A numerical approximation for solutions of Hammerstein integral equations in L^p spaces

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Abstract. In this work, we give conditions that guarantee the existence and the uniqueness of the solution of the Hammerstein integral equation in the L^p space and under such assumptions the successive approximation converges almost everywhere to the solution of the equation. Finally, we treat numerical exemples to confirm these results.

1. Introduction

Some phenomena which appear in many areas of scientific fields such as plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modelled by nonlinear integral equations in particular Hammerstein integral equations $[1, 4, 6, 13]$. A broad class of analytical solutions methods and numerical solutions methods were used in handling these problems $[3, 5, 7]$. Also this type of equations occur of scattering and radiation of surface water wave, where due to the Green's function we can transform any ordinary differential equation of the second order with boundary conditions into an Hammerstein integral equation of the general form

$$\varphi(t_0) = \int_0^1 k(t, t_0) l(t, \varphi(t)) dt, \tag{1}$$

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where $k(t,t_0)$ is a map from $[a,b]\times[a,b]$, into $\mathbb{R}$, $l(t,\varphi(t)$ a nonlinear map from $[a,b]\times\mathbb{R}$, into $\mathbb{R}$ and the unknown $\varphi(t)$ is defined on $[a,b]$. The equation (1) can be put in the form of a nonlinear functional equation

$$\varphi + KL\varphi(t) = 0, \tag{2}$$

with the linear and nonlinear mappings K and L respectively given by

$$K\psi(t_0) = \int_0^1 k(t,t_0)\psi(t)dt,\ \ L\varphi(t) = l(t,\varphi(t)). \tag{3}$$

In this work we ensure that under weaker conditions the Niemitskyi operator L is well-defined on the space $L^q([a,b])$ of functions on the interval $[a,b]$, and that for each element φ of $L^p([a,b])$, the superposition operator L lies in the space $L^q([a,b])$. Also the linear operator K maps the space $L^q([a,b])$ into the space $L^p([a,b])$ and therefore the composition KL of the two operators is well-defined and maps $L^p([a,b])$ into itself.

1) the function$l(t,\varphi(t))$ is strongly measurable in t and continuous in φ

2) $\|l(t,\varphi(t))\| \le a_0(t)+b_0\|\varphi\|$ for $t\in[a,b]$ and $\varphi\in\mathbb{R}$, where $a_0\in L^q([a,b])$ such that $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $b_0\ge 0$.

Let us recall that the existence theorems for solutions of (1) with a kernel $k(t,t_0)\in L^p([a,b]\times[a,b])$ were proved in the papers [8,12]. Obviously, in this paper the kernel $k(t,t_0)$ is not necessarily integrable in $L^p([a,b]\times[a,b])$.

2. Main Results

Theorem 1

Suppose that the functions $k(t_0,t)$ and $l(t,\varphi(t))$ satisfy the following conditions

(A1) *The kernel $k(t_0,t)$ is measurable on $[a,b]\times[a,b]$ and such that*

$$\left(\int_a^b |k(t_0,t)|^\sigma\, dt_0\right)^{\frac{1}{\sigma}} \le M_1,\ \text{ for all } t\in[a,b],$$

where $\sigma<p$ and $\sigma,p>1$.

(A2) *The kernel $k(t_0,t)$ is measurable on $[a,b]\times[a,b]$ and such that*

$$\left(\int_a^b |k(t_0,t)|^{\frac{p-\sigma}{p-1}}\, dt\right)^{\frac{p-1}{p-\sigma}} \le M_2,\ \ \text{for all } t_0\in[a,b].$$

(A3) *The function* $l(t,\varphi(t))$ *is a nonlinear map from* $[a,b]\times\mathbb{R}$, *into* $\mathbb{R}$ *satisfying the Carathéodory condition and such that*

$$|l(t,\varphi(t))| \le a_0(t) + b_0\,|\varphi(t)|^{\frac{p}{q}},$$

where $a_0(t)\in L^q([a,b],\mathbb{R})$, $b_0>0$ and $\frac{1}{p}+\frac{1}{q}=1$

Under conditions **(A1)**, **(A2)**, **(A3)** *the operator*

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt, \tag{4}$$

is a map from L^p *into* L^p.

