omega'$  in  $\Omega$ ,  $f$  admits a representative  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega')$  with  $f_\varepsilon \in \mathcal{O}(\Omega')$ ,  $\varepsilon \in (0, 1]$ . In the same paper it is shown that  $\mathcal{G}_H(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{O}(\Omega)$ . Holomorphic generalized functions can be well understood through the next theorem.

**Theorem 3.2.** Let  $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1]}$  and suppose that for every point  $z_0 \in \Omega$  there exist  $r_\varepsilon > 0$ ,  $0 < \varepsilon \leq 1$  such that

- (i)  $\bar{\partial} f_\varepsilon|_{D(z_0, r_\varepsilon)} = 0$ ,  $0 < \varepsilon \leq 1$ , i.e. the restrictions to the discs vanish.
- (ii)  $\exists \eta > 0, \exists a > 0, \exists \varepsilon_0 \in (0, 1]$ ,  $|f_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^{-a}$ ,  $n \in \mathbb{N}, \varepsilon \in (0, \varepsilon_0)$ .

Then  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$  and  $[(f_\varepsilon)_\varepsilon] \in \mathcal{G}_H(\Omega)$ .

This leads to the result that  $\mathcal{G}_H(\Omega) \subset \mathcal{G}^\infty(\Omega)$ , where the algebra of regular generalized functions  $\mathcal{G}^\infty(\Omega)$  was defined in Section ??

We collect properties of holomorphic generalized functions in the next theorem.

**Theorem 3.3.** Let  $g \in \mathcal{G}_H(\Omega)$ .

- (i)  $g$  admits a representative  $(g_\varepsilon)_\varepsilon$  such that  $g_\varepsilon \in \mathcal{O}(\Omega)$ ,  $\varepsilon \in (0, 1]$ .
- (ii)  $g = 0$  if and only if for any open set  $\Omega' \subset \subset \Omega$  there exists a representative  $(g_\varepsilon)_\varepsilon \in \mathcal{O}(\Omega')^{(0,1]}$  such that

$$\begin{aligned} \forall z_0 \in \Omega', \forall a > 0, \exists \eta > 0, \exists \varepsilon_0 \in (0, 1], \exists C > 0, \\ |g_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^a, \quad n \in \mathbb{N}, \varepsilon \in (0, \varepsilon_0). \end{aligned} \quad (3.1)$$

(iii) Let  $z_0 \in \Omega$ ,  $\eta > 0$ . Then  $f$  vanishes in the disc  $V = D(z_0, \frac{1}{\eta})$  if and only if there exists a representative  $(g_\varepsilon)_\varepsilon$  of  $f|_V$  in  $\mathcal{O}(V)^{(0,1]}$  such that

$$\forall a > 0, \exists \varepsilon_0 \in (0, 1] : |g_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^a; \quad n \in \mathbb{N}, \varepsilon \in (0, \varepsilon_0).$$

As a consequence, we obtain a simple proof that on a connected open set  $\Omega$  an  $f \in \mathcal{G}_H(\Omega)$  vanishes on  $\Omega$  if and only if it vanishes on a non-void open subset of  $\Omega$ . This is an important result of Khelif and Scarpalezos [22]. It shows a main property of holomorphic generalized functions. For their analysis, standard points are enough while for Colombeau type generalized functions, generalized points are essential [27] (see also [23] for more general aspects of this fact.)

The existence of a global holomorphic representative of  $f \in \mathcal{G}_H(\Omega)$  depends on  $\Omega \subset \mathbb{C}^d$  is still an open problem although such representation for appropriate domains  $\Omega \subset \mathbb{C}^n$  can be constructed. In the one dimensional case we have

**Theorem 3.4.** *If  $f \in \mathcal{G}_H(\mathbb{C})$ , then there exists one of its representative  $(f_\varepsilon)_\varepsilon$  consisting of entire functions on  $\mathbb{C}$ .*

#### 4. Real analytic generalized functions [35]

Now we are considering  $\omega$ , an open set in  $\mathbb{R}^d$ .

**Definition 4.1.** *Let  $x_0 \in \omega$ . A generalized function  $f \in \mathcal{G}(\omega)$  is said to be real analytic at  $x_0$  if there exist an open ball  $B = B(x_0, r)$  in  $\omega$  containing  $x_0$  and  $(g_\varepsilon)_\varepsilon \in \mathcal{E}_M(B)$  such that*

$$\begin{aligned} (i) \quad & f|_B = [(g_\varepsilon)_\varepsilon] \text{ in } \mathcal{G}(B); \\ (ii) \quad & (\exists \eta > 0)(\exists a > 0)(\exists \varepsilon_0 \in (0, 1)) \\ & \sup_{x \in B} |\partial^\alpha g_\varepsilon(x)| \leq \eta^{|\alpha|+1} \alpha! \varepsilon^{-a}, \quad 0 < \varepsilon < \varepsilon_0, \quad \alpha \in \mathbb{N}^d. \end{aligned}$$

*It is said that  $f$  is real analytic in  $\omega$  if  $f$  is real analytic at each point of  $\omega$ . The space of all generalized functions which are real analytic in  $\omega$  is denoted by  $\mathcal{G}_A(\omega)$ .*

*The analytic singular support,  $\text{singsupp}_{g_a} f$ , is the complement of the set of points  $x \in \omega$  where  $f$  is real analytic.*

It follows from the definition that  $\mathcal{G}_A$  is a subsheaf of  $\mathcal{G}$ .