Proof. From the condition **(A3)**, we can write

$$|l(t,\varphi(t))|^q \le \left(|a_0(t)| + b_0\,|\varphi(t)|^{\frac{p}{q}}\right)^q,$$

and therefore

$$\|l(t,\varphi(t))\|_q = \left(\int_a^b |l(t,\varphi(t))|^q\,dt\right)^{\frac{1}{q}} \le \left(\int_a^b \left(|a_0(t)| + b_0\,|\varphi(t)|^{\frac{p}{q}}\right)^q dt\right)^{\frac{1}{q}}.$$

Using Minkovski's inequality, it comes

$$\begin{aligned}\|l(t,\varphi(t))\|_q &\le c\left(\left(\int_a^b |a_0(t)|^q\right)^{\frac{1}{q}} + \left(\int_a^b b_0^q\,|\varphi(t)|^p\right)^{\frac{1}{q}}\right)\\ &\le c\left(\|a_0(t)\|_q + b_0\,\|\varphi(t)\|_p^{\frac{p}{q}}\right).\end{aligned}$$

Hence the operator $l(t,\varphi(t))$ is a continuous element of $L^q([a,b],\mathbb{R})$ [4].
However, on the space $L^p([a,b],\mathbb{R})$ we consider,

$$A\varphi(t_0) = \int_a^b k(t_0,t)l(t,\varphi(t))dt,$$

where following [1], we have

$$\begin{aligned}|A\varphi(t_0)| &= \left|\int_a^b k(t_0,t)l(t,\varphi(t))dt\right|,\\ &\le \int_a^b |k(t_0,t)l(t,\varphi(t))|\,dt,\\ &= \int_a^b \left(|k(t_0,t)|^\sigma\,|l(t,\varphi(t))|^q\right)^{\frac{1}{p}}\,|k(t_0,t)|^{1-\frac{\sigma}{p}}\,|l(t,\varphi(t))|^{1-\frac{q}{p}}\,dt,\end{aligned}$$

$$\leq \left(\int_a^b |k(t_0,t)|^\sigma |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{p}} \left(\int_a^b |k(t_0,t)|^{p-\sigma} \, dt\right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t))|^q \, dt\right)^{\frac{p-q}{pq}},$$

$$|A\varphi(t_0)| \leq M_2^{\frac{(p-\sigma)}{p}} \|l(t,\varphi(t))\|^{\frac{(p-q)}{p}} \left(\int_a^b |k(t_0,t)|^\sigma |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{p}},$$

or again,

$$|A\varphi(t_0)|^p \leq \left(M_2^{\frac{(p-\sigma)}{p}} \|l(t,\varphi(t))\|^{\frac{(p-q)}{p}} \left(\int_a^b |k(t_0,t)|^\sigma |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{p}}\right)^p$$

$$\left(\int_a^b |A\varphi(t_0)|^p \, dt_0\right)^{\frac{1}{p}}$$

$$\leq M_2^{\frac{(p-\sigma)}{p}} \|l(t,\varphi(t))\|^{\frac{(p-q)}{p}} \left(\int_a^b \int_a^b |k(t_0,t)|^\sigma |l(t,\varphi(t))|^q \, dt dt_0\right)^{\frac{1}{p}}$$

$$\leq M_2^{\frac{(p-\sigma)}{p}} \|l(t,\varphi(t))\|^{1-\frac{q}{p}} \left(\int_a^b |k(t_0,t)|^\sigma \, dt_0\right)^{\frac{1}{p}} \left(\int_a^b |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{p}}$$

$$\leq M_2^{\frac{(p-\sigma)}{p}} \|l(t,\varphi(t))\|^{1-\frac{q}{p}} M_1^{\frac{\sigma}{p}} \|l(t,\varphi(t))\|^{\frac{q}{p}}$$

$$\|A\varphi(t_0)\|_p \leq M_2^{\frac{(p-\sigma)}{p}} M_1^{\frac{\sigma}{p}} \|l(t,\varphi(t))\|_q.$$

Hence, the operator $A\varphi(t_0)$ is well defined from L^p to L^p by interpolation.
L^p solution

Consider the nonlinear integral equation

$$\varphi(t_0) = \int_a^b k(t_0,t) l(t,\varphi(t)) dt :$$

We would like to know what conditions one require on $k(t_0,t)$ and $l(t,\varphi(t))$ in order for this equation to have a solution $\varphi(t) \in L^p([a,b])$.

Theorem 2

Suppose that the functions $k(t_0,t)$ and $l(t,\varphi(t))$ satisfy the following conditions

(B1) *The kernel $k(t_0,t)$ belongs to the space L^p for all $t_0 \in [a,b]$*

$$\left(\int_a^b |k(t_0,t)|^p \, dt\right)^{\frac{1}{p}} \le N_1(t_0), \quad \forall t_0 \in [a,b].$$

(B2) *the function $l(t,\varphi(t))$ belongs to the space L^q for all $t \in [a,b]$*

$$\left(\int_a^b |l(t,\varphi(t))|^q \, dt\right)^{\frac{1}{q}} \le C,$$

and satisfying the Lipschitz condition

$$|l(t,\varphi_1(t)) - l(t,\varphi_2(t))| \le L(t)\, |\varphi_1(t) - \varphi_2(t)|\,,$$

with the function $L(t)$ belongs to the space $L^{\frac{pq}{p-q}}$ with $q \le p$,

$$\left(\int_a^b |L(t)|^{\frac{pq}{p-q}} \, dt\right)^{\frac{p-q}{pq}} \le N_2.$$

Under assumptions **(B1)** *and* **(B2)***, the successive approximation*

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t) l(t,\varphi_n(t)) dt,$$

converges almost everywhere to the solution of the equation (1) *provided*

$$N_2^p \int_a^b N_1^p(t) dt = N^p < 1.$$

Proof.

For this method we put $\varphi_0(t)$ as an identically null function and successively

$$\varphi_{n+1}(t_0) = \int_a^b k(t_0,t) l(t,\varphi_n(t)) dt, \quad n = 0,1,2,...,n..,$$

and therefore, we obtain

$$\begin{aligned}
|\varphi_{n+1}(t_0) - \varphi_n(t_0)| &\le \int_a^b |k(t_0,t)|\, |l(t,\varphi_n(t)) - l(t,\varphi_{n-1}(t))|\, dt, \\
|\varphi_{n+1}(t_0) - \varphi_n(t_0)| &\le \int_a^b |k(t_0,t)|\, L(t)\, |\varphi_n(t) - \varphi_{n-1}(t)|\, dt,
\end{aligned}$$

$$\le \left(\int_a^b |k(t_0,t)|^p \, dt\right)^{\frac{1}{p}} \left(\int_a^b |L(t)|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \left(\int_a^b |\varphi_n(t) - \varphi_{n-1}(t)|^p \, dt\right)^{\frac{1}{p}}.$$

Hence

$$|\varphi_{n+1}(t_0) - \varphi_n(t_0)|^p \le N_1^p(t_0) N_2^p \int_a^b |\varphi_{n+1}(t) - \varphi_n(t)|^p \, dt, \tag{5}$$

using the condition $\varphi_0(t) = 0$, we get

$$|\varphi_1(t_0)|^p \le N_1^p(t_0) \left(\int_a^b |l(t,0)|^q \, dt \right)^{\frac{p}{q}} = N_1^p(t_0) C^p,$$

and from (5), it comes

$$\begin{aligned} |\varphi_2(t_0) - \varphi_1(t_0)|^p &\le N_1^p t(_0) N_2^p \int_a^b N_1^p(t_0) C^p dt_0 = C^p N^p N_1^p(t_0), \\ |\varphi_3(t_0) - \varphi_2(t_0)|^p &\le N_1^p t(_0) N_2^p \int_a^b C^p N_1^p(t_0) N^p dt_0 = C^p N^{2p} N_1^p(t_0), \end{aligned}$$

more generally

$$|\varphi_{n+1}(t_0) - \varphi_n(t_0)|^p \le C^p N^{2np} N_1^p(t_0),$$

or again after simplification

$$|\varphi_{n+1}(t_0) - \varphi_n(t_0)| \le C N^{2n} N_1(t_0).$$

This expression gives that the sequence $\varphi_n(t_0)$ taken by the series

$$\varphi_1(t_0) + (\varphi_2(t_0) - \varphi_1(t_0)) + ... + (\varphi_p(t_0) - \varphi_{p-1}(t_0)) + ...,$$

has the majorant

$$C N_1(t_0)(1 + N + N^2 + ...N^{p-1} + ..$$

Naturally, this series converges. Hence the sequence $\varphi_n(t_0)$ converges to the solution of the equation (1).

3. Numerical Experiments

In this section we describe some of the numerical experiments performed in solving the Hammerstein integral equations (1). In all cases, the interval is $[0,1]$ and we chose the right hand side $f(t)$ in such way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with the method is correct $[2,9,10,11]$.