Using Stirling's formula it is seen that condition (ii) in Definition 4.1 is equivalent to

$$\begin{aligned} (iii) \quad & (\exists \eta > 0)(\exists a > 0)(\exists \varepsilon_0 \in (0, 1)) \\ & \sup_{x \in B} |\partial^\alpha g_\varepsilon(x)| \leq \eta(\eta|\alpha|)^{|\alpha|} \varepsilon^{-a}, \quad 0 < \varepsilon < \varepsilon_0, \quad \alpha \in \mathbb{N}_0^d. \end{aligned}$$

The use of Taylor expansion and condition (ii) of Definition 4.1 imply that  $(g_\varepsilon)_\varepsilon$  admits a holomorphic extension in a complex ball  $B = \{z \in \mathbb{C}^d; |z - x_0| < r\}$  which is independent of  $\varepsilon$ . Consequently we get a holomorphic extension  $G$  of  $[(g_\varepsilon)_\varepsilon]$  and then  $f|B = G|B$ . (It is clear from the context whether  $B$  is a complex or real ball.)

The existence of a global real analytic representative of  $f \in \mathcal{G}_A(\mathbb{R}^d)$ , as in the case of analytic generalized functions is also an open problem. In the case  $d = 1$  we have a positive answer, there exist a global representative, i.e. a representative a real analytic generalized function on  $\mathbb{R}$  consisting of real analytic functions defined on  $\mathbb{R}$ .

Moreover, we have a similar situation for real analytic generalized functions as for holomorphic, they are determined by their values at standard points. We have

**Theorem 4.2.** *a) Let  $\Omega$  be an open set of  $\mathbb{C}^p, p > 1$  and  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_H(\Omega)$  such that  $f(x) = 0$  for every  $x \in \Omega$  ( $(f_\varepsilon(x))_\varepsilon \in \mathcal{N}(\Omega)$ ). Then  $f \equiv 0$ .*

*b) Let  $\omega$  be an open set of  $\mathbb{R}^d, d \in \mathbb{N}$  and  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_A(\omega)$  such that  $f(x) = 0$  for every  $x \in \omega$ . Then  $f \equiv 0$ .*

The singular support of an  $f \in \mathcal{G}(\omega)$  is defined as the complement of the union of open sets in  $\omega$  where  $f \in \mathcal{G}^\infty(\omega)$ . In a similar way one defines the notion  $\text{singsupp}_{ga} f$ .

**Theorem 4.3.** [35] *Let  $f \in \mathcal{E}'(\omega)$  and  $f_\varepsilon = f * \phi_\varepsilon, \varepsilon \in (0, 1)$  be its regularization by a net  $\phi_\varepsilon = \varepsilon^{-1} \phi(\cdot/\varepsilon), \varepsilon < 1$ , where  $(\phi_\varepsilon)_\varepsilon$  is a net of mollifiers. Then*

$$\text{singsupp}_a f = \text{singsupp}_{ga} [(f_\varepsilon)_\varepsilon].$$

## 5. Harmonic generalized function [36]

We denote by  $\text{Har}(\Omega)$  the space of harmonic functions in  $\Omega$ .

**Definition 5.1.** *We call a generalized function  $G \in \mathcal{G}(\Omega)$  harmonic generalized function, if  $\Delta G = 0$  holds in  $\mathcal{G}(\Omega)$ . The linear space of harmonic generalized functions in  $\Omega$  is denoted by  $\mathcal{G}_{\text{Har}}(\Omega)$ .*

We have shown that  $\mathcal{G}_{\text{Har}}$  is a closed subsheaf of the sheaf of  $\tilde{\mathbb{C}}$ -modules  $\mathcal{G}$ .

Moreover we have the next important result

**Theorem 5.2.** *Every harmonic generalized function  $G \in \mathcal{G}(\Omega)$  admits a global harmonic representative  $(G_\varepsilon)_\varepsilon$ , that is, for each  $\varepsilon \leq 1$ ,  $G_\varepsilon$  is harmonic.*



We call the  $(G_\varepsilon)_\varepsilon$  a global harmonic representative of  $G$ . For the main results of our analysis we use global harmonic representatives.

Obviously,  $\Omega \rightarrow \mathcal{G}_A(\Omega)$  is a subsheaf of the sheaf of algebras  $\mathcal{G}^\infty$ . Moreover, every harmonic generalized function is a real analytic generalized function. Since  $\mathcal{G}_{Har}(\Omega)$  is a submodule of  $\mathcal{G}_A(\Omega)$  the following consequence is immediate (see the previous section).

**Theorem 5.3.** *Let  $\Omega$  be a connected open subset of  $\mathbb{R}^d$  and  $f \in \mathcal{G}_{Har}(\Omega)$ . If there exists  $A \subset \Omega$  of positive Lebesgue measure ( $\mu(A) > 0$ ) such that  $f(x) = 0$  for every  $x \in A$ , then  $f \equiv 0$ .*

For the generalized maximum principle, we need additional notation. Let  $K$  be a compact set of  $\Omega$ ,  $\tilde{x}_0 \in \tilde{\Omega}_c$  be supported by  $K$  and let  $r > 0$  such that  $K + B(0, r) \subset \subset \Omega$ . With such an  $\tilde{x}_0$ , we denote by  $B(\tilde{x}_0, r)$  the following subset of  $\tilde{\Omega}_c$

$$B(\tilde{x}_0, r) = \{\tilde{t} = [(t_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c; |x_{0,\varepsilon} - t_\varepsilon| \leq r, \varepsilon \leq 1\}. \quad (5.1)$$

We call the set  $B(\tilde{x}_0, r) \subseteq \tilde{\Omega}_c$  a semi-ball in  $\Omega$ . Note that by now we distinguish between balls  $B(x_0, r) \subset \mathbb{K}$ , balls  $\tilde{B}(\tilde{x}_0, r)$  in  $\tilde{\mathbb{K}}$  and semi-balls  $B(\tilde{x}_0, r)$ .

**Theorem 5.4** (Maximum principle). *Let  $G$  be a real-valued harmonic generalized function in an open set  $\Omega$ . Then the following holds:*

- (1) *Let  $r > 0$  and  $\tilde{x}_0 \in \tilde{\Omega}_c$  with representative  $(x_{0,\varepsilon})_\varepsilon$  be given such that  $B(x_{0,\varepsilon}, 2r) \subset \subset \Omega$ ,  $\varepsilon \leq 1$  (that is the semi-ball  $B(\tilde{x}_0, 2r) \subset \tilde{\Omega}_c$ ). Suppose  $G(\tilde{x}_0) \geq G(\tilde{t})$  for each  $\tilde{t} \in B(\tilde{x}_0, r)$ . Then  $G$  is a constant generalized function in  $B(\tilde{x}_0, r)$ .*
- (2) *Let  $\Omega$  be connected. If there exists a compactly supported point  $\tilde{x}_0 \in \tilde{\Omega}_c$  such that  $G(\tilde{x}_0) \geq G(\tilde{t})$ ,  $\tilde{t} \in \tilde{\Omega}_c$ , then  $G$  is a constant generalized function in  $\Omega$ .*

**Proposition 5.5.** *Let  $G = [(G_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$  such that for every  $\tilde{x} = [(x_\varepsilon)_\varepsilon] \in \tilde{\Omega}_c$  and every  $R > 0$  such that the semi-ball  $\tilde{B}(\tilde{x}, R) \subset \tilde{\Omega}_c$ ,*

$$[(G_\varepsilon(\tilde{x}))_\varepsilon] = \left[ \left( \frac{1}{V_R} \int_{B(x_\varepsilon, R)} G_\varepsilon(t) dt \right)_\varepsilon \right]. \quad (5.2)$$

*Then  $G \in \mathcal{G}_{Har}(\Omega)$ .*

**Theorem 5.6.** *Let  $\Omega$  be connected.*

(i) *With the assumptions of Theorem 5.4 (i),  $G$  is a constant generalized function in  $\Omega$ .*

(ii) If there exists a nearly standard point  $\tilde{x}_0 \in \tilde{\Omega}_{ns}$  such that  $G(\tilde{x}_0) \geq G(\tilde{t})$  for each nearly standard point  $\tilde{t} \in \tilde{\Omega}_{ns}$ , then  $G$  is a constant generalized function in  $\Omega$ .

**Corollary 5.7.** (i) Let  $u$  be a complex harmonic generalized function in a connected open set  $\Omega$ . If  $|u|$  has a maximum  $\tilde{M} \in \tilde{\mathbb{R}}$  at  $\tilde{x}_0 \in \tilde{\Omega}_c$ , then  $u \equiv \tilde{A} = u(\tilde{x}_0) \in \tilde{\mathbb{C}}$ ,  $|\tilde{A}| = \tilde{M}$ .

(ii) Let  $G \in \mathcal{G}_{Har}(\Omega)$  be a non-constant real valued generalized function. Then

- (1)  $G$  does not attain its maximum inside  $\Omega$ , that is at a generalized point  $\tilde{t}_0 \in \tilde{\Omega}_c$ .
- (2) Let  $\Omega' \subset \subset \Omega$  be open. Then the maximum of  $G$  in  $\overline{\Omega'}$  is attained at a generalized point supported by the boundary of  $\Omega'$ .

For the generalizations of Liouville's theorem for harmonic generalized functions we need to repeat several notions for  $u \in \mathcal{G}(\mathbb{R}^d)$ .

We call  $u$ :

(i) non-negative, if for each compact set  $K$  there exists a representative  $(u_\varepsilon)_\varepsilon$  of  $u$  such that for each  $\varepsilon > 0$ ,  $\inf_{x \in K} u_\varepsilon(x) \geq 0$ .

(ii) strictly positive, if for each representative  $(u_\varepsilon)_\varepsilon$  of  $u$  and for each compact set  $K$  there exists constants  $m$  and  $\varepsilon_0$  such that for each  $\varepsilon < \varepsilon_0$ ,  $\inf_{x \in K} u_\varepsilon(x) \geq \varepsilon^m$ .

(iii) A harmonic generalized function  $u$  is said to be globally non-negative, if it admits a global harmonic representative  $(G_\varepsilon)_\varepsilon$  so that  $G_\varepsilon$  is non-negative for each  $\varepsilon \leq 1$ .

(iv) We call a harmonic generalized function  $u$  H-non-negative (and write  $u \geq_H 0$ ), if it admits a global harmonic representative  $(G_\varepsilon)_\varepsilon$  with the following property:

$$(\forall m > 0)(\forall a > 0)(\exists \varepsilon_{a,m} \in (0, 1])(\forall \varepsilon < \varepsilon_{a,m})(\forall t : |t| < \frac{1}{\varepsilon^m})(u_\varepsilon(t) + \varepsilon^a \geq 0) \quad (5.3)$$

Furthermore, a harmonic generalized function  $u$  is said to be H-bounded from above (resp. below) by  $\tilde{c} \in \tilde{\mathbb{R}}$ , if for a representative  $(c_\varepsilon)_\varepsilon$  of  $\tilde{c}$ , the global harmonic representative  $(G_\varepsilon)_\varepsilon - (c_\varepsilon)_\varepsilon$  satisfies condition (5.3). A harmonic generalized function  $u$  is said to be H-bounded if it is H-bounded from above and from below.