In each table, φ represents the given exact solution of the Hammerstein equation and $\widetilde{\varphi}$ corresponds to the approximate solution of the equation produced by the iterative method.

Example 1

Consider the Hammerstein integral equation

$$10\varphi(t_0) - \int_0^1 \exp(t^4 + t_0^4)(\varphi(t))^3 dt = 10t_0 - \frac{1}{4}(e-1)\exp(t_0^4),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation after N=10 iterations.

Points of t	Exact solution	Approx solution	Error	Error [3]
0.000000	0.000000e+000	2.209477e-005	2.209477e-005	2.1462e-003
0.250000	2.500000e-001	2.500222e-001	2.218125e-005	2.1546e-003
0.500000	5.000000e-001	5.000235e-001	2.351976e-005	2.2846e-003
0.750000	7.500000e-001	7.500303e-001	3.031818e-005	2.9450e-003
0.100000	1.000000e+000	1.000060e+000	6.005982e-005	2.8340e-003

Table 1. The exact and approximate solutions of example 1 in some arbitrary points, and the error compared with the ones treated in [3].

Example 2

Consider the Hammerstein integral equation

$$20\varphi(t_0) - \int_0^1 \sin(\exp(t)+t_0)\exp(\varphi(t))dt = 20t_0 + \cos(\exp(1)+t_0) - \cos(1+t_0),$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = t.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation after N=10 iterations.

Points of t	Exact solution	Approx solution	Error	Error [3]
0.000000	0.000000e+000	-3.184043e-006	3.184043e-006	1.6361e-005
0.250000	2.500000e-001	2.499964e-001	3.602525e-006	4.3978e-005
0.500000	5.000000e-001	4.999962e-001	3.797019e-006	6.8861e-005
0.750000	7.500000e-001	7.499962e-001	3.755433e-006	8.9462e-005
0.100000	1.000000e+000	9.999965e-001	3.480352e-006	1.0450e-004

Table 2. The exact and approximate solutions of example 2 in some arbitrary points, and the error compared with the ones treated in [3].

Example 3

Consider the Hammerstein integral equation

$$\varphi(t_0) - \int_0^1 tt_0(\varphi(t))^3 dt = \frac{1}{t_0^2+1} - \frac{3}{16}t_0,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \frac{1}{t^2+1}.$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation after N=10 iterations.

Points of t	Exact solution	Approx solution	Error	Error [5]
0.000000	1.000000e+000	1.000000e+000	0.000000e+000	0.000000e+000
0.200000	9.615385e-001	9.615348e-001	3.642846e-006	1.194620e-004
0.400000	8.620690e-001	8.620617e-001	7.285693e-006	2.389660e-004
0.600000	7.352941e-001	7.352832e-001	1.092854e-005	3.581180e-004
0.800000	6.097561e-001	6.097415e-001	1.457139e-005	4.780980e-004

Table 3. The exact and approximate solutions of example 3 in some arbitrary points, and the error compared with the ones treated in [5].

Example 4

Consider the Hammerstein integral equation

$$\varphi(t_0) - \int_0^1 \sin(t+t_0)\ln(\varphi(t))dt = \exp(t_0) - 0.382\sin(t_0) - 0.301\cos(t_0), \quad 0 \le t_0 \le 1,$$

where the function $f(t_0)$ is chosen so that the solution $\varphi(t)$ is given by

$$\varphi(t) = \exp(t).$$

The approximate solution $\widetilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the successive approximation after N=10 iterations.

Points of t	Exact solution	Approx solution	Error	Error [8]
0.000000	1.000000e+000	1.000195e+000	1.953229e-004	0.000000e+000
0.200000	1.221403e+000	1.221559e+000	1.567282e-004	1.940000e-004
0.400000	1.491825e+000	1.491937e+000	1.118852e-004	5.410000e-004
0.600000	1.822119e+000	1.822181e+000	6.258175e-005	3.360000e-004
0.800000	2.225541e+000	2.225552e+000	1.078332e-005	2.890000e-004

Table 4. The exact and approximate solutions of example 4 in some arbitrary points, and the error compared with the ones treated in [8].

4. Conclusion

In this work we remark the convergence of the successive approximation method to the exact solution with a considerable accuracy for the Hammerstein integral equation under conditions of the theorems cited above: This numerical results show that the accuracy improves with increasing of the number of iterations. Finally we confirm that the theorems cited above lead us to the good approximation of the exact solution.

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