**Theorem 5.8.** A harmonic generalized function  $u$  in  $\mathbb{R}^d$  which is H-bounded from below is a constant.

A direct consequence is that every  $H$ -bounded harmonic generalized function  $u \in \mathcal{G}(\mathbf{R}^d)$  is a constant.

Now we give the definition of isolated singularity of harmonic generalized function.

Let  $\Omega$  be an open set of  $\mathbb{R}^d$  and  $x_0 \in \Omega$ . A generalized function  $G \in \mathcal{G}(\Omega \setminus \{x_0\})$  (resp.  $G \in \mathcal{G}_{Har}(\Omega \setminus \{x_0\})$ ) is said to have an isolated (resp. isolated harmonic) singularity at  $x_0$ . Moreover, if there exists  $F \in \mathcal{G}(\Omega)$  (resp.  $F \in \mathcal{G}_{Har}(\Omega)$ ) such that  $F|_{\Omega \setminus \{x_0\}} = G$ , then it is said that  $G$  has a removable (resp. harmonic removable) singularity.

Theorem 5.9 below states assertions on harmonic generalized functions in pierced domains. First we need a definition of  $H$ -boundedness in a neighbourhood of  $x_0$  which corresponds to a  $H$ -boundedness at infinity.

Let  $G \in \mathcal{G}_{Har}(\Omega \setminus \{x_0\})$  and let  $B(x_0, R) \subset \Omega$ . It is said that it is  $H$ -bounded in a neighbourhood of  $x_0$  if there exists  $M = [(M_\varepsilon)_\varepsilon] > 0$  and a global harmonic representative  $(G_\varepsilon)_\varepsilon$  in  $\Omega \setminus \{x_0\}$  such that for every  $m \in \mathbb{N}$  there exists  $\varepsilon_m \in (0, 1]$  such that

$$|G_\varepsilon(x)| < M_\varepsilon, x \in \{\varepsilon^m < |x - x_0| < R, \varepsilon < \varepsilon_m\}.$$

**Theorem 5.9.** *Let  $G \in \mathcal{G}_{Har}(\Omega \setminus \{x_0\})$ . The following holds:*

- (1) *Assume additionally that  $G \in \mathcal{G}(\Omega)$ , and that for every sharp neighbourhood  $V$  of  $x_0$   $G$  has a representative  $(G_\varepsilon)_\varepsilon$  so that for every  $\varepsilon \leq 1$ ,  $G_\varepsilon$  is harmonic outside  $V_\varepsilon$ , where  $V = [(V_\varepsilon)_\varepsilon]$ . Then  $G \in \mathcal{G}_{Har}(\Omega)$ .*
- (2) *If  $G$  is  $H$ -bounded at  $x_0$ , then  $G$  extends uniquely to an element of  $\mathcal{G}_{Har}(\Omega)$ .*

## 6. New spaces defined by integration in $\varepsilon$ [37]

Following [6], we will consider representatives  $(f_\varepsilon)_\varepsilon, (\varepsilon, x) \mapsto f_\varepsilon(x)$  which continuously depend on  $\varepsilon$  or (moreover) smoothly depend on  $\varepsilon \in (0, 1]$  (always smooth in  $x$ ). The notation  $_{co}$  stands for the continuous parametrization, while  $_{sm}$  stands for the smooth parametrization. It is obvious that

$$\mathcal{E}_{M,sm}(\Omega) \subset \mathcal{E}_{M,co}(\Omega) \subset \mathcal{E}_M(\Omega), \mathcal{N}_{sm}(\Omega) \subset \mathcal{N}_{co}(\Omega) \subset \mathcal{N}(\Omega),$$

Furthermore, it is shown in [6] that

$$\mathcal{G}_{co}(\Omega) = \mathcal{G}_{sm}(\Omega) \subset \mathcal{G}(\Omega),$$

where the last inclusion is strict. The same relations hold for generalized complex (and real) numbers.

Moreover, we will consider representatives  $(f_\varepsilon)_\varepsilon$  which are measurable functions with respect to  $\varepsilon$  :

for every fixed  $x \in \Omega$ ,  $(0, 1) \ni \varepsilon \mapsto f_\varepsilon(x) \in \mathbb{C}$  is measurable.

Let  $p \in [1, \infty]$ . The definitions of algebras  $\mathcal{E}_{L_{loc}^p, M}(\Omega) = \mathcal{E}_M(\Omega)$  and  $\mathcal{N}_{L_{loc}^p}(\Omega) = \mathcal{N}(\Omega)$  from Section 2 can be formulated by measurable representatives with respect to  $\varepsilon$ . We will denote them by the symbols  $\mathcal{E}_{M, me}(\Omega)$ ,  $\mathcal{N}_{me}(\Omega)$  and their quotient by  $\mathcal{G}_{me}(\Omega)$ , where we assume measurability dependence. The next proposition is also from [6].

**Proposition 6.1.** *The following strict embeddings hold*

$$\mathcal{E}_{M, co}(\Omega) \subset \mathcal{E}_{M, me}(\Omega); \mathcal{N}_{co}(\Omega) \subset \mathcal{N}_{me}(\Omega); \mathcal{G}_{co}(\Omega) \subset \mathcal{G}_{me}(\Omega) \subset \mathcal{G}(\Omega).$$

Let  $p \in [1, \infty]$ ,  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_{M, me}(\Omega)$  and  $\omega \subset\subset \Omega$ . Then it is clear that  $(0, 1) \ni \varepsilon \mapsto \|f_\varepsilon(\cdot)\|_{L^p(\omega)} \in \mathbb{R}$  is a measurable function.

**Example 6.2.** Let  $A \subset (0, 1]$  be the well known non-measurable Vitaly set. Let  $f_\varepsilon = 1, \varepsilon \in A$ ,  $f_\varepsilon = 0, \varepsilon \in (0, 1] \setminus A$ . Then  $[f_\varepsilon]$  shows that  $\mathcal{G}_{me}(\Omega)$  is strictly contained in  $\mathcal{G}(\Omega)$ .

Let  $q \in [1, \infty)$  and  $p \in [1, \infty]$ . We say that a net of  $(f_\varepsilon) \in \mathcal{E}(\Omega)^{(0,1)}$  belongs to  $\mathcal{E}_{q, L_{loc}^p}(\Omega)$ , respectively to  $\mathcal{N}_{q, L_{loc}^p}(\Omega)$  if it is measurable, locally bounded on  $(0, 1]$ , with respect to  $\varepsilon$ , for every fixed  $x \in \Omega$  and it satisfies the growth estimates

$$(\forall k \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\exists s \in \mathbb{R})\left(\int_0^1 \varepsilon^{qs} \|f_\varepsilon\|_{W^{k,p}(\omega)}^q d\varepsilon / \varepsilon < \infty\right), \quad (6.1)$$

respectively,

$$(\forall k \in \mathbb{N}_0)(\forall \omega \subset\subset \Omega)(\forall s \in \mathbb{R})\left(\int_0^1 \varepsilon^{qs} \|f_\varepsilon\|_{W^{k,p}(\omega)}^q d\varepsilon / \varepsilon < \infty\right). \quad (6.2)$$

By the Sobolev lemma, it follows that these spaces of nets are independent of the value of  $p$ . We therefore set

$$\mathcal{E}_{q, me}(\Omega) := \mathcal{E}_{q, me, L_{loc}^\infty}(\Omega), \quad \mathcal{N}_{q, me}(\Omega) := \mathcal{N}_{q, me, L_{loc}^\infty}(\Omega) \quad \text{and}$$

$$\mathcal{G}_q(\Omega) := \mathcal{E}_{q, me}(\Omega) / \mathcal{N}_{q, me}(\Omega).$$

We shall call the elements of  $\mathcal{E}_{q, me}(\Omega)$  nets of smooth functions with  $L^q$ -moderate growth, while the ones of  $\mathcal{N}_q(\Omega)$  will be refereed as  $L^q$ -negligible nets. In the same way we define  $\mathcal{E}_{q, sm}(\Omega)$ ,  $\mathcal{N}_{q, sm}(\Omega)$ , and  $\mathcal{G}_{q, sm}(\Omega)$  as well as the spaces with the continuous representatives with respect to  $\varepsilon$  (with subindex *co*). Moreover, we have shown:

**Proposition 6.3.** *Let  $q \in [1, \infty)$ . Every  $f \in \mathcal{G}_q(\Omega)$  has a representative  $(f_\varepsilon)_\varepsilon$  for which the function  $(x, \varepsilon) \mapsto f_\varepsilon(x) \in C^\infty(\Omega \times (0, 1))$ .*

Thus, we have

$$\mathcal{G}_q(\Omega) = \mathcal{G}_{q,me}(\Omega) = \mathcal{G}_{q,co}(\Omega) = \mathcal{G}_{q,sm}(\Omega), q \in [1, \infty). \quad (6.3)$$

Hence we may always use nets which are smooth with respect to  $\varepsilon$ .

When  $q = \infty$ , we can define two different spaces associated to the  $q = \infty$  (cf. Proposition 6.1):

$$\mathcal{G}_{\infty,me}(\Omega) = \mathcal{G}_{me}(\Omega) \quad \text{and} \quad \mathcal{G}_{\infty,co}(\Omega) = \mathcal{G}_{co}(\Omega) = \mathcal{G}_{\infty,sm}(\Omega) = \mathcal{G}_{sm}(\Omega)$$

but Proposition 6.3 does not hold for these two spaces. We shall therefore make a choice for the index  $q = \infty$ . Our convention is:

$$\mathcal{G}_\infty(\Omega) := \mathcal{G}_{co}(\Omega) = \mathcal{G}_{sm}(\Omega).$$

Summarizing, Proposition holds for  $\mathcal{G}_p(\Omega)$  for every  $p \in [1, \infty]$ . Furthermore, without loss of generality, we will assume in the sequel that:

*all the representatives are continuous with respect to  $\varepsilon$ ,  $q \in [1, \infty]$ .*

We also write from now on

$$\mathcal{E}_q(\Omega) := \mathcal{E}_{q,co,L_{loc}^p}(\Omega), \quad \mathcal{N}_q(\Omega) := \mathcal{N}_{q,co,L_{loc}^p}(\Omega), \quad p \in [1, \infty].$$

It is worth mentioning that  $\mathcal{D}'(\Omega)$  is embedded into each of the spaces  $\mathcal{G}_q(\Omega)$  in the same way that it is embedded into the Colombeau algebra of generalized functions.

Note also that  $\mathcal{E}_{q'}(\Omega) \subsetneq \mathcal{E}_q(\Omega)$  and  $\mathcal{N}_{q'}(\Omega) \subsetneq \mathcal{N}_q(\Omega)$  if  $q' > q$ . These assertions are shown by Example 6.4 below. This implies that there exists a canonical linear mapping  $\mathcal{G}_{q'}(\Omega) \rightarrow \mathcal{G}_q(\Omega)$ ,  $q' > q$ . As a matter of fact, this mapping is not injective, as the next elementary example shows.

**Example 6.4.** Consider a net given by  $f_\varepsilon(x) = n^{-2}e^{n/q}$ ,  $x \in \mathbb{R}$  if  $\varepsilon \in [n^{-1} - e^{-n}, n^{-1} + e^{-n}]$  and  $n \geq 4$  and  $f_\varepsilon(x) = 0$  otherwise. Then,  $(f_\varepsilon) \in \mathcal{E}_q(\mathbb{R})$ ,  $q' \leq q$  but  $(f_\varepsilon) \notin \mathcal{E}_{q'}(\mathbb{R})$  if  $q' > q$ .

The space  $\mathcal{G}_q(\Omega)$ ,  $1 \leq q < \infty$ , is not an algebra. This follows from Example 6.4 since  $(f_\varepsilon^2)_\varepsilon$  does not belong to  $\mathcal{E}_q(\mathbb{R})$ . Nevertheless, pointwise multiplication on the representative induces a well defined mapping on the corresponding quotients, which operates according to

$$(f, g) \in \mathcal{G}_{q'}(\Omega) \times \mathcal{G}_q(\Omega) \mapsto f \cdot g \in \mathcal{G}_r(\Omega), \quad \frac{1}{q'} + \frac{1}{q} = \frac{1}{r},$$

as a consequence of Hölder's inequality. In particular, we obtain the ensuing result.

**Theorem 6.5.** *Let  $q \in [1, \infty]$ . The space  $\mathcal{G}_p(\Omega)$  is a module over the algebra  $\mathcal{G}_\infty(\Omega) (= \mathcal{G}_{co}(\Omega))$  under the natural multiplication. Furthermore, it is a differential module, i.e.,*

$$(f \cdot g)^{(\alpha)} = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} f^{(\beta)} \cdot g^{(\alpha-\beta)}.$$

Analogously to the generalized numbers  $\tilde{\mathbb{R}}$  and  $\tilde{\mathbb{C}}$ , one can define the new sets of generalized numbers  $\tilde{\mathbb{R}}_q$  and  $\tilde{\mathbb{C}}_q$ ,  $q \in [1, \infty]$ . Denote by  $\mathcal{E}_{0,q}$ , resp.,  $\mathcal{N}_{0,q}$  spaces of continuous with respect to  $\varepsilon$  nets,  $(r_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]}$  with the property

$$(\exists a \in \mathbb{R}) \left( \int_0^1 \varepsilon^{aq} |r_\varepsilon|^q \frac{d\varepsilon}{\varepsilon} < \infty \right)$$

resp.,

$$(\forall b < 0) \left( \int_0^1 \varepsilon^{bq} |r_\varepsilon|^q \frac{d\varepsilon}{\varepsilon} < \infty \right).$$

Then  $\tilde{\mathbb{C}}_q = \mathcal{E}_{0,q}/\mathcal{N}_{0,q}$ ;  $\tilde{\mathbb{R}}_q$  is defined with the real nets above.

The sets  $\tilde{\mathbb{R}}_{co} = \tilde{\mathbb{R}}_\infty$  and  $\tilde{\mathbb{C}}_{co} = \tilde{\mathbb{C}}_\infty$  are rings, and  $\tilde{\mathbb{R}}_q$  and  $\tilde{\mathbb{C}}_q$  become modules over them, respectively.

In particular,  $\mathcal{G}_q(\Omega)$  becomes a module over the ring of generalized constants  $\tilde{\mathbb{C}}_{co}$ .

## 7. Besov type spaces of generalized functions [37]

In the sequel we will consider representatives of generalized functions consisting of continuous functions with respect to  $\varepsilon \in (0, 1]$ , as discussed in the previous section.

Let  $p \in [1, \infty]$ ,  $q \in [1, \infty]$ . We consider  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_q(\Omega)$  such that for given  $k \in \mathbb{N}$  and  $s \in \mathbb{R}$  there holds

$$(\forall \omega \subset \subset \Omega) \left( \int_0^1 \varepsilon^{sq} \|f_\varepsilon\|_{W^{k,p}(\omega)}^q d\varepsilon / \varepsilon < \infty \right). \quad (7.1)$$

We say that a net  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_q(\Omega)$  belongs to  $\mathcal{E}_{q,L_{loc}^p}^{k,-s}(\Omega)$  if (7.1) holds. Special attention will be devoted in Section 9 to Zygmund type spaces in the case  $q = \infty$ .

Furthermore, for  $k = \infty$ , we put

$$\mathcal{E}_{q,L_{loc}^p}^{\infty,-s}(\Omega) = \bigcap_{k \in \mathbb{N}} \mathcal{E}_{q,L_{loc}^p}^{k,-s}(\Omega).$$

The spaces of the following definition will be vital in our study of Besov type regularity.

**Definition 7.1.** Let  $s \in \mathbb{R}$ ,  $k \in \mathbb{N}_0 \cup \{\infty\}$ ,  $q \in [1, \infty]$ , and  $p \in [1, \infty]$ . Then  $\mathcal{G}_{q, L_{loc}^p}^{k, -s}(\Omega)$  is the quotient space

$$\mathcal{G}_{q, L_{loc}^p}^{k, -s}(\Omega) = \mathcal{E}_{q, L_{loc}^p}^{k, -s}(\Omega) / \mathcal{N}_q(\Omega). \quad (7.2)$$

We have  $\mathcal{G}_{q, L_{loc}^p}^{k, -s}(\Omega) \subset \mathcal{G}_q(\Omega)$  for any  $p \in [1, \infty]$ . Note that the definition does not depend on the representatives.

We list some properties of these vector spaces of generalized functions in the next proposition.

**Proposition 7.2.** Let  $s \in \mathbb{R}$ ,  $k \in \mathbb{N}_0 \cup \{\infty\}$ ,  $q \in [1, \infty]$  and  $p \in [1, \infty]$ .

- (i)  $\mathcal{G}_{q, L_{loc}^p}^{k, -s}(\Omega) \subseteq \mathcal{G}_{q, L_{loc}^p}^{k_1, -s_1}(\Omega)$  if and only if  $k \geq k_1$  and  $s \leq s_1$ .
- (ii) Let  $P(D)$  be a differential operator of order  $m \leq k$  with constant coefficients. Then  $P(D) : \mathcal{G}_{q, L_{loc}^p}^{k, -s}(\Omega) \rightarrow \mathcal{G}_{q, L_{loc}^p}^{k-m, -s}(\Omega)$ .
- (iii) Let  $\infty > r > q$ ,  $\rho < s$ . Then  $\mathcal{G}_{q, L_{loc}^p}^{m, -s}(\Omega) \subset \mathcal{G}_{r, L_{loc}^p}^{m, -\rho}(\Omega)$ .
- (iv)  $\mathcal{G}_{q, L_{loc}^p}^{\infty, -s}(\Omega) = \mathcal{G}_{q, L_{loc}^\infty}^{\infty, -s}(\Omega)$ ,

## 8. Characterization of Besov regularity of distributions [37]

We defined in [38] non-degenerate wavelets and generalized Littlewood-Paley (LP) pairs of order  $\alpha \in \mathbb{R}$ . Here we will simplify the exposition considering special (LP) pairs defined as follows. A (LP) pair is  $(\phi_1, \psi_1)$ , where  $\hat{\phi}_1, \hat{\psi}_1 \in \mathcal{D}(\mathbb{R}^d)$ ,  $\hat{\phi}_1 \equiv 1$  in a ball  $B(0, r)$ ,  $\text{supp } \hat{\psi}_1 \subset B(0, r_1) \setminus B(0, r_2)$ ,  $r_1 > r > r_2$ ,  $\hat{\psi}_1 \equiv 1$  in a neighborhood of  $S(0, r)$  ( $B(0, \rho)$  denotes an open ball whose boundary is the sphere  $S(0, \rho)$ ).

Clearly this definition is satisfied by the special (LP) pair  $(\varphi, \psi)$  from Section 2. In the sequel, we will assume that the mollifier  $\phi$  is chosen so that

$$(\varphi * \phi, \psi * \phi) \quad (8.1)$$

makes an (LP) pair.

**Proposition 8.1.** Let  $(\phi_1, \psi_1)$  be a Littlewood-Paley pair as above. Define  $\|u\|_{(1), q, p}^s$  with this pair, see (2.4). Then, for  $k = |\alpha|$ ,

$$\|\psi_\varepsilon * u^{(\alpha)}\|_{L^p(\mathbb{R}^d)} \leq C \|(\psi_1)_\varepsilon * u^{(\alpha)}\|_{L^p(\mathbb{R}^d)}.$$

In particular the norms  $\|u\|_{q, p}^s$  and  $\|u\|_{(1), q, p}^s$  are equivalent.

The space  $B_{q,L^p}^s(\mathbb{R}^d) \cap \mathcal{E}'(\Omega)$  is naturally embedded into  $\mathcal{G}_q(\Omega)$ , through convolution with a mollifier:  $T \mapsto T_\varepsilon = T * \phi_\varepsilon|_\Omega, \varepsilon \in (0, 1)$ .

**Proposition 8.2.** *Let  $f \in \mathcal{E}'(\mathbb{R}^d)$  such that*

$$\iota(f) \in \iota(\mathcal{D}'(\mathbb{R}^d)) \cap \mathcal{G}_{q,L_{loc}^p}^{0,s}(\mathbb{R}^d).$$

*Then  $f \in B_{q,L^p}^s$ .*

**Theorem 8.3.** *Let  $s > 0$  and  $k \in \mathbb{N}$ . Then,*

$$\mathcal{G}_{q,L_{loc}^p}^{k,-s}(\mathbb{R}^d) \cap \iota(\mathcal{E}'(\mathbb{R}^d)) \supset \iota(B_{q,L^p}^{k-s_0})$$

*for any  $s_0 < s$ .*

Recall that a net  $(f_\varepsilon)_\varepsilon \in \mathcal{E}^{(0,1)}(\Omega)$ , or the generalized function  $f = [(f_\varepsilon)_\varepsilon]$ , is strongly associated to  $T \in \mathcal{D}'(\Omega)$  if there exists  $b > 0$  such that

$$(\forall \rho \in \mathcal{D}(\Omega)) (\langle T - f_\varepsilon, \rho \rangle = o(\varepsilon^b), \varepsilon \rightarrow 0). \quad (8.2)$$

With  $o(1)$  instead  $o(\varepsilon^b)$  in (8.2), one has the notion of weak association. We introduce a new concept of association.

**Definition 8.4.** *Let  $T \in \mathcal{D}'(\mathbb{R}^d)$ ,  $(f_\varepsilon)_\varepsilon \in \mathcal{E}_q(\mathbb{R}^d)$ . We say that  $(f_\varepsilon)_\varepsilon$  is strongly  $q$ -associated to  $T$  if there exists  $b > 0$  such that*

$$(\forall \rho \in \mathcal{D}(\mathbb{R}^d)) \left( \int_0^1 \varepsilon^{-bq} |\langle T - f_\varepsilon, \rho \rangle|^q d\varepsilon / \varepsilon < \infty \right). \quad (8.3)$$

Clearly the strong association implies the  $q$ -association and the converse does not hold. Moreover the weak association and the  $q$  associations are not comparable.

**Theorem 8.5.** *Let  $T \in \mathcal{E}'(\mathbb{R}^d)$  and  $[(f_\varepsilon)_\varepsilon] \in \mathcal{G}_{q,\infty}^{k,s}(\mathbb{R}^d)$  for some  $k \in \mathbb{N}$  and every  $s > 0$ . Assume that  $T$  and  $(f_\varepsilon)_\varepsilon$  are strongly  $q$ -associated,  $q \geq 1$ . Then  $\iota(T) \in \mathcal{G}_{q,L_{loc}^\infty}^{k,s}$  for every  $s > 0$ . In particular,  $T \in B_{q,\infty}^{k+s}$  for every  $s > 0$ .*

## 9. Zygmund regularity through association [32]

In this section we present the local regularity of distributions in connection with the Zygmund type classes  $\mathcal{G}_{\infty,L_{loc}^\infty}^{k,-s}(\Omega)$  (see (7.1) with  $q = \infty$ )

The next theorem provides a precise characterization of those distributions that belong to  $\mathcal{G}_{\infty,L_{loc}^\infty}^{k,-s}(\Omega)$ , denoted in the sequel as  $\mathcal{G}^{k,-s}(\Omega)$ , they turn out to be elements of a Zygmund space. We only consider the case  $s > 0$ , since for  $s \leq 0$ , one has  $\mathcal{G}^{k,-s}(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \{0\}$ .



**Theorem 9.1.** *Let  $s > 0$ . We have  $\mathcal{G}^{k,-s}(\Omega) \cap \iota(\mathcal{D}'(\Omega)) = \iota(C_{*,loc}^{k-s}(\Omega))$ .*

This implies that for  $r \in \mathbb{R}$  and any non-negative integer  $k > r$ ,

$$\iota(C_{*,loc}^r(\Omega)) = \mathcal{G}^{k,r-k}(\Omega) \cap \iota(\mathcal{D}'(\Omega)).$$

Consequently, we immediately have  $\iota(\mathcal{D}'(\Omega)) \cap \mathcal{G}^\infty(\Omega) = \iota(C^\infty(\Omega))$ .

We return to the strong association but with the more general rate of approximation in (8.2). Let  $R : (0, 1] \rightarrow \mathbb{R}_+$  be a positive function such that  $R(\varepsilon) = o(1)$ ,  $\varepsilon \rightarrow 0$ . We write  $T - f_\varepsilon = O(R(\varepsilon))$  in  $\mathcal{D}'(\Omega)$  if

$$(\forall \rho \in \mathcal{D}(\Omega)) (\langle T - f_\varepsilon, \rho \rangle = O(R(\varepsilon)), \varepsilon < 1).$$

We now present our results concerning the regularity analysis through association.

**Theorem 9.2.** *Let  $T \in \mathcal{D}'(\Omega)$  and let  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$  be associated to it. Assume that  $f \in \mathcal{G}^\infty(\Omega)$ . If  $(f_\varepsilon)_\varepsilon$  approximates  $T$  with convergence rate:*

$$(\exists b > 0)(T - f_\varepsilon = O(\varepsilon^b) \text{ in } \mathcal{D}'(\Omega)). \quad (9.1)$$

*Then  $T \in C^\infty(\Omega)$ .*

**Theorem 9.3.** *Let  $T \in \mathcal{D}'(\Omega)$  and let  $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$  be a net of smooth functions associated to it. Furthermore, let  $k \in \mathbb{N}$ . Assume that either of following pair of conditions hold:*

(i)  $f \in \mathcal{G}^{k,-a}(\Omega)$ ,  $\forall a > 0$ , namely,

$$(\forall a > 0)(\forall \omega \subset \subset \Omega)(\forall \alpha \in \mathbb{N}^d, |\alpha| \leq k)(\sup_{x \in \omega} |f_\varepsilon^{(\alpha)}(x)| = O(\varepsilon^{-a})), \quad (9.2)$$

*and the convergence rate of  $(f_\varepsilon)_\varepsilon$  to  $T$  is as in (9.1).*

(ii)  $f \in \mathcal{G}^{k,-s}(\Omega)$  for some  $s > 0$ , and there is a rapidly decreasing function  $R : (0, 1] \rightarrow \mathbb{R}_+$ , i.e.,  $(\forall a > 0)(\lim_{\varepsilon \rightarrow 0} \varepsilon^{-a} R(\varepsilon) = 0)$ , such that

$$T - f_\varepsilon = O(R(\varepsilon)) \text{ in } \mathcal{D}'(\Omega). \quad (9.3)$$

*Then,  $T \in C_{*,loc}^{k-\eta}(\Omega)$  for every  $\eta > 0$ .*

